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Buckling analysis of linearly tapered micro-columns based on strain gradient elasticity

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Abstract. The buckling problem of linearly tapered micro-columns is investigated on the basis of modified strain gradient elasticity theory. Bernoulli-Euler beam theory is used to model the non-uniform micro column. Rayleigh-Ritz solution method is utilized to obtain the critical buckling loads of the tapered cantilever micro-columns for different taper ratios. Some comparative results for the cases of rectangular and circular cross-sections are presented in graphical and tabular form to show the differences between the results obtained by modified strain gradient elasticity theory and those achieved by modified couple stress and classical theories. From the results, it is observed that the differences between critical buckling loads achieved by classical and those predicted by non-classical theories are considerable for smaller values of the ratio of the micro-column thickness (or diameter) at its bottom end to the additional material length scale parameters and the differences also increase due to increasing of the taper ratio.

Keywords: buckling; strain gradient theory; size effect; Rayleigh-Ritz method; non-uniform micro-column

1. Introduction

One of the main features that required of engineering structures is to be economical in addition to being safety, functional and aesthetic. Therefore, structural members with variable cross-section or material properties like beams, columns, plates and shells are frequently used in many applications of civil, mechanical and aerospace engineering. As a result of using these type structural members, strength and structural efficiency may be increased while total cost and weight may be reduced. Buckling analysis of non-uniform or non-homogenous beams/columns has been investigated by many researches (Timoshenko and Gere 1961, Gere and Carter 1962, Elishakoff and Bert 1988, Eisenberger 1991, Wang *et al.* 2005, Darbandi *et al.* 2010).

Recently, micro- and nano-sized structures and devices such as biosensors, atomic force microscope, microactuators, and nano probes, in which their cross-sections can be either constant or variable along the longitudinal direction, have been widely used in microelectromechanical (MEMS) and nanoelectromechanical systems (NEMS). In these applications, it is observed that the size effect has a major role on static and dynamic behavior of material (Fleck *et al.* 1994, Lam *et al.* 2003). The size effect cannot be taken into consideration by classical continuum theories due to lack of any material length scale parameters. Then, higher-order (nonclassical) continuum theories,

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which include least one additional material length scale parameter in addition to classical ones, have been proposed to predict the size dependence of these small-scale structures.

Using nonlocal elasticity theory (Eringen 1983), Lim and Wang (2007) and Lim (2009, 2010) studied static bending problem of nanobeams and obtained governing equation and boundary conditions with the aid of a variational principle. In the last decade, on the basis of the nonlocal elasticity theory, many studies have been carried out for static and dynamic analysis of small-sized structures such as carbon nanotubes (Sudak 2003, Zhang *et al.* 2004, 2006, Liu *et al*. 2005, Wang and Liew 2007, Zhang and Shen 2007, Liew *et al.* 2008, Wang *et al.* 2008, Demir *et al.* 2010, Lim *et al.* 2010, Şimşek 2011), microtubules (Civalek and Akgöz 2010, Civalek *et al.* 2010, Civalek and Demir 2011,Shen 2010) and graphene sheets (Shen *et al.* 2010a, b). Zhang *et al.* (2006) investigated buckling analysis of double-walled carbon nanotubes on an elastic medium by energy method with considering effect of van der Waals forces.

Modified couple stress theory is a higher-order continuum theory, has been elaborated by Yang *et al.* (2002) which contains a new higher-order equilibrium relation for moments of couples besides conventional (classical) equilibrium relations for forces and moments of forces. Furthermore, this convenient theory involves only one additional material length scale parameter and a symmetric couple stress tensor. Based on this theory, Şimşek (2010) investigated dynamic analysis of an embedded microbeam carrying a moving microparticle. Recently, free vibration analysis of axially functionally graded tapered microbeams is studied with modified couple stress theory (Akgöz and Civalek 2013a).

Modified strain gradient theory is another higher-order continuum theory, was developed by Lam *et al.* (2003) which includes two higher-order stress components in addition to classical and couple stresses. This theory has been employed to analyze for static and dynamic behaviors of linear homogenous microbeams by Akgöz and Civalek (2011a, b, 2012, 2013b).

