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Asymmetric transient thermal stress of a functionally graded hollow cylinder with piecewise power law

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Abstract. This paper is concerned with the theoretical treatment of transient thermoelastic problems involving a functionally graded hollow cylinder with piecewise power law due to asymmetrical heating from its surfaces. The thermal and thermoelastic constants of each layer are expressed as power functions of the radial coordinate, and their values continue on the interfaces. The exact solution for the two-dimensional temperature change in a transient state, and thermoelastic response of a hollow cylinder under the state of plane strain is obtained herein. Some numerical results for the temperature change and the stress distributions are shown in figures. Furthermore, the influence of the functional grading on the thermal stresses is investigated.

Keywords: thermal stress problem; functionally graded material; hollow cylinder; piecewise power law; transient state; asymmetrical heating; plane strain problem

1. Introduction

Functionally graded materials (FGMs) are nonhomogeneous material systems that two or more different material ingredients change continuously and gradually, and are used as constituents of the beam, strip, plate and shell types. The concept of FGMs was proposed as a new material which is adaptable for a super-high-temperature environment at first in Japan. In recent years, the concept of FGMs has been applied in many industrial fields such as engineering, chemical plant, electronics, energy conversion, optics, biomaterials and so on in addition to the aerospace field (Miyamoto *et al*. 1999, Ichikawa 2001). FGMs subjected to several thermal loading consist of metals and ceramics as their constituents, and are remarkable heat-resistant materials for relaxation of thermal stress. Therefore, it is necessary to analyze the thermal stress problems for FGMs. Because the governing equations for the temperature field and the associate thermoelastic field of FGMs become of nonlinear form in generally, the analytical treatment is difficult. It is well-known that thermal stress distributions in a transient state can show large values compared with the one in a steady state. Therefore, the transient thermoelastic problems for FGMs become important. The analytical treatment of the transient thermoelastic problems is more difficult.

As the analytical treatment of the thermoelastic problems of FGMs, there are two pieces of

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treatment mainly. One is introducing the theory of laminated composites, which have a number of homogeneous layers along the thickness direction. Using the theory of laminated composites, the transient thermal stress problems of several analytical models (Tanigawa *et al*. 1989, Ootao and Tanigawa 1994, Sugano *et al*. 1996, Ootao and Tanigawa 1999) were analyzed theoretically.

The other analytical treatment is the exact analysis under the assumption that the material properties are given in the specific functions containing the variable of the thickness coordinate without using the laminated composite model. Examples of exact transient thermal stress analysis for FGM plate type structures are as follows. Sugano (1987) analyzed exactly one-dimensional transient thermal stresses of nonhomogeneous plate where the thermal conductivity and Young's modulus vary exponentially, whereas Poisson's ratio and the coefficient of linear thermal expansion vary arbitrarily in the thickness direction. Vel and Batra (2003) analyzed the three-dimensional transient thermal stresses of the functionally graded rectangular plate. Ootao and Tanigawa (2005) analyzed the transient thermal stress problems of a functionally graded rectangular plate, where the thermal conductivity, the coefficient of linear thermal expansion and Young's modulus vary exponentially in the thickness direction, due to nonuniform heat supply.

On the other hand, examples of exact analysis for FGM shell type structures are as follows. Obata and Noda (1994) analyzed one-dimensional thermal stress problem of functionally graded hollow cylinder and hollow sphere using a perturbation method. Zimmerman and Lutz (1999) presented the exact solution for one-dimensional thermal stresses of functionally graded cylinder whose elastic modulus and coefficient of linear thermal expansion vary linearly with the radius. Ye *et al*. (2001) presented the exact solution for the axisymmetric thermoelastic problem of a uniformly heated functionally graded transversely isotropic cylindrical shell, assuming that the modulus of elasticity and the coefficient of linear thermal expansion vary with the power product form of radial coordinate variable. Tarn (2001) presented the exact solutions for functionally graded anisotropic cylinders subjected to thermal and mechanical loads. Jabbari *et al*. (2003) presented the exact solutions for thermal stresses of functionally graded hollow cylinder whose material properties vary with the power product form of radial coordinate variable due to nonaxisymmetric loads. Poultangari *et al*. (2008) obtained the two-dimensional exact solutions for thermal stresses of functionally graded sphere whose material properties vary with the power product form of radial coordinate variable. Jabbari *et al*. (2007) presented the analytical solution for three-dimensional thermal stresses in a short length functionally graded hollow cylinder whose material properties vary with the power product form of radial coordinate variable. You *et al*. (2007) analyzed the thermoelasatic problem of functionally graded cylindrical vessels under internal pressure and uniform temperature by a simple and accurate method. Peng and Li (2009) analyzed the thermoelastic problem of functionally graded annulus with arbitrary gradient. Vel (2011) analyzed the thermoelastic problem of functionally graded anisotropic hollow cylinders, whose thermoelastic constants are expressed as Taylor's series. These papers, however, only treated the thermoelasic problems under steady temperature distribution.

As a transient thermoelastiv problem of FGM shell type structures, one-dimensional and two-dimensional solutions for transient thermal stresses of a functionally graded hollow cylinder whose material properties vary with the power product form of radial coordinate variable were obtained by Ootao and Tanigawa (2006, 2009). Zhao *et al*. (2006) analyzed the one-dimensional transient thermo-mechanical behavior of a functionally graded solid cylinder, whose thermoelastic constants vary exponentially through the thickness. Shao *et al*. (2007) analyzed one-dimensional transient thermo-mechanical behavior of functionally graded hollow cylinders, whose thermoelastic constants are expressed as Taylor's series.

However, these studies discuss the thermoelastic problems of one-layered FGM models, which have the big limitation of nonhomogeneity. On the other hand, the arbitrary nonhomogeneity can be expressed in the theory of laminated composites approximately. But the theory of laminated composites has a weak point that the material properties are discontinuous on such interface. Guo and Noda (2007) proposed a piecewise-exponential model, for the crack problems in FGMs with arbitrary material properties which are continuous on each interface in order to improve the ordinary theory of laminated composites. Ootao (2010) analyzed the transient thermoelastic problem in the FGM hollow cylinder by a piecewise-power model when material properties can be expressed by piecewise power law. To the author's knowledge, however, the two-dimensional analysis for transient thermoelastic problems of FGM shell type structures with piecewise-power law has not been reported.

From the viewpoint of above mentioned, we analyze the transient thermoelastic analysis for a functionally graded hollow cylinder with piecewise power law due to asymmetrical surface heating to guarantee arbitrary nonhomogeneity of material properties.

2. Analysis

The infinite long, functionally graded hollow cylinder consists of many layers whose material properties are expressed by piecewise power law of position. The thermal and thermoelastic constants of each layer are expressed as power functions of the radial coordinate, and their values continue on the interfaces. The hollow cylinder's inner and outer radii are defined r_a and r_b, respectively. Moreover, r_i is the outer radius of ith layer. Throughout this article, indices i (=1,2,…, N) are associated with the ith layer from the inner side of a functionally graded hollow cylinder.

2.1 Heat conduction problem

We assume that the functionally graded hollow cylinder is initially at zero temperature and is heated from the inner and outer surfaces by surrounding media with relative heat transfer coefficients (heat transfer coefficient/thermal conductivity) h_a and h_b. We denote the temperatures of the surrounding media by the functions $T_a f_a(\theta)$ and $T_b f_b(\theta)$. Then the temperature distribution shows a two-dimensional distribution in $r-\theta$ plane, and the transient heat conduction equation for the ith layer is taken in the following form

$$c_i(r)\rho_i(r)\frac{\partial T_i}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}\left(\lambda_{ti}(r)r\frac{\partial T_i}{\partial r}\right)+\frac{1}{r^2}\frac{\partial}{\partial\theta}\left(\lambda_{ti}(r)\frac{\partial T_i}{\partial\theta}\right);i=1,\cdots,N \tag{1}$$

The thermal conductivity λ_{ti} and the heat capacity per unit volume $c_i\rho_i$ in each layer are assumed to take the following forms

$$\lambda_{ti}(r)=\lambda_{ti}^0(r/r_{i-1})^{m_i} \tag{2}$$

$$c_i(r)\rho_i(r)=c_i^0\rho_i^0(r/r_{i-1})^{k_i} \tag{3}$$

where

$$m_i = \frac{\ln(\overline{\lambda}^0_{t,i+1} / \overline{\lambda}^0_{ti})}{\ln(\overline{r}_i / \overline{r}_{i-1})}, \quad k_i = \frac{\ln(\overline{c}^0_{i+1} \overline{\rho}^0_{i+1} / \overline{c}^0_i \overline{\rho}^0_i)}{\ln(\overline{r}_i / \overline{r}_{i-1})} \quad ; i = 1, \cdots, N \tag{4}$$

In Eq. (4), $\overline{r}_0$ and $\overline{r}_N$ are

$$\overline{r}_0 = \overline{r}_a, \quad \overline{r}_N = 1 \tag{5}$$

Substituting the Eqs. (2) and (3) into the Eq.(1), the transient heat conduction equations in dimensionless form are

$$\frac{\partial \overline{T}_i}{\partial \tau} = \overline{\kappa}_i^0 \overline{r}_{i-1}^{k_i - m_i} [(m_i + 1) \overline{r}^{m_i - k_i - 1} \frac{\partial \overline{T}_i}{\partial \overline{r}} + \overline{r}^{m_i - k_i} \frac{\partial^2 \overline{T}_i}{\partial \overline{r}^2} + \overline{r}^{m_i - k_i - 2} \frac{\partial^2 \overline{T}_i}{\partial \theta^2}] \; ; i = 1, 2, \cdots, N \tag{6}$$

The initial and thermal boundary conditions in dimensionless form are

$$\tau = 0; \quad \overline{T}_i = 0 \quad ; \quad i = 1, 2, \cdots, N \tag{7}$$

$$\overline{r} = \overline{r}_a \, ; \; \frac{\partial \overline{T}_1}{\partial \overline{r}} - H_a \overline{T}_1 = -H_a \overline{T}_a f_a(\theta) \tag{8}$$

$$\overline{r} = \overline{r}_i \, ; \; \overline{T}_i = \overline{T}_{i+1} \quad ; \quad i = 1, 2, \cdots, N - 1 \tag{9}$$

$$\overline{r} = \overline{r}_i ; \overline{\lambda}_{ti} \frac{\partial \overline{T}_i}{\partial \overline{r}} = \overline{\lambda}_{t,i+1} \frac{\partial \overline{T}_{i+1}}{\partial \overline{r}} \; ; \; i = 1, 2, \cdots, N - 1 \tag{10}$$

$$\overline{r} = 1; \; \frac{\partial \overline{T}_N}{\partial \overline{r}} + H_b \overline{T}_N = H_b \overline{T}_b f_b(\theta) \tag{11}$$

In Eqs. (4)-(11), we introduced the following dimensionless values

$$(\overline{T}_i, \overline{T}_a, \overline{T}_b) = (T_i, T_a, T_b) / T_0, \; (\overline{r}, \overline{r}_i, \overline{r}_a) = (r, r_i, r_a) / r_b, \; \tau = \lambda_{t0} t / (c_0 \rho_0 r_b{}^2),$$

$$\overline{\kappa}_i^0 = \lambda_{ti}^0 / (c_i^0 \rho_i^0), \; (\overline{\lambda}_{ti}, \overline{\lambda}_{ti}^0) = (\lambda_{ti}, \lambda_{ti}^0) / \lambda_{t0}, \; (H_a, H_b) = (h_a, h_b) r_b \tag{12}$$

where T_i is the temperature change; t is time; and T_0, λ_{t0} and $c_0\rho_0$ are typical values of temperature, thermal conductivity, and heat capacity per unit volume, respectively. For the sake of simplicity of analysis, we assume that the temperature functions $f_a(\theta)$ and $f_b(\theta)$ are symmetrical with respect to $\theta = 0$, and expand the functions into the following Fourier's series forms

$$\begin{Bmatrix} f_a(\theta) \\ f_b(\theta) \end{Bmatrix} = \sum_{q=0}^{\infty} \begin{Bmatrix} a_q \\ b_q \end{Bmatrix} \cos q\theta \tag{13}$$

$$\begin{Bmatrix} a_q \\ b_q \end{Bmatrix} = \frac{\varepsilon_q}{\pi} \int_0^{\pi} \begin{Bmatrix} f_a(\theta) \\ f_b(\theta) \end{Bmatrix} \cos q\theta \, d\theta \,, \varepsilon_q = \begin{cases} 1 \;; q = 0 \\ 2 \;; q = 1,2,3,\cdots \end{cases} \tag{14}$$

Introducing the Laplace transformation with respect to the variable τ and the method of separation of variables, the solution of Eq. (6) can be obtained so as to satisfy conditions (7)-(11). This solution is shown as follows

$$\begin{aligned} \bar{T}_i(\bar{r},\theta,\tau) &= \frac{1}{F_0}(\bar{A}'_{i0} + \bar{B}'_{i0}\bar{r}^{-m_i}) + \sum_{q=1}^{\infty} \frac{1}{F_q}(\bar{A}'_{iq}\bar{r}^{\xi_{i1}} + \bar{B}'_{iq}\bar{r}^{\xi_{i2}})\cos q\theta \\ &+ \sum_{q=0}^{\infty}\sum_{j=1}^{\infty} \frac{2\bar{r}^{-m_i/2}}{\mu_{1j}\Delta'_q(\mu_{1j})} \exp[-\frac{\mu_{1j}^2}{4} \cdot \frac{\bar{\kappa}_1^0}{\bar{r}_a^{m_1-k_1}}(2-m_1+k_1)^2\tau] \\ &\times [\bar{A}_{iq} J_{\gamma_i}(\Omega_i \mu_{1j} \bar{r}^{1-\frac{m_i-k_i}{2}}) + \bar{B}_{iq} Y_{\gamma_i}(\Omega_i \mu_{1j} \bar{r}^{1-\frac{m_i-k_i}{2}})] \cos q\theta \end{aligned} \tag{15}$$

where $J_x(\,)$ and $Y_x(\,)$ are the Bessel functions of the first and second kind of order x, respectively. And Δ_q and F_q are the determinants of $2N \times 2N$ matrix $[a_{kl}^q]$ and $[e_{kl}^q]$, respectively; the coefficients $\bar{A}_{iq}$ and $\bar{B}_{iq}$ are defined as the determinant of the matrix similar to the coefficient matrix $[a_{kl}^q]$, in which the (2i-1)th column or 2ith column is replaced by the constant vector $\{c_k^q\}$, respectively. Similarly, the coefficients $\bar{A}'_{iq}$ and $\bar{B}'_{iq}$ are defined as the determinant of the matrix similar to the coefficient matrix $[e_{kl}^q]$, in which the (2i-1)th column or 2ith column is replaced by the constant vector $\{c_k^q\}$, respectively. The nonzero elements of the coefficient matrices $[a_{kl}^q]$, $[e_{kl}^q]$ and the constant vector $\{c_k^q\}$ are given as

$$\begin{aligned} a_{1,1}^q = \bar{r}_a^{-\frac{m_1}{2}} \Bigg\{ & \left[(1-\frac{m_1-k_1}{2})(\gamma_1 - \frac{m_1}{2-m_1+k_1})\bar{r}_a^{-1} - H_a \right] J_{\gamma_1}(\mu_1 \bar{r}_a^{1-\frac{m_1-k_1}{2}}) \\ & - \mu_1 (1-\frac{m_1-k_1}{2}) \bar{r}_a^{-\frac{m_1-k_1}{2}} J_{\gamma_1+1}(\mu_1 \bar{r}_a^{-\frac{m_1-k_1}{2}}) \Bigg\}, \end{aligned}$$

$$a_{1,2}^q = \bar{r}_a^{-\frac{m_1}{2}} \Bigg\{ \left[(1-\frac{m_1-k_1}{2})(\gamma_1 - \frac{m_1}{2-m_1+k_1})\bar{r}_a^{-1} - H_a \right] Y_{\gamma_1}(\mu_1 \bar{r}_a^{1-\frac{m_1-k_1}{2}})$$

$$-\mu_1(1-\frac{m_1-k_1}{2})\bar{r}_a^{-\frac{m_1-k_1}{2}}Y_{\gamma_1+1}(\mu_1\bar{r}_a^{-\frac{m_1-k_1}{2}})\Bigg\},$$

$$a^q_{2N,2N-1}=\left[(1-\frac{m_N-k_N}{2})(\gamma_N-\frac{m_N}{2-m_N+k_N})+H_b\right]J_{\gamma_N}(\mu_N)$$

$$-\mu_N(1-\frac{m_N-k_N}{2})J_{\gamma_N+1}(\mu_N),$$

$$a^q_{2N,2N}=\left[(1-\frac{m_N-k_N}{2})(\gamma_N-\frac{m_N}{2-m_N+k_N})+H_b\right]Y_{\gamma_N}(\mu_N)$$

$$-\mu_N(1-\frac{m_N-k_N}{2})Y_{\gamma_N+1}(\mu_N) \tag{16}$$

$$a^q_{2i,2i-1}=\bar{r}_i^{-\frac{m_i}{2}}J_{\gamma_i}(\mu_i\bar{r}_i^{1-\frac{m_i-k_i}{2}}),\quad a^q_{2i,2i}=\bar{r}_i^{-\frac{m_i}{2}}Y_{\gamma_i}(\mu_i\bar{r}_i^{1-\frac{m_i-k_i}{2}}),$$

$$a^q_{2i,2i+1}=-\bar{r}_i^{-\frac{m_{i+1}}{2}}J_{\gamma_{i+1}}(\mu_{i+1}\bar{r}_i^{1-\frac{m_{i+1}-k_{i+1}}{2}}),\quad a^q_{2i,2i+2}=-\bar{r}_i^{-\frac{m_{i+1}}{2}}Y_{\gamma_{i+1}}(\mu_{i+1}\bar{r}_i^{1-\frac{m_{i+1}-k_{i+1}}{2}}),$$

$$a^q_{2i+1,2i-1}=\bar{r}_i^{-\frac{m_i}{2}}\Bigg[(1-\frac{m_i-k_i}{2})(\gamma_i-\frac{m_i}{2-m_i+k_i})\bar{r}_i^{-1}J_{\gamma_i}(\mu_i\bar{r}_i^{1-\frac{m_i-k_i}{2}})$$

$$-\mu_i(1-\frac{m_i-k_i}{2})\bar{r}_i^{-\frac{m_i-k_i}{2}}J_{\gamma_i+1}(\mu_i\bar{r}_i^{1-\frac{m_i-k_i}{2}})\Bigg],$$

$$a^q_{2i+1,2i}=\bar{r}_i^{-\frac{m_i}{2}}\Bigg[(1-\frac{m_i-k_i}{2})(\gamma_i-\frac{m_i}{2-m_i+k_i})\bar{r}_i^{-1}Y_{\gamma_i}(\mu_i\bar{r}_i^{1-\frac{m_i-k_i}{2}})$$

$$-\mu_i(1-\frac{m_i-k_i}{2})\bar{r}_i^{-\frac{m_i-k_i}{2}}Y_{\gamma_i+1}(\mu_i\bar{r}_i^{1-\frac{m_i-k_i}{2}})\Bigg],$$

$$a^q_{2i+1,2i+1}=-\bar{r}_i^{-\frac{m_{i+1}}{2}}\Bigg[(1-\frac{m_{i+1}-k_{i+1}}{2})(\gamma_{i+1}-\frac{m_{i+1}}{2-m_{i+1}+k_{i+1}})\bar{r}_i^{-1}J_{\gamma_{i+1}}(\mu_{i+1}\bar{r}_i^{1-\frac{m_{i+1}-k_{i+1}}{2}})$$