The objective of this study is to analyze the buckling problem of linearly tapered micro-columns in conjunction with Bernoulli-Euler beam and modified strain gradient elasticity theory. Rayleigh-Ritz solution method is utilized to obtain the critical buckling loads for different taper ratios. Some comparative results are presented in graphical and tabular form to show the differences between the results obtained by modified strain gradient elasticity theory and those achieved by modified couple stress and classical elasticity theories.

2. Formulation

The modified strain gradient elasticity theory proposed by Lam *et al.* (2003) in which contains a new additional equilibrium equation besides the classical equilibrium equations and also three material length scale parameters besides two classical ones for isotropic linear elastic materials. The strain energy U in a linear elastic isotropic material occupying region Ω based on the modified strain gradient elasticity theory can be written by (Lam *et al.* 2003)

$$U = \frac{1}{2}\int_{\Omega}\left(\sigma_{ij}\varepsilon_{ij} + p_i\gamma_i + \tau_{ijk}^{(1)}\eta_{ijk}^{(1)} + m_{ij}^{s}\chi_{ij}^{s}\right) dv \qquad (1)$$

$$\varepsilon_{ij} = \frac{1}{2}\left(u_{i,j} + u_{j,i}\right) \qquad (2)$$

$$\varepsilon_{mm,i} = \gamma_i \tag{3}$$

$$\eta_{ijk}^{(1)} = \frac{1}{3}\left(\varepsilon_{jk,i} + \varepsilon_{ki,j} + \varepsilon_{ij,k}\right) - \frac{1}{15}\left[\left(\delta_{ij}(\varepsilon_{mm,k} + 2\varepsilon_{mk,m})\right.\right.$$
$$\left.\left. + \delta_{jk}(\varepsilon_{mm,i} + 2\varepsilon_{mi,m}) + \delta_{ki}(\varepsilon_{mm,j} + 2\varepsilon_{mj,m})\right)\right] \tag{4}$$

$$\chi_{ij}^{(s)} = \frac{1}{2}\left(\theta_{i,j} + \theta_{j,i}\right) \tag{5}$$

$$\theta_i = \frac{1}{2} e_{ijk} u_{k,j} \tag{6}$$

where u_i, θ_i, ε_{ij}, γ_i, $\eta_{ijk}^{(1)}$ and χ_{ij}^{s} denote the components of the displacement vector $\mathbf{u}$, the rotation vector $\boldsymbol{\theta}$, the strain tensor $\boldsymbol{\varepsilon}$, the dilatation gradient vector $\boldsymbol{\gamma}$, the deviatoric stretch gradient tensor $\boldsymbol{\eta}^{(1)}$ and the symmetric rotation gradient tensor $\boldsymbol{\chi}^{s}$, respectively. Also, δ is the symbol of Kronecker delta and e_{ijk} is the permutation symbol. Furthermore, the components of the classical stress tensor $\boldsymbol{\sigma}$ and the higher-order stress tensors $\mathbf{p}$, $\boldsymbol{\tau}^{(1)}$ and $\mathbf{m}^{s}$ defined as (Lam *et al.* 2003)

$$\sigma_{ij} = \lambda \varepsilon_{mm} \delta_{ij} + 2G\varepsilon_{ij} \tag{7}$$

$$p_i = 2Gl_0^2 \gamma_i \tag{8}$$

$$\tau_{ijk}^{(1)} = 2Gl_1^2 \eta_{ijk}^{(1)} \tag{9}$$

$$m_{ij}^{s} = 2Gl_2^2 \chi_{ij}^{s} \tag{10}$$

where l_0, l_1, l_2 are additional material length scale parameters related to dilatation gradients, deviatoric stretch gradients and rotation gradients, respectively. Furthermore, λ and G are the Lamé constants defined as

$$\lambda = \frac{Ev}{(1+v)(1-2v)}, \quad G = \frac{E}{2(1+v)} \tag{11a, b}$$

Using above equations into Eq. (1) with neglecting Poisson's effect and some mathematical manipulations, we obtain an expression for the strain energy U

$$U = \frac{1}{2}\int_0^L \left[B.(w'')^2 + D.(w''')^2\right] dx \tag{12}$$

where

$$w'' = \frac{d^2 w}{dx^2}, \quad w''' = \frac{d^3 w}{dx^3} \tag{13a, b}$$

$$B = EI + 2GAl_0^2 + \frac{8}{15}GAl_1^2 + GAl_2^2, \quad D = I\left(2Gl_0^2 + \frac{4}{5}Gl_1^2\right) \tag{14a, b}$$