$$\left. - \mu_{i+1}(1-\frac{m_{i+1}-k_{i+1}}{2})\bar{r}_i^{-\frac{m_{i+1}-k_{i+1}}{2}} J_{\gamma_{i+1}+1}(\mu_{i+1}\bar{r}_i^{1-\frac{m_{i+1}-k_{i+1}}{2}})\right] ,$$

$$a^q_{2i+1,2i+2} = -\bar{r}_i^{-\frac{m_{i+1}}{2}}\left[(1-\frac{m_{i+1}-k_{i+1}}{2})(\gamma_{i+1}-\frac{m_{i+1}}{2-m_{i+1}+k_{i+1}})\bar{r}_i^{-1} Y_{\gamma_{i+1}}(\mu_{i+1}\bar{r}_i^{1-\frac{m_{i+1}-k_{i+1}}{2}})\right.$$

$$\left. - \mu_{i+1}(1-\frac{m_{i+1}-k_{i+1}}{2})\bar{r}_i^{-\frac{m_{i+1}-k_{i+1}}{2}} Y_{\gamma_{i+1}+1}(\mu_{i+1}\bar{r}_i^{1-\frac{m_{i+1}-k_{i+1}}{2}})\right] ;\ i = 1,2,\cdots,N-1 \tag{17}$$

$$e^0_{1,1} = -H_a,\ e^0_{1,2} = -(m_1\bar{r}_a^{-m_1-1} + H_a\bar{r}_a^{-m_1}),\ e^0_{2N,2N-1} = H_b,\ e^0_{2N,2N} = H_b - m_N \tag{18}$$

$$e^0_{2i,2i-1} = 1,\ e^0_{2i,2i} = \bar{r}_i^{-m_i},\ e^0_{2i,2i+1} = -1,\ e^0_{2i,2i+2} = -\bar{r}_i^{-m_{i+1}},$$

$$e^0_{2i+1,2i} = -m_i\bar{r}_i^{-m_i-1},\ e^0_{2i+1,2i+2} = m_{i+1,1}\bar{r}_i^{-m_{i+1}-1};\ i = 1,2,\cdots,N-1 \tag{19}$$

$$e^q_{1,1} = \xi_{11}\bar{r}_a^{\xi_{11}-1} - H_a\bar{r}_a^{-\xi_{11}},\ e^q_{1,2} = \xi_{12}\bar{r}_a^{\xi_{12}-1} - H_a\bar{r}_a^{-\xi_{12}},$$

$$e^q_{2N,2N-1} = \xi_{N1} + H_b,\ e^q_{2N,2N} = \xi_{N2} + H_b\ ;\ q = 1,2,3,\cdots \tag{20}$$

$$e^q_{2i,2i-1} = \bar{r}_i^{\xi_{i1}},\ e^q_{2i,2i} = \bar{r}_i^{\xi_{i2}},\ e^q_{2i,2i+1} = -\bar{r}_i^{\xi_{i+1,1}},\ e^q_{2i,2i+2} = -\bar{r}_i^{\xi_{i+1,2}},$$

$$e^q_{2i+1,2i-1} = \xi_{i1}\bar{r}_i^{\xi_{i1}-1},\ e^q_{2i+1,2i} = \xi_{i2}\bar{r}_i^{\xi_{i2}-1},\ e^q_{2i+1,2i+1} = -\xi_{i+1,1}\bar{r}_i^{\xi_{i+1,1}-1},$$

$$e^q_{2i+1,2i+2} = -\xi_{i+1,2}\bar{r}_i^{\xi_{i+1,2}-1}\quad ;\quad i = 1,2,\cdots,N-1\ ,\ q = 1,2,3,\cdots \tag{21}$$

$$c^q_1 = -H_a\bar{T}_a a_q,\ c^q_{2N} = H_b\bar{T}_b b_q\ ;\ q = 0,1,2,3,\cdots \tag{22}$$

In Eq. (15), ς_{i1}, ς_{i2}, γ_i, Ω_i and $\Delta'(\mu_{1j})$ are

$$\varsigma_{i1},\varsigma_{i2} = \frac{-m_i \pm \sqrt{m_i^2+4q^2}}{2},\ \gamma_i = \frac{\sqrt{m_i^2+4q^2}}{|2-m_i+k_i|},$$

$$\Omega_i = \sqrt{\frac{\bar{\kappa}_1^0}{\bar{\kappa}_i^0}\cdot\frac{\bar{r}_{i-1}^{m_i-k_i}}{\bar{r}_a^{m_1-k_1}}\left(\frac{2-m_1+k_1}{2-m_i+k_i}\right)^2},\ \Delta'(\mu_{1j}) = \left.\frac{d\Delta}{d\mu_1}\right|_{\mu_1=\mu_{1j}} \tag{23}$$

and μ_{1j} the jth positive root of the following transcendental equation

$$\Delta(\mu_1) = 0 \tag{24}$$

In Eqs. (16) and (17), the relations between μ_i and μ_1 are

$$\mu_i = \Omega_i \mu_1 \,; \quad i = 2, \cdots, N \tag{25}$$

2.2 Thermoelastic problem

We now analyze the transient thermal stress of a functionally graded hollow cylinder as a plane strain problem. The displacement-strain relations are expressed in dimensionless form as follows

$$\bar{\varepsilon}_{rri} = \bar{u}_{ri},_{\bar{r}} \,, \quad \bar{\varepsilon}_{\theta\theta i} = \bar{r}^{-1}(\bar{u}_{\theta i},_{\theta} + \bar{u}_{ri}) \,, \quad \bar{\varepsilon}_{r\theta i} = [\bar{r}^{-1}(\bar{u}_{ri},_{\theta} - \bar{u}_{\theta i}) + \bar{u}_{\theta i},_{\bar{r}}\,]/2 \,,$$

$$\bar{\varepsilon}_{zzi} = \bar{\varepsilon}_{rzi} = \bar{\varepsilon}_{\theta zi} = 0 \quad ; i = 1, \cdots, N \tag{26}$$

where a comma denotes partial differentiation with respect to the variable that follows. The stress-strain relation in dimensionless form is given by the following relation

$$\begin{Bmatrix} \bar{\sigma}_{rri} \\ \bar{\sigma}_{\theta\theta i} \\ \bar{\sigma}_{zzi} \\ \bar{\sigma}_{r\theta i} \end{Bmatrix} = \frac{\bar{E}_i}{(1+\nu_i)(1-2\nu_i)} \begin{bmatrix} 1-\nu_i & \nu_i & 0 \\ \nu_i & 1-\nu_i & 0 \\ \nu_i & \nu_i & 0 \\ 0 & 0 & 1-2\nu_i \end{bmatrix} \begin{Bmatrix} \bar{\varepsilon}_{rri} \\ \bar{\varepsilon}_{\theta\theta i} \\ \bar{\varepsilon}_{r\theta i} \end{Bmatrix}$$

$$- \frac{\bar{\alpha}_i \bar{E}_i \bar{T}_i}{1-2\nu_i} \begin{Bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{Bmatrix} \tag{27}$$

The Young's modulus E_i, the coefficient of linear thermal expansion α_i and Poisson's ratio ν_i are assumed to take the following forms

$$\bar{E}_i(\bar{r}) = \bar{E}_i^0 (\bar{r}/\bar{r}_{i-1})^{l_i} \,, \bar{\alpha}_i(\bar{r}) = \bar{\alpha}_i^0 (\bar{r}/\bar{r}_{i-1})^{b_i} \,, \nu_i = const\,(\nu_i \neq \nu_{i+1}) \tag{28}$$

where

$$l_i = \frac{\ln(\bar{E}_{i+1}^0/\bar{E}_i^0)}{\ln(\bar{r}_i/\bar{r}_{i-1})} \,, \quad b_i = \frac{\ln(\bar{\alpha}_{i+1}^0/\bar{\alpha}_i^0)}{\ln(\bar{r}_i/\bar{r}_{i-1})} \qquad ; i = 1, \cdots, N \tag{29}$$

The equilibrium equations are expressed in dimensionless form as follows:

$$\bar{\sigma}_{rri},_{\bar{r}} + \bar{r}^{-1}\bar{\sigma}_{r\theta i},_{\theta} + \bar{r}^{-1}(\bar{\sigma}_{rri} - \bar{\sigma}_{\theta\theta i}) = 0 \tag{30}$$

$$\bar{\sigma}_{r\theta i},_{\bar{r}} + \bar{r}^{-1}\bar{\sigma}_{\theta\theta i},_{\theta} + 2\bar{r}^{-1}\bar{\sigma}_{r\theta i} = 0 \tag{31}$$

Substituting Eqs. (26)-(28) into Eqs. (30) and (31), the displacement equations of equilibrium are written as

$$\bar{u}_{ri},_{\bar{r}\bar{r}} + (l_i + 1)\bar{r}^{-1}\bar{u}_{ri},_{\bar{r}} + (\frac{\nu_i l_i}{1-\nu_i} - 1)\bar{u}_{ri}\bar{r}^{-2} + \frac{1-2\nu_i}{2(1-\nu_i)}\bar{r}^{-2}\bar{u}_{ri},_{\theta\theta} + \frac{1}{2(1-\nu_i)}\bar{r}^{-1}\bar{u}_{\theta i},_{\bar{r}\theta}$$

$$+ [\frac{\nu_i}{1-\nu_i} l_i - \frac{3-4\nu_i}{2(1-\nu_i)}]\bar{r}^{-2}\bar{u}_{\theta i},_{\theta} = \frac{(1+\nu_i)\bar{\alpha}_i^0}{(1-\nu_i)\bar{r}_{i-1}^{b_i}}[(l_i + b_i)\bar{r}^{b_i - 1}\bar{T}_i + \bar{r}^{b_i}\bar{T}_i,_{\bar{r}}] \tag{32}$$

$$\bar{u}_{ri},_{\bar{r}\theta} + [(1-2\nu_i)l_i + 2(1-\nu_i)]\bar{r}^{-1}\bar{u}_{ri},_{\theta} + (1-2\nu_i)[(l_i+1)(\bar{u}_{\theta i},_{\bar{r}} - \bar{r}^{-1}\bar{u}_{\theta i}) + \bar{r}\bar{u}_{\theta i},_{\bar{r}\bar{r}}]$$

$$+ 2(1-\nu_i)\bar{r}^{-1}\bar{u}_{\theta\theta},_{\theta} = \frac{2(1+\nu_i)\bar{\alpha}_i^0}{\bar{r}_{i-1}^{b_i}}\bar{r}^{b_i}\bar{T}_i,_{\theta} \tag{33}$$