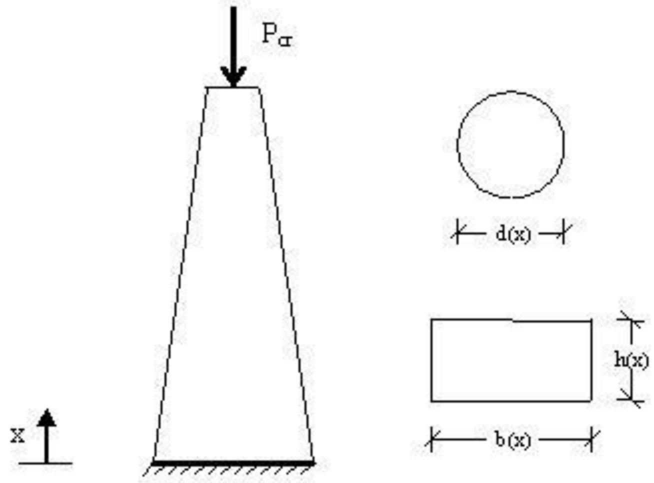


Fig. 1 Geometry and cross-sections of a tapered cantilever micro-column

I and A are the moment of inertia and cross section area of the micro-column, respectively.

3. Buckling problem of a tapered micro-column

Consider a tapered micro-column, in which its thickness, $h(x)$, and width, $b(x)$ for rectangular cross-section and its diameter, $d(x)$ for circular cross-section, are linearly varied along longitudinal direction, as shown in Fig. 1. $h(x)$, $b(x)$ and $d(x)$ can be defined by

$$h(x) = h_1 - (h_1 - h_2)\frac{x}{L} \tag{15a}$$

$$b(x) = b_1 - (b_1 - b_2)\frac{x}{L} \tag{15b}$$

$$d(x) = d_1 - (d_1 - d_2)\frac{x}{L} \tag{15c}$$

where h_1, h_2, b_1, b_2 and d_1, d_2 are the thicknesses, widths and diameters of the micro-column at its lower and upper ends, respectively. Eqs. (15a)-(15c) can be rewritten as

$$h(x) = h_1\left(1 - \alpha\frac{x}{L}\right) \tag{16a}$$

$$b(x) = b_1\left(1 - \beta\frac{x}{L}\right) \tag{16b}$$

$$d(x) = d_1\left(1 - \eta\frac{x}{L}\right) \tag{16c}$$

where α, β and η are called as taper ratios and are given by

$$\alpha = 1 - h_2 / h_1 \tag{17a}$$

$$\beta = 1 - b_2 / b_1 \tag{17b}$$

$$\eta = 1 - d_2 / d_1 \tag{17c}$$

As an instance, if α, β and η are equal to zero, the model will become a uniform beam/column. From Eqs. (16a)-(16c), the usual second moment of cross-sectional area for rectangular and circular cross-sections can be expressed as

$$I^r(x) = \frac{b(x)h(x)^3}{12} = \frac{b_1 h_1^3}{12}\left(1-\beta\frac{x}{L}\right)\left(1-\alpha\frac{x}{L}\right)^3 = I_1^r\left(1-\beta\frac{x}{L}\right)\left(1-\alpha\frac{x}{L}\right)^3 \tag{18a}$$

$$I^c(x) = \frac{\pi\ d(x)^4}{64} = \frac{\pi\ d_1^4}{64}\left(1-\eta\frac{x}{L}\right)^4 = I_1^c\left(1-\eta\frac{x}{L}\right)^4 \tag{18b}$$

Similarly, the cross-sectional area can be written for rectangular and circular cross-sections

$$A^r(x) = b(x)h(x) = b_1 h_1\left(1-\beta\frac{x}{L}\right)\left(1-\alpha\frac{x}{L}\right) = A_1^r\left(1-\beta\frac{x}{L}\right)\left(1-\alpha\frac{x}{L}\right) \tag{19a}$$

$$A^c(x) = \frac{\pi\ d(x)^2}{4} = \frac{\pi\ d_1^2}{4}\left(1-\eta\frac{x}{L}\right) = A_1^c\left(1-\eta\frac{x}{L}\right) \tag{19b}$$