In Eqs. (26)-(33), the following dimensionless values are introduced

$$\bar{\sigma}_{kli} = \frac{\sigma_{kli}}{\alpha_0 E_0 T_0}, \quad \bar{\varepsilon}_{kli} = \frac{\varepsilon_{kli}}{\alpha_0 T_0}, \quad (\bar{\alpha}_i, \bar{\alpha}_i^0) = \frac{(\alpha_i, \alpha_i^0)}{\alpha_0},$$

$$(\bar{E}_i, \bar{E}_i^0) = \frac{(E_i, E_i^0)}{E_0}, \quad (\bar{u}_{ri}, \bar{u}_{\theta i}) = \frac{(u_{ri}, u_{\theta i})}{\alpha_0 T_0 r_b} \tag{34}$$

where σ_{kli} are the stress components, ε_{kli} are the strain tensor, (u_{ri}, $u_{\theta i}$) are the displacement components and α_0 and E_0 are the typical values of the coefficient of linear thermal expansion and Young's modulus, respectively. If the inner and outer surfaces are traction free, and the interfaces of the each layer are perfectly bonded, then the boundary conditions of inner and outer surfaces and the conditions of continuity on the interfaces can be represented as follows

$$\bar{r} = \bar{r}_a; \ \bar{\sigma}_{rr1} = 0, \ \bar{\sigma}_{r\theta 1} = 0,$$

$$\bar{r} = \bar{r}_i; \ \bar{\sigma}_{rri} = \bar{\sigma}_{rr,i+1}, \ \bar{\sigma}_{r\theta i} = \bar{\sigma}_{r\theta,i+1}, \ \bar{u}_{ri} = \bar{u}_{r,i+1}, \ \bar{u}_{\theta i} = \bar{u}_{\theta,i+1} \ ; i = 1,2,\cdots,N-1,$$

$$\bar{r} = 1; \ \bar{\sigma}_{rrN} = 0, \bar{\sigma}_{r\theta N} = 0 \tag{35}$$

We assume the solutions of Eqs. (32) and (33) in the following form.

$$\bar{u}_{ri} = \sum_{q=0}^{\infty}[U_{rcqi}(\bar{r}) + U_{rpqi}(\bar{r})]\cos q\theta, \qquad \bar{u}_{\theta i} = \sum_{q=1}^{\infty}[U_{\theta cqi}(\bar{r}) + U_{\theta pqi}(\bar{r})]\sin q\theta \tag{36}$$

In Eq. (36), the first term on the right side gives the homogeneous solution and the second term of right side gives the particular solution. We now consider the homogeneous solution and introduce the following equation

$$\bar{r} = \exp(s) \tag{37}$$

Substituting the first term on the right side of Eq. (36) into the homogeneous equations of Eqs. (32) and (33), and later changing a variable with the use of Eq. (37), we have

$$[(\bar{D}^2 + l_i\bar{D}) + \frac{\nu_i}{1-\nu_i}l_i - 1 - \frac{1-2\nu}{2(1-\nu_i)}q^2]U_{rcqi}$$

$$+[\frac{1}{2(1-\nu_i)}\bar{D} + \frac{\nu_i}{1-\nu_i}l_i - \frac{3-4\nu_i}{2(1-\nu_i)}]qU_{\theta cqi} = 0 \tag{38}$$

$$-[\bar{D} + (1-2\nu_i)(l_i+1) + 2(1-\nu_i)]qU_{rcqi}$$

$$+\{(1-2\nu_i)(\bar{D}^2 + l_i\bar{D}) - [(1-2\nu_i)(l_i+1) + 2(1-\nu_i)q^2]\}U_{\theta cqi} = 0 \tag{39}$$

where

$$\bar{D} = \frac{d}{ds} \tag{40}$$

We show U_{rcqi} and $U_{\theta cqi}$ as follows

$$(U_{rcqi}, U_{\theta cqi}) = (U^0_{rcqi}, U^0_{\theta cqi})\exp(\lambda_i s) \tag{41}$$

Substituting Eq. (41) into Eqs. (38) and (39), the condition that nontrivial solutions of $(U^0_{rcqi}, U^0_{\theta cqi})$ for $q \geq 2$ exist leads to the following equation

$$\lambda_i^4 + 2l_i\lambda_i^3 - [2q^2 + 2 + l_i(\frac{1-2\nu_i}{1-\nu_i} - l_i)]\lambda_i^2 - l_i(2q^2 + 2 + \frac{1-2\nu_i}{1-\nu_i}l_i)\lambda_i$$

$$+(q^2-1)^2 + (q^2-1)\frac{l_i}{1-\nu_i}[\nu_i l_i - (1-2\nu_i)] = 0 \tag{42}$$

From Eq. (42), there might be four real roots, two real roots and one pair of conjugate complex roots, or two pairs of conjugate complex roots.

Case 1: real roots for λ_i

Given J_{iR} real roots for λ_i, $U_{rcqi}(\bar{r})$ and $U_{\theta cqi}(\bar{r})$ are given by the following expressions

$$U_{rcqi}(\bar{r}) = \sum_{J=1}^{J_{iR}} F^{(i)}_{qJ}\bar{r}^{\lambda_{iJ}},$$

$$U_{\theta cqi}(\bar{r}) = \sum_{J=1}^{J_{iR}} M_{qJ}^{(i)}(\lambda_{iJ}) F_{qJ}^{(i)} \bar{r}^{\lambda_{iJ}} \tag{43}$$

where

$$M_{qJ}^{(i)}(\lambda_{iJ}) = \frac{[\lambda_{iJ} + l_i(1-2\nu_i) + 3 - 4\nu_i]q}{(1-2\nu_i)(\lambda_{iJ}^2 + l_i\lambda_{iJ}) - [(l_i+1)(1-2\nu_i) + 2(1-\nu_i)q^2]} \tag{44}$$

In Eq. (43), $F_{qJ}^{(i)}$ are unknown constants.

Case 2: complex roots for λ_i

If the complex root for λ_i is expressed by $\lambda_{iJ} = \alpha_{iJ} \pm j\beta_{iJ}$, and given J_{iI} pairs of complex roots for λ_i, $U_{rcqi}(\bar{r})$ and $U_{\theta cqi}(\bar{r})$ are given by the following expressions

$$\begin{aligned} U_{rcqi}(\bar{r}) &= \sum_{J=1}^{J_{iI}} \left[C_{1J}^{(i)} \bar{r}^{\alpha_{iJ}} \cos(\beta_{iJ} \ln \bar{r}) + C_{2J}^{(i)} \bar{r}^{\alpha_{iJ}} \sin(\beta_{iJ} \ln \bar{r}) \right], \\ U_{\theta cqi}(\bar{r}) &= \sum_{J=1}^{J_{iI}} \Big\{ C_{1J}^{(i)} \bar{r}^{\alpha_{iJ}} [\Gamma_{iJ} \cos(\beta_{iJ} \ln \bar{r}) - \Omega_{iJ} \sin(\beta_{iJ} \ln \bar{r})] \\ &\quad + C_{2J}^{(i)} \bar{r}^{\alpha_{iJ}} \left[\Omega_{iJ} \cos(\beta_{iJ} \ln \bar{r}) + \Gamma_{iJ} \sin(\beta_{iJ} \ln \bar{r}) \right] \Big\} \end{aligned} \tag{45}$$

where

$$\Gamma_{iJ} = R_e[M_{qJ}^{(i)}\big|_{\lambda_{iJ} = \alpha_{iJ} + j\beta_{iJ}}], \quad \Omega_{iJ} = I_m[M_{qJ}^{(i)}\big|_{\lambda_{iJ} = \alpha_{iJ} + j\beta_{iJ}}] \tag{46}$$

In Eq. (46), j, $R_e[\ ]$ and $I_m[\ ]$ are imaginary unit $j = \sqrt{-1}$, real part and imaginary part, respectively. Furthermore, in Eq. (45), $C_{1J}^{(i)}$ and $C_{2J}^{(i)}$ are unknown constants.

On the other hand, substituting Eq. (41) into Eqs. (38) and (39), the condition that nontrivial solutions of $(U_{rcqi}^0, U_{\theta cqi}^0)$ for $q = 1$ exist leads to the following equation

$$\lambda_i(\lambda_i + l_i)(\lambda_i^2 + l_i\lambda_i - \frac{1-2\nu_i}{1-\nu_i} l_i - 4) = 0 \tag{47}$$

We now introduce the following expression

$$D_{i1} = l_i^2 + \frac{4l_i(1-2\nu_i)}{1-\nu_i} + 16 \tag{48}$$

When D_{i1} is positive and l_i is not zero, there are $\lambda_i = 0$, $\lambda_{i1} = -l_i$ and two distinct real roots as follows

$$\lambda_{i2} = \frac{-l_i + \sqrt{D_{i1}}}{2}, \quad \lambda_{i3} = \frac{-l_i - \sqrt{D_{i1}}}{2} \tag{49}$$

$U_{rcqi}(\bar{r})$ and $U_{\theta cqi}(\bar{r})$ for $\lambda_i = 0$ can be expressed as follows

$$U_{rcqi}(\bar{r}) = G_{r0}^{(i)}, \ U_{\theta cqi}(\bar{r}) = -G_{r0}^{(i)} \tag{50}$$

where $G_{r0}^{(i)}$ is a unknown constant. $U_{rcqi}(\bar{r})$ and $U_{\theta cqi}(\bar{r})$ for λ_{iJ} ($J = 1,2,3$) can be expressed as follows

$$U_{rcqi}(\bar{r}) = \sum_{J=1}^{3} F_{qJ}^{(i)} \bar{r}^{\lambda_{iJ}}, \ U_{\theta cqi}(\bar{r}) = \sum_{J=1}^{3} M_{qJ}^{(i)}(\lambda_{iJ}) F_{qJ}^{(i)} \bar{r}^{\lambda_{iJ}} \tag{51}$$

where $F_{qJ}^{(i)}$ are unknown constants. When D_{i1} is negative and l_i is not zero, there are $\lambda_i = 0$, $\lambda_i = -l_i$ and one pair of conjugate complex roots. When D_{i1} is zero and l_i is not zero, there are $\lambda_i = 0$, $\lambda_i = -l_i$ and $\lambda_i = -l_i / 2$ (double root). The details are omitted here for the sake of brevity.

In the case of $q = 0$, the deformation is axisymmetric. $U_{rcqi}(\bar{r})$ for $q = 0$ can be expressed as follows

$$U_{rcqi}(\bar{r}) = \sum_{J=1}^{2} F_{0J}^{(i)} \bar{r}^{\lambda_{iJ}} \tag{52}$$

where

$$\lambda_{i1} = \frac{-l_i + \sqrt{D_{i0}}}{2}, \ \lambda_{i2} = \frac{-l_i - \sqrt{D_{i0}}}{2} \tag{53}$$

$$D_{i0} = l_i^2 + 4\left(1 - \frac{\nu_i}{1-\nu_i} l_i\right) \tag{54}$$

In order to obtain the particular solution, we use the series expansions of the Bessel functions. Since the order γ_i of the Bessel function in Eq. (15) is not integer in general, Eq. (15) can be written as the following expression.

$$\bar{T}_i(\bar{r}, \theta, \tau) = \sum_{q=0}^{\infty} \bar{T}_{iq}(\bar{r}, \tau) \cos q\theta \tag{55}$$

where

$$\overline{T}_{iq}(\overline{r},\tau) = a'_{i0q}\overline{r}^{\xi_{i1}} + b'_{i0q}\overline{r}^{\xi_{i2}} + \sum_{n=0}^{\infty}[a_{inq}(\tau)\overline{r}^{\omega_{1i}} + b_{inq}(\tau)\overline{r}^{\omega_{2i}}] \tag{56}$$

$$a'_{i0q} = \frac{\overline{A}'_{iq}}{F_q},\ b'_{i0q} = \frac{\overline{B}'_{iq}}{F_q},$$

$$a_{inq}(\tau) = \sum_{j=1}^{\infty}\frac{2}{\mu_{1j}\Delta'(\mu_{1j})}\exp[-\frac{\mu_{1j}^2}{4}\cdot\frac{\overline{\kappa}_1^0}{\overline{r}_a^{m_1-k_1}}(2-m_1+k_1)^2\tau]\ (\overline{A}_{iq} + \overline{B}_{iq}\frac{\cos\gamma_i\pi}{\sin\gamma_i\pi})$$

$$\times\frac{(-1)^n}{n!\Gamma(\gamma_i+n+1)}(\frac{\Omega_i\mu_{1j}}{2})^{2n+\gamma_i},$$

$$b_{inq}(\tau) = -\sum_{j=1}^{\infty}\frac{2}{\mu_{1j}\Delta'(\mu_{1j})}\exp[-\frac{\mu_{1j}^2}{4}\cdot\frac{\overline{\kappa}_1^0}{\overline{r}_a^{m_1-k_1}}(2-m_1+k_1)^2\tau]\ \overline{B}_{iq}\frac{1}{\sin\gamma_i\pi}$$

$$\times\frac{(-1)^n}{n!\Gamma(-\gamma_i+n+1)}(\frac{\Omega_i\mu_{1j}}{2})^{2n-\gamma_i},$$

$$\omega_{1i} = \frac{1}{2}[(2-m_i+k_i)(2n+\gamma_i)-m_i],$$

$$\omega_{2i} = \frac{1}{2}[(2-m_i+k_i)(2n-\gamma_i)-m_i] \tag{57}$$

We assume $U_{rpqi}(\overline{r})$ and $U_{\theta pqi}(\overline{r})$ of the particular solutions as follows

$$U_{rpqi}(\overline{r}) = B'_{1iq}\overline{r}^{\xi_{i1}+b_i+1} + B'_{2iq}\overline{r}^{\xi_{i2}+b_i+1} + \sum_{n=0}^{\infty}(B_{1inq}\overline{r}^{\omega_{1i}+b_i+1} + B_{2inq}\overline{r}^{\omega_{2i}+b_i+1}),$$

$$U_{\theta pqi}(\overline{r}) = C'_{1iq}\overline{r}^{\xi_{i1}+b_i+1} + C'_{2iq}\overline{r}^{\xi_{i2}+b_i+1} + \sum_{n=0}^{\infty}(C_{1inq}\overline{r}^{\omega_{1i}+b_i+1} + C_{2inq}\overline{r}^{\omega_{2i}+b_i+1}) \tag{58}$$

Substituting Eqs. (55), (56), (58) and the second term of right side of Eq. (36) into Eqs. (32) and (33), and later comparing the coefficients of functions with regard to $\overline{r}$ respectively, the constants B'_{1iq}, B'_{2iq}, C'_{1iq}, C'_{2iq}, B_{1inq}, B_{2inq}, C_{1inq} and C_{2inq} can be obtained.

Then, the stress components can be evaluated by substituting Eq. (36) into Eq. (26), and later into Eq. (27). Since the terms with the constant $G_{r0}^{(i)}$ is corresponding to the translation of rigid body, the next condition is added.

$$G_{r0}^{(N)} = 0 \tag{59}$$

The unknown constants in Eqs. (43), (45), (50), (51) and (52) are determined so as to satisfy the boundary condition (35).

3. Numerical results

We consider the functionally graded materials composed of titanium alloy (Ti-6Al-4V) and zirconium oxide (ZrO_2). The materials of the inner and outer surfaces are titanium alloy 100% and zirconium oxide 100%, respectively. We assume that the hollow cylinder is partially heated from the outer surface by surrounding media. In this formulation, the material properties of the interfaces can be decided using all manner of rule of mixtures. In the interests of simplicity, we apply the simplest linear law of mixture. The material properties g_i of the interface between ith layer and $(i+1)$th layer are assumed as follows

$$g_i = g_a + (g_b - g_a) f_i,\ 0 \le f_i \le 1\ ; i = 1,2,\cdots,N-1 \tag{60}$$

where g_a is the material property of the inner surface, and g_b is the material property of the outer surface. The numerical parameters of heat conduction, shape and f_i are presented as follows

$$H_b = 1.0,\ H_a = H_b \bar{\lambda}_{tb} / \bar{\lambda}_{ta}\ ,\ \bar{T}_a = 0\ ,\ \bar{T}_b = 1.0\ ,\ \bar{r}_a = 0.7\ ,$$

$$f_b(\theta) = (1 - \theta^2 / \theta_b^2) H(\theta_b - |\theta|)\ ,\ \theta_b = 30^\circ \tag{61}$$

Material 1: $N = 2\ ,\ \bar{r}_1 = 0.85\ ,\ f_1 = 0.1, 0.5, 0.9$ (62)

Material 2: $N = 2\ ,\ \bar{r}_1 = 0.73, 0.85, 0.97\ ,\ f_1 = 0.5$ (63)

Material 3: $N = 3\ ,\ \bar{r}_1 = 0.8,\ \bar{r}_2 = 0.9,\ f_1 = 0.1\ ,\ f_2 = 0.2,\ 0.5,\ 0.9$ (64)

Material 4: $N = 1$ (65)

The material constants for titanium alloy (Ti-6Al-4V) are taken as

$$\kappa = 2.61 \times 10^{-6}\ \mathrm{m^2/s}\ ,\ c = 537.7\ \mathrm{J/(kg \cdot K)}\ ,\ \rho = 4420\ \mathrm{kg/m^3}\ ,$$

$$\lambda_t = 6.2\ \mathrm{W/(m \cdot K)}\ ,\ \alpha = 8.9 \times 10^{-6}\ \mathrm{1/K}\ ,\ E = 105.8\ \mathrm{GPa}\ ,\ \nu = 0.3 \tag{66}$$

for zirconium oxide (ZrO_2)

$$\kappa = 1.06 \times 10^{-6}\ \mathrm{m^2/s}\ ,\ c = 461.4\ \mathrm{J/(kg \cdot K)}\ ,\ \rho = 3657\ \mathrm{kg/m^3}\ ,$$

$$\lambda_t = 1.78 \times 10\ \mathrm{W/(m \cdot K)}\ ,\ \alpha = 8.7 \times 10^{-6}\ \mathrm{1/K}\ ,\ E = 116.4 \times 10\ \mathrm{GPa}\ ,\ \nu = 0.3 \tag{67}$$

The typical values of material properties such as κ_0, λ_{t0}, α_0 and E_0 used to normalize the numerical data, are based on those of zirconium oxide.