where superscripts r and c denotes rectangular and circular cross-sections, respectively. Also, I_1 and A_1 are the usual second moment of cross-sectional area and the cross-sectional area of the micro-column at x=0, respectively. The strain energy U in Eq. (12) can be rewritten for the linearly tapered micro-column shown in Fig. 1 as following

$$\overline{U} = \frac{1}{2}\int_0^L \left[\overline{B}.(w'')^2 + \overline{D}.(w''')^2\right] dx \tag{20}$$

where

$$\overline{B} = EI(x) + 2GA(x)l_0^2 + \frac{8}{15}GA(x)l_1^2 + GA(x)l_2^2 \tag{21a}$$

$$\overline{D} = I(x)\left(2Gl_0^2 + \frac{4}{5}Gl_1^2\right) \tag{21b}$$

$I(x)$ and $A(x)$ are the moment of inertia and cross-section area of the non-uniform micro-column, respectively.

4. Solution method

The total potential energy of the micro-column is

$$\Pi = \overline{U} + \overline{V} \tag{22}$$

where $\overline{U}$ is the strain energy of the non-uniform micro-column and $\overline{V}$ is the potential of external applied buckling load as

$$\overline{V} = -\frac{P}{2}\int_0^L (w')^2\, dx \tag{23}$$

where P is an axial tip load and

$$w' = \frac{dw}{dx} \tag{24}$$

The total potential energy of the micro-column, Π, is rewritten as

$$\Pi = \overline{U} + \overline{V} = \frac{1}{2}\int_0^L \left[\overline{B}.(w'')^2 + \overline{D}.(w''')^2\right] dx - \frac{P}{2}\int_0^L (w')^2 dx \tag{25}$$

Applying Rayleigh-Ritz method with assumed trial function $w(x)$ as

$$w(x) = \sum_{i=1}^{n} c_i f_i(x) \tag{26}$$

where c_i are constants and $f_i(x)$ is the acceptable function which is necessary to satisfy only the essential (geometric) boundary conditions, but not required for satisfying the natural (force) boundary conditions. For a cantilever beam/column as shown in Fig. 1, the classical boundary conditions are

$$w(0) = 0,\ w'(0) = 0,\ V(L) = 0,\ M(L) = 0 \tag{27}$$

where V and M are shear force and bending moment resultants, respectively. For non-classical boundary conditions, two enable boundary conditions are considered at fixed end as

$$M^h(0) = 0 \tag{28a}$$

$$w''(0) = 0 \tag{28b}$$

and non-classical boundary condition at free end can be written as

$$M^h(L) = 0 \tag{29}$$

where M^h is higher-order moment resultant. In the present study, two acceptable functions are chosen for depending on the above boundary conditions as following

$$f_i^1(x) = \left(\frac{x}{L}\right)^{i+1},\ i = 1,2,\ldots,10 \tag{30a}$$

$$f_i^2(x) = \left(\frac{x}{L}\right)^{i+2},\ i = 1,2,\ldots,10 \tag{30b}$$

Eq. (30a) satisfies all essential boundary conditions in Eqs. (27), (28a) and (29), while Eq. (30b) satisfies all essential boundary conditions in Eqs. (27), (28b) and (29). The stationary points of the total potential energy of the micro-column are the solutions which satisfy equilibrium as

$$\frac{\partial \Pi}{\partial c_i} = 0\,,\ i = 1,2,\ldots,10 \tag{31}$$

Eq. (31) involves ten linear homogenous algebraic equations with constant coefficients. For a non-trivial solution, the determinant of coefficients matrix must be equal to zero. As a result, ten roots of this eigenvalue problem can be assessed and the smallest of them indicates the critical buckling load.

5. Conclusions

In this section, in order to illustrate size effects on buckling behavior of tapered micro-columns, some numerical results are provided for linearly tapered micro-columns that subjected to an axial concentrated compressive tip load. The micro-column is considered as one end is clamped (at x=0) and other end is free (at x=L). The cross-section of the micro-column is both rectangular and circular, in which its thickness and width (or diameter) decrease linearly along the longitudinal axis, as shown in Fig. 1. For illustration purpose, following material constants and geometric properties are taken into consideration: E=1.44 GPa, ν=0.38, $L=20h_1$, $b_1=2h_1$, $d_1=h_1$, $l_0=l_1=l_2=l$. Furthermore, the cases of material length scale parameters $l_0=l_1=0$, $l_2=l$ and $l_0=l_1=l_2=0$ represent the micro-column model based on modified couple stress theory and classical theory, respectively.