The numerical results for Material 1 are shown in Figs. 1-4. Fig. 1 shows the variations of temperature change. The variation on the heated surface ($\bar{r} = 1$) is shown in Fig. 1(a) and the variation in the radial direction on the middle cross-section ($\theta = 0$) is shown in Fig. 1(b). As shown in Fig. 1(a), the temperature rise can clearly be seen in the heated region ($0^\circ \le \theta \le 30^\circ$). As shown in Fig. 1, the temperature rises as the time proceeds and is greatest in a steady state. From Fig. 1, it can be seen that the temperature change on the heated surface increases when the

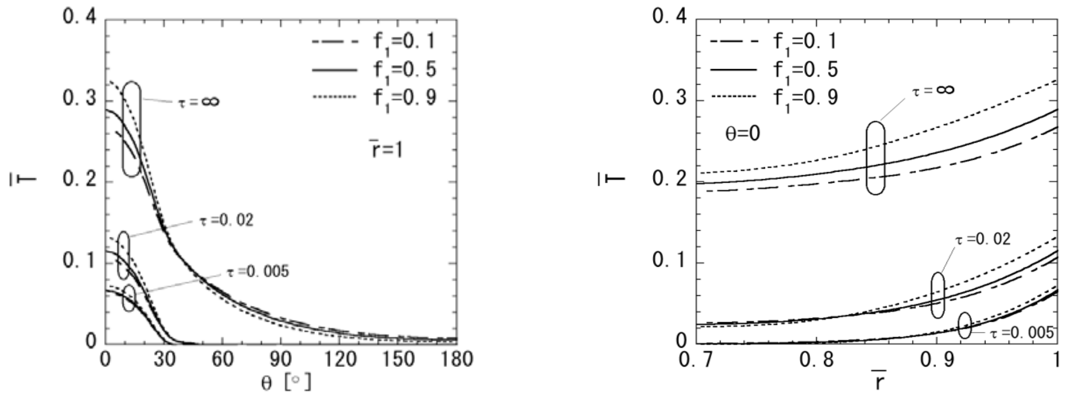

Fig. 1 Temperature change (Material 1, $N = 2$, $\bar{r}_1 = 0.85$): (a) variation in the heated surface ($\bar{r} = 1.0$) and (b) variation in the radial direction ($\theta = 0$)

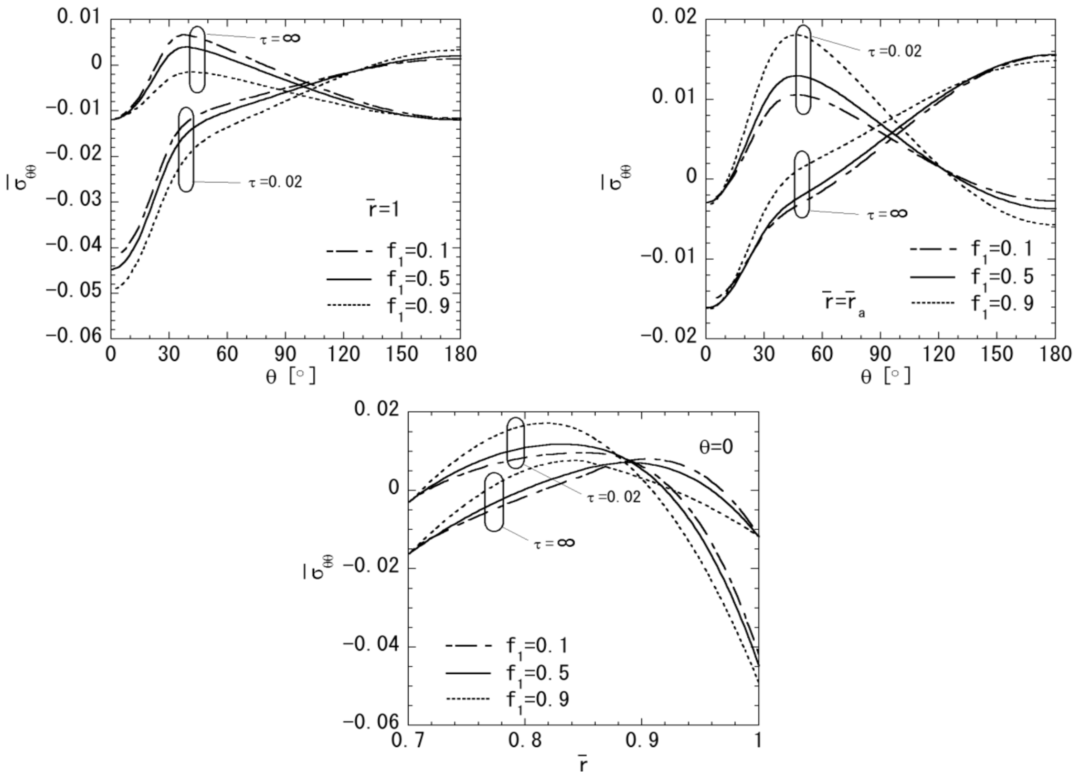

Fig. 2 Variation of thermal stress $\bar{\sigma}_{\theta\theta}$ (Material 1, $N = 2$, $\bar{r}_1 = 0.85$): (a) variation in the heated surface ($\bar{r} = 1.0$), (b) variation in the inner surface ($\bar{r} = 0.7$) and (c) variation in the radial direction ($\theta = 0$)

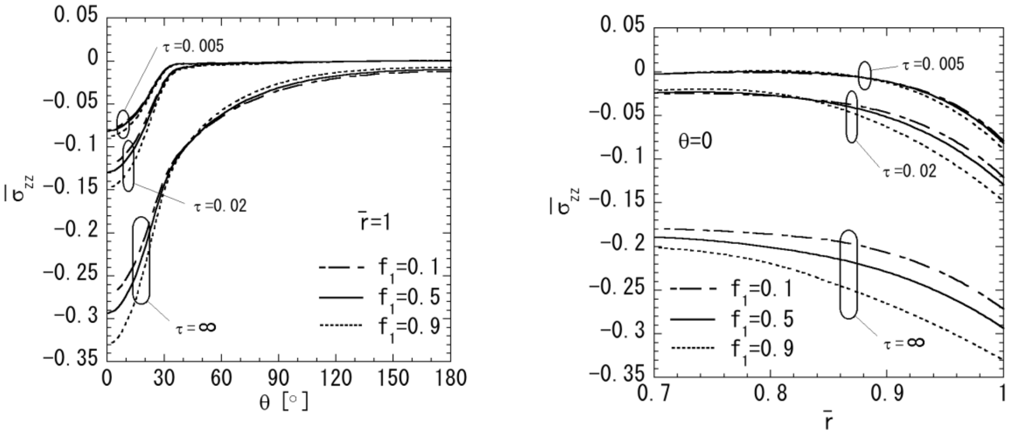

Fig. 3 Variation of thermal stress $\overline{\sigma}_{zz}$ (Material 1, N = 2, $\overline{r}_1$ = 0.85): (a) variation in the heated surface ($\overline{r}$ = 1.0) and (b) variation in the radial direction (θ = 0)

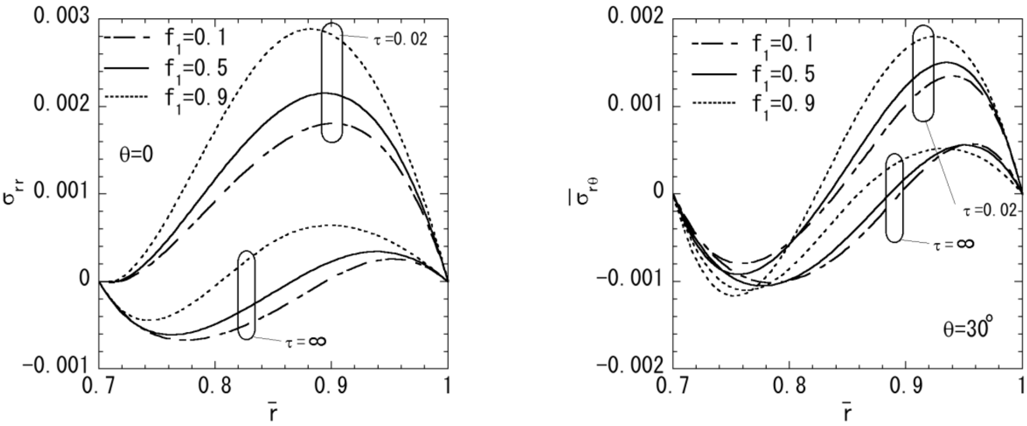

Fig. 4 Variation of thermal stresses in the radial direction (Material 1, N = 2, $\overline{r}_1$ = 0.85): (a) normal stress $\overline{\sigma}_{rr}$ (θ = 0) and (b) shearing stress $\overline{\sigma}_{r\theta}$ ($\theta = 30^{\circ}$)

parameter f_1 increases. Fig. 2 shows the variations of thermal stress $\overline{\sigma}_{\theta\theta}$. The variation on the heated surface ($\overline{r} = 1$) is shown in Fig. 2(a), the variation on the inner surface ($\overline{r} = \overline{r}_a$) is shown in Fig. 2(b), and the variation in the radial direction on the middle cross section (θ = 0) is shown in Fig. 2(c). From Fig. 2(a), the thermal stress $\overline{\sigma}_{\theta\theta}$ shows compressive stress on the outer surface, and the maximum compressive stress occurs in a transient state. From Fig. 2(b), the large tensile stress occurs in the inner surface. From Fig. 2(c), the large tensile stress occurs inner part of the hollow cylinder and large compressive stress occurs on the outer surface. As shown in Fig. 2, the maximum tensile stress and the maximum compressive stress decrease when the parameter f_1 decreases except on the heated surface in the steady state. Fig. 3 shows the variations of normal stress $\overline{\sigma}_{zz}$. The variation on the heated surface ($\overline{r} = 1$) is shown in Fig. 3(a), and the variation in

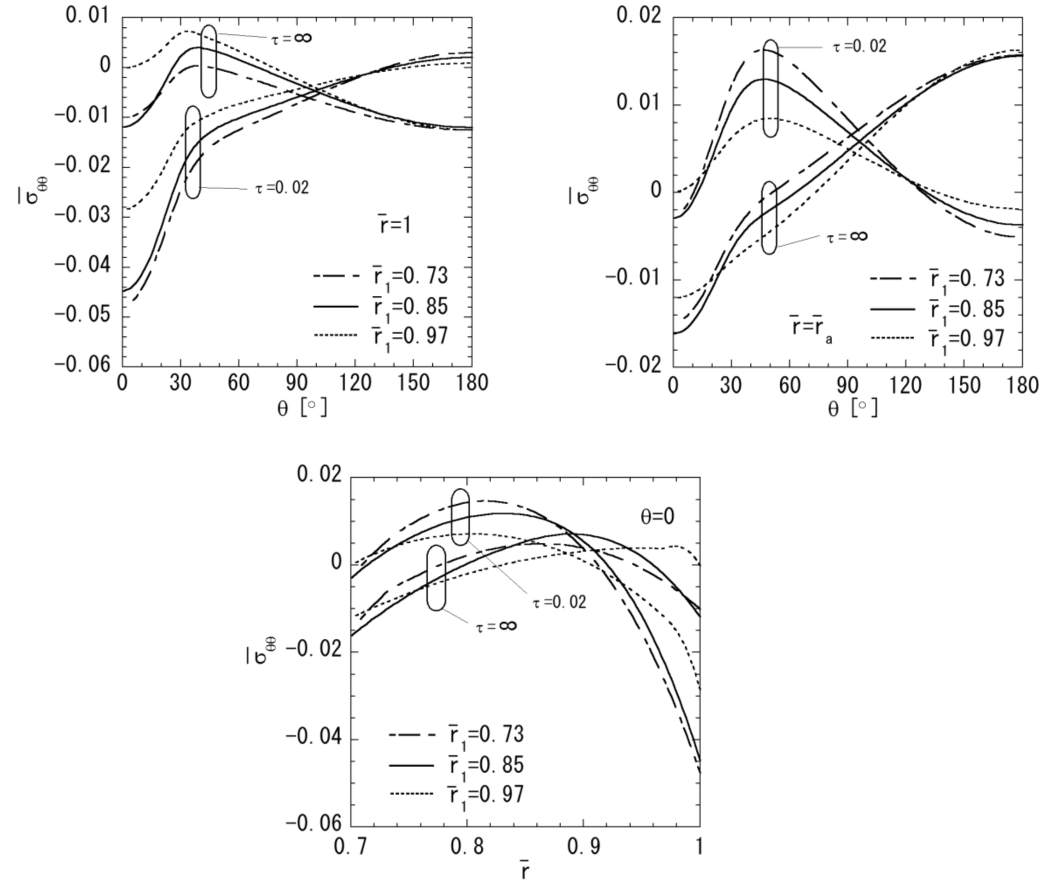

Fig. 5 Variation of thermal stress $\overline{\sigma}_{\theta\theta}$ (Material 2, $N = 2, f_1 = 0.5$): (a) variation in the heated surface ($\overline{r}_1 = 1.0$), (b) variation in the inner surface ($\overline{r} = 0.7$) and (c) variation in the radial direction ($\theta = 0$)

the radial direction on the middle cross section ($\theta = 0$) is shown in Fig. 3(b). From Fig. 3, the thermal stress $\overline{\sigma}_{zz}$ shows compression almost. The absolute value of thermal stress $\overline{\sigma}_{zz}$ rises as the time proceeds and is greatest in a steady state. The maximum compressive stress of $\overline{\sigma}_{zz}$ decreases when the parameter f_1 decreases. Fig. 4 shows the variations of thermal stresses in the radial direction. The variations of normal stress $\overline{\sigma}_{rr}$ on the middle cross section ($\theta = 0$) and shearing stress $\overline{\sigma}_{r\theta}$ at the edge ($\theta = 30^{\circ}$) of the heated region are shown in Figs. 4(a) and 4(b), respectively. From Fig. 4(a), the maximum tensile stress occurs in a transient state, the maximum tensile stress decreases when the parameter f_1 decreases. From Fig. 4(b), the maximum shearinge stress occurs in a transient state, the maximum stress decreases when the parameter f_1 decreases.

The numerical results for Material 2 are shown in Figs. 5-7. Fig. 5 shows the variations of thermal stress $\overline{\sigma}_{\theta\theta}$. The variation on the heated surface ($\overline{r} = 1$) is shown in Fig. 5(a), the variation on the inner surface ($\overline{r} = \overline{r}_a$) is shown in Fig. 5(b), and the variation in the radial direction on the middle cross section ($\theta = 0$) is shown in Fig. 5(c). Fig. 6 shows the variations of normal stress $\overline{\sigma}_{zz}$.

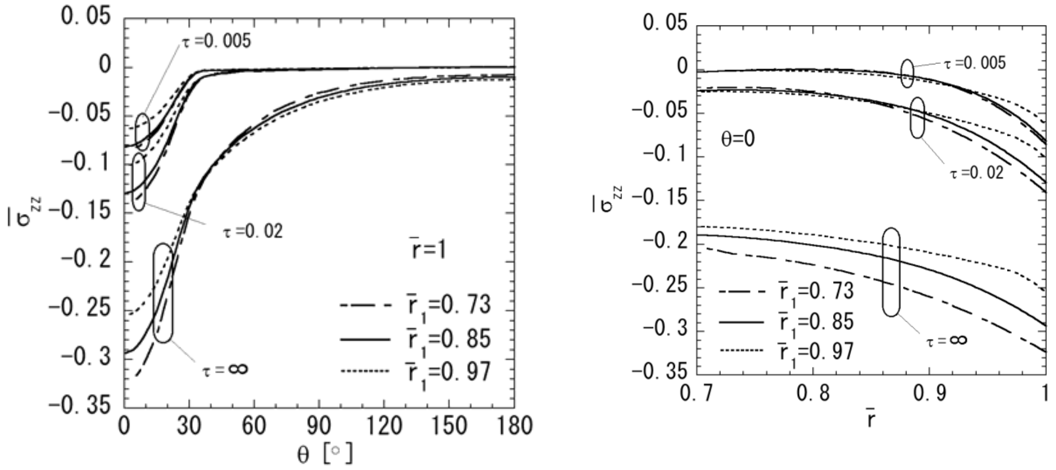

Fig. 6 Variation of thermal stress $\overline{\sigma}_{zz}$ (Material 2, $N = 2, f_1 = 0.5$): (a) variation in the heated surface ($\overline{r} = 1.0$) and (b) variation in the radial direction ($\theta = 0$)

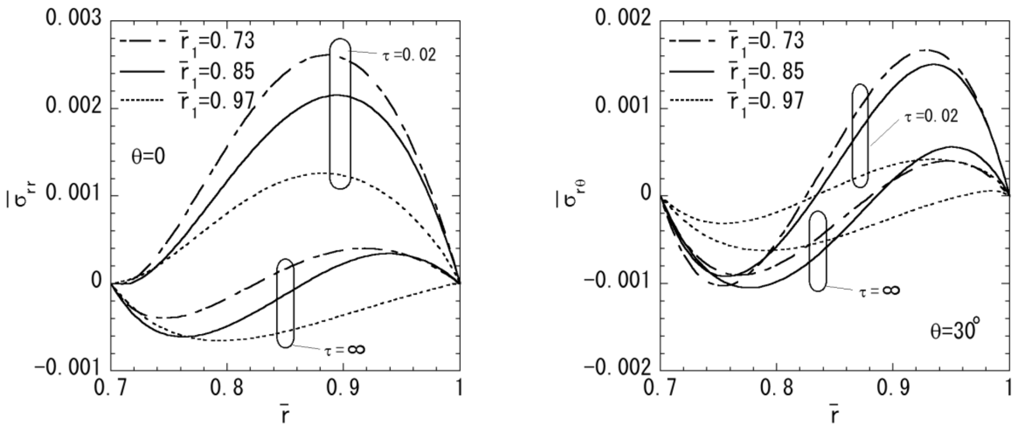

Fig. 7 Variation of thermal stresses in the radial direction (Material 2, $N = 2, f_1 = 0.5$): (a) normal stress $\overline{\sigma}_{rr}$ ($\theta = 0$) and (b) shearing stress $\overline{\sigma}_{r\theta}$ ($\theta = 30^{\circ}$)

The variation on the heated surface ($\overline{r} = 1$) is shown in Fig. 6(a), and the variation in the radial direction on the middle cross section ($\theta = 0$) is shown in Fig. 6(b). Fig. 7 shows the variations of thermal stresses in the radial direction. The variations of normal stress $\overline{\sigma}_{rr}$ on the middle cross section ($\theta = 0$) and shearing stress $\overline{\sigma}_{r\theta}$ at the edge ($\theta = 30^{\circ}$) of the heated region are shown in Figs. 7(a) and 7(b), respectively. As shown in Figs. 5-7, it can be seen that the values decrease when the outer radius of first layer $\overline{r}_1$ increases without distinct of time, except the normal stress $\overline{\sigma}_{\theta\theta}$ on the heated surface in the steady state..