In the tables and figures, the results that achieved by classical theory, modified couple stress theory and modified strain gradient elasticity theory, are represented as CT, MCST, MSGT-1 and MSGT-2, respectively and also the results for the cases of rectangular and circular cross section are presented respectively in a and b of each figures. It also should be noted that the results of CT, MCST and MSGT-1 are obtained by using Eq. (30a) while the results of MSGT-2 are achieved from Eq. (30b). Some benchmark results are presented in Tables 1-3.

Table 1 Comparison of non-dimensional critical buckling loads, P_{cr}^* ($P_{cr}^* = P_{cr}L^2/EI_1$) for the case of the thickness, $h(x)$, varies linearly while the width, b, is constant along longitudinal axis of the micro-column ($l_0=l_1=l_2=0$, β=0)

Taper ratio, α	Present	Wang *et al.* (2005)	Darbandi*et al.* (2010)
0	2.4674	2.467	2.47
0.1	2.2464	2.246	-
0.2	2.0233	-	2.02
0.3	1.7977	1.798	-
0.4	1.5691	-	1.57
0.5	1.3364	1.336	-
0.6	1.0985	-	1.10
0.7	0.8533	0.853	-
0.8	0.5968	-	0.60
0.9	0.3215	0.321	-

Table 2 Non-dimensional critical buckling loads, P_{cr}^{*} for the case of both $h(x)$ and $b(x)$ vary linearly throughout the micro-column (rectangular cross-section) ($\alpha=\beta=\eta$, $l_0=l_1=l_2=l=h_1$)

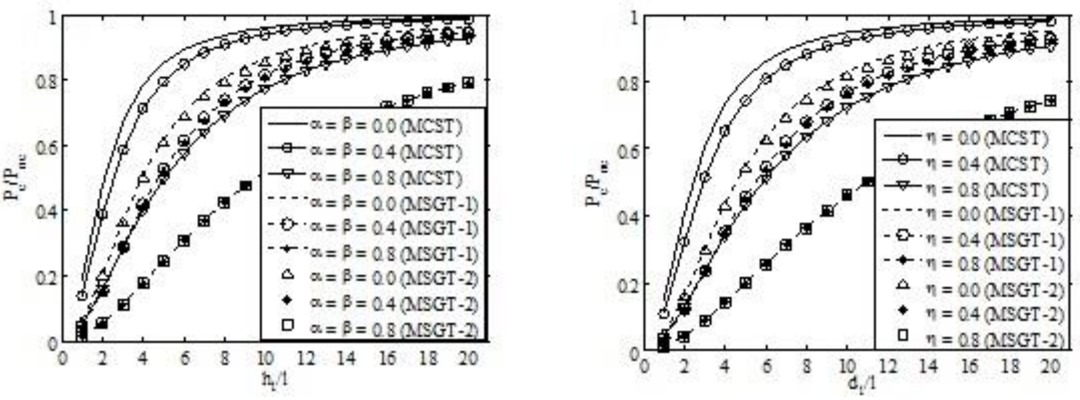
Taper ratio, η	CT	MCST	MSGT-1	MSGT-2
0	2.4674	13.1952	40.3875	41.5274
0.2	1.8835	11.3224	35.2055	36.0447
0.4	1.3093	9.4163	29.8100	30.3868
0.6	0.7566	7.4384	24.0483	24.4008
0.8	0.2644	5.2769	17.5125	17.6787

Table 3 Non-dimensional critical buckling loads, P_{cr}^{*} for the case of the diameter, $d(x)$, varies linearly along longitudinal axis of the micro-column (circular cross-section) ($l_0=l_1=l_2=l=d_1$)

Taper ratio, η	CT	MCST	MSGT-1	MSGT-2
0	2.4674	16.7712	53.0226	54.3880
0.2	1.8835	14.4642	46.3045	47.3112
0.4	1.3093	12.1003	39.2868	39.9799
0.6	0.7566	9.6255	31.7650	32.1895
0.8	0.2644	6.8886	23.1951	23.3959

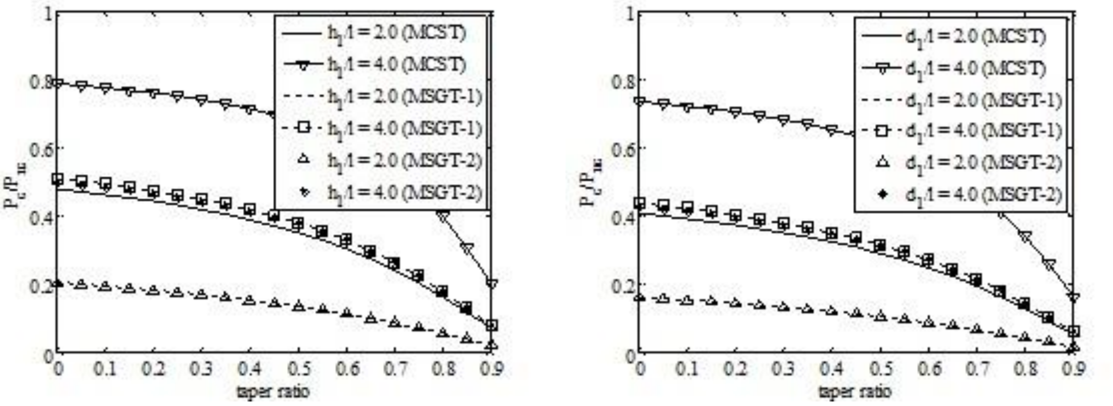
Non-dimensional critical buckling loads, P_{cr}^{*} ($P_{cr}^{*}=P_{cr}L^2/EI_1$), for the case of the thickness, $h(x)$, varies linearly while the width, b, is constant along longitudinal axis of the rectangular micro-column are given in Table 1. The present results are compared with the previous results obtained by analytical and approximate methods for demonstrating the validity and suitability of the current analysis. It can clearly be said that there is good agreement between the results. It also should be noted that when taper ratio increases, this agreement gradually decreases. It is observed from the table that non-dimensional critical buckling loads decrease due to the increase in taper ratio. Table 2 and 3 exhibit that values of the non-dimensional critical buckling loads, P_{cr}^{*}, for the case of both $h(x)$ and $b(x)$ or $d(x)$ vary linearly throughout the micro-column for rectangular and circular cross-sections, respectively. It is remarkable that non-dimensional critical buckling load values predicted by non-classical models for circular cross-section are greater than those values for rectangular cross-section while the results obtained by classical model equal to each other with the same taper ratio. It is also notable that the critical buckling load values obtained by MSGT-2 are bigger than those obtained by CT, MCST and MSGT-1. It can easily be said that the micro-column modeled by MSGT-1 and MSGT-2 are stiffer than that modeled by CT and MCST.

The normalized critical buckling loads versus the ratio of the micro-column thickness (or diameter) at its bottom end to the additional material length scale parameters for different taper ratios are depicted in Fig. 2. From both Fig. 2(a) and (b), it is found that when taper ratio increases, the normalized critical buckling loads decrease for constant values of h_1/l or d_1/l. Furthermore, an increase of h_1/l or d_1/l gives rise to a decrease of the differences between critical buckling loads achieved by CT and those predicted by MCST, MSGT-1 and MSGT-2. However, the additional material length scale parameters are more significant on buckling behavior for smaller value of the micro-column thickness. In other words, when the internal material length scale parameters of micro-sized columns are comparable to its thickness, the use of a higher-order continuum theory



(a) rectangular (b) circular

Fig. 2 Normalized critical buckling loads versus the ratio of the micro-column thickness (or diameter) at its bottom end to the additional material length scale parameters for different taper ratios



(a) rectangular (b) circular

Fig. 3 Normalized critical buckling loads with respect to various values of taper ratio with different h_1/l and d_1/l

becomes required. It can be clearly seen that the results obtained by CT, MCST, MSGT-1 and MSGT-2 are nearly equal to each other for higher values of h_1/l and d_1/l as this ratio is greater than 20.

The normalized critical buckling loads with respect to various values of taper ratio with different values of the ratio of the micro-column thickness (or diameter) at its bottom end to the additional material length scale parameters are plotted in Fig. 3. The figure illustrates that due to the increase in the values of h_1/l and d_1/l also increases the values of normalized critical buckling load for same taper ratios. In addition, an increase in taper ratio leads to an increase in differences between critical buckling loads predicted by classical and non-classical models. It is remarkable that the amount of this increment rises rapidly when taper ratio is greater than 0.7.

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