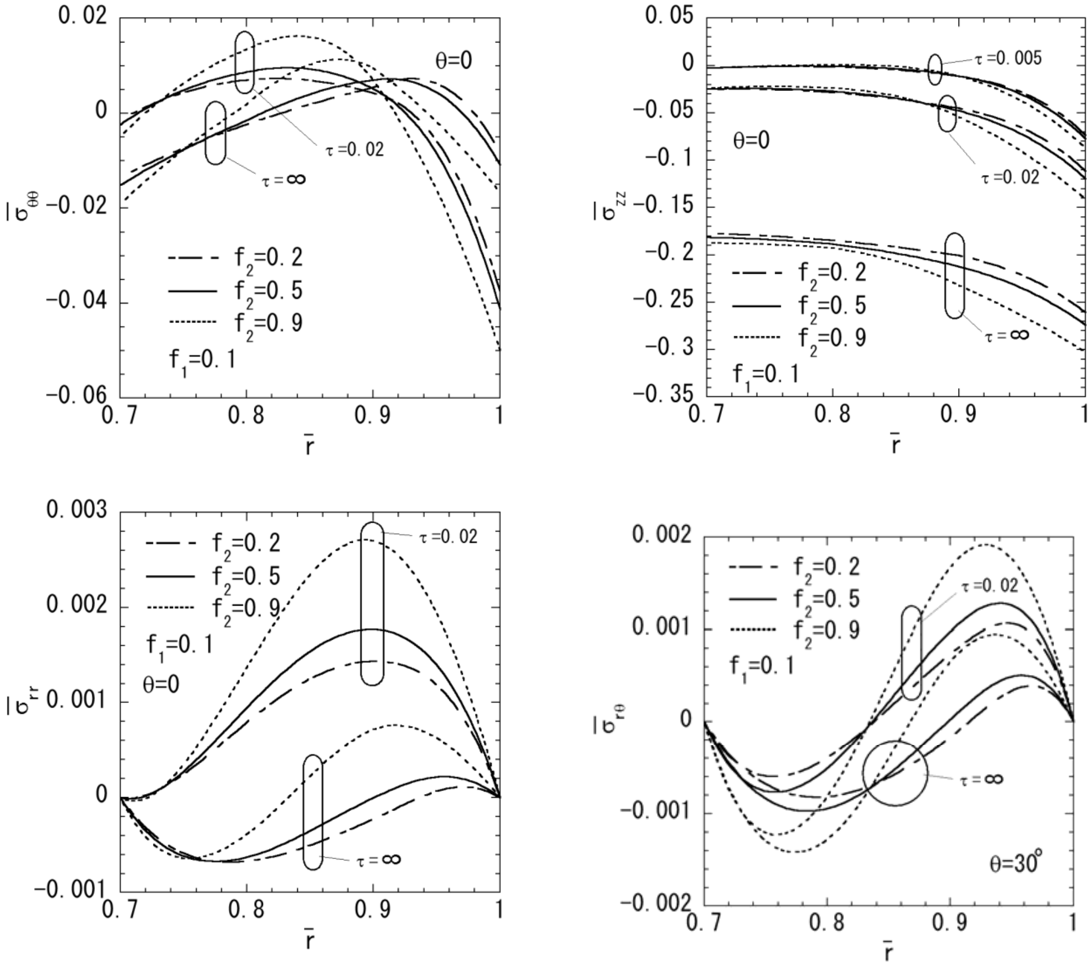

Fig. 8 Variation of thermal stresses in the radial direction (Material 3, N = 3, $\bar{r}_1$ = 0.8, $\bar{r}_2$ = 0.9, f_1 =0.1): (a) normal stress $\overline{\sigma}_{\theta\theta}$ (θ = 0), (b) normal stress $\overline{\sigma}_{zz}$ (θ = 0), (c) normal stress $\overline{\sigma}_{rr}$ (θ = 0), and (d) shearing stress $\overline{\sigma}_{r\theta}$ ($\theta = 30^{\circ}$).

The numerical results for Material 3 are shown in Fig. 8. Figs. 8 (a), (b) and (c) show the variations of thermal stresses $\overline{\sigma}_{\theta\theta}$, $\overline{\sigma}_{zz}$ and $\overline{\sigma}_{rr}$ along the radial direction on the middle cross section (θ = 0), respectively. Fig. 8(d) shows the variation of shearing stress $\overline{\sigma}_{r\theta}$ in the radial direction at the edge ($\theta = 30^{\circ}$) of the heated region. As shown in Fig. 8, it can be seen that the values decrease when the parameter f_2 decreases without distinct of time.

In order to assess the influence of the functional grading, the numerical results for Material 4, i.e., one-layered FGM model, are shown in Fig. 9. Figs. 9 (a), (b) and (c) show the variations of thermal stresses $\overline{\sigma}_{\theta\theta}$, $\overline{\sigma}_{zz}$ and $\overline{\sigma}_{rr}$ along the radial direction on the middle cross section (θ = 0), respectively. Fig. 9(d) shows the variation of shearing stress $\overline{\sigma}_{r\theta}$ in the radial direction at the

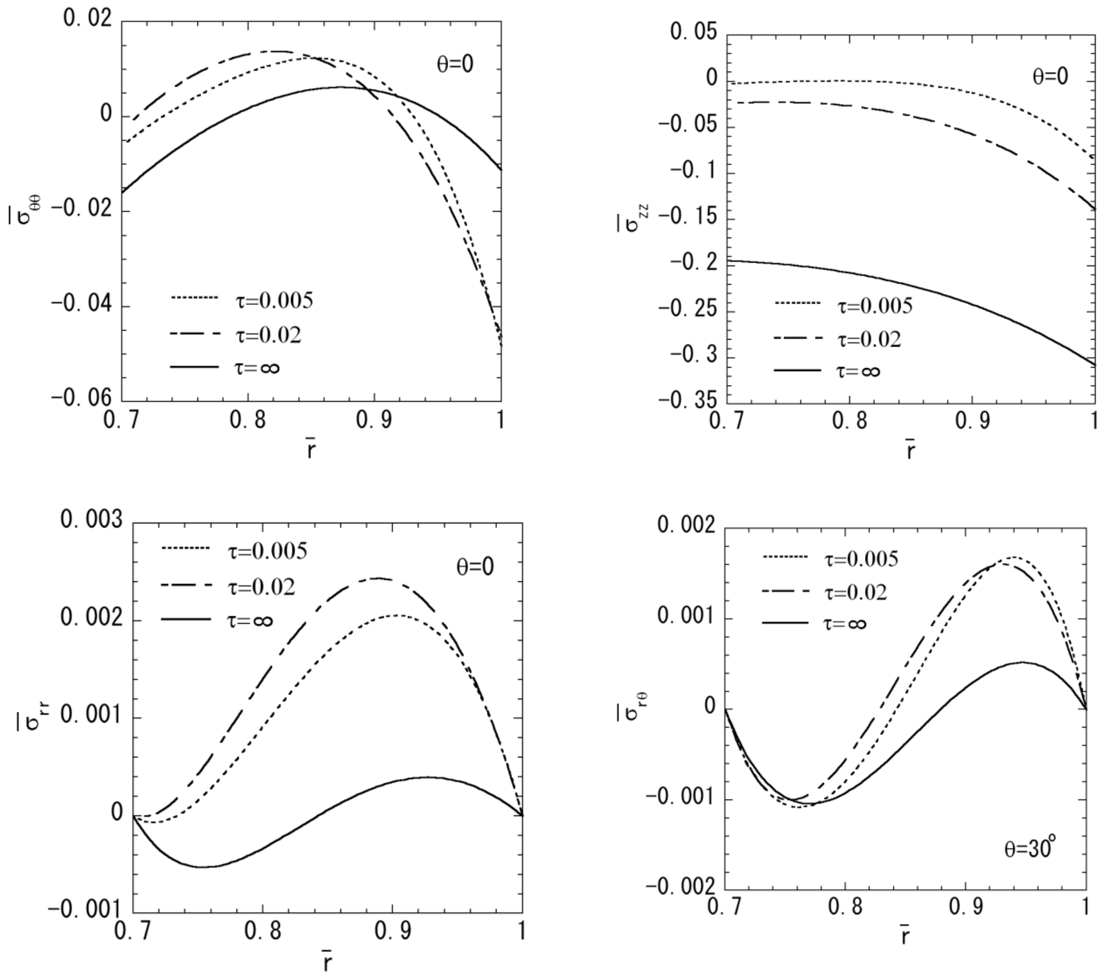

Fig. 9 Results for one-layered model (Material 4, N = 1): (a) normal stress $\overline{\sigma}_{\theta\theta}$ (θ = 0), (b) normal stress $\overline{\sigma}_{zz}$ (θ = 0), (c) normal stress $\overline{\sigma}_{rr}$ (θ = 0), and (d) shearing stress $\overline{\sigma}_{r\theta}$ ($\theta = 30^{\circ}$)

edge ($\theta = 30^{\circ}$) of the heated region. In comparison with the numerical results for Materails 1, 2 and 3, it is possible to decrease the maximum values of thermal stresses $\overline{\sigma}_{\theta\theta}$, $\overline{\sigma}_{zz}$, $\overline{\sigma}_{rr}$ and $\overline{\sigma}_{r\theta}$ using the multilayered FGM model with piecewise power law .

4. Conclusions

In the present article, we analyzed the transient thermoelastic problem involving a functionally graded hollow cylinder with piecewise power law due to asymmetrical heating from its surfaces. The thermal and thermoelastic constants of each layer are expressed as power functions of the radial coordinate in the radial direction, and their values continue on the interfaces. We obtained the exact solution for the transient two-dimensional temperature and transient thermoelastic response of a functionally hollow cylinder with piecewise power law under the state of plane strain.

As an illustration, we carried out numerical calculations for the functionally graded materials composed of titanium alloy (Ti-6Al-4V) and zirconium oxide (ZrO_2) and examined the behaviors in the transient state for the temperature change, the thermal stress distributions. Furthermore, the influence of the functional grading on the thermal stresses is investigated.

It is difficult to obtain the exact solution of the asymmetric transient thermal stress problem involving a functionally graded hollow cylinder using the arbitrary manner of rule of mixtures in all areas to the radial direction. Though the material properties in each layer are expressed as power functions of the radial coordinate, the material properties of the interfaces can be decided using all manner of rule of mixtures. By increasing a number of layers, the manner of rule of mixtures can be applied to all areas to the radial direction approximately.

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