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Exact solution for free vibration of curved beams with variable curvature and torsion

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Abstract. For the purpose of investigating the free vibration response of the spatial curved beams, the governing equations are derived in matrix formats, considering the variable curvature and torsion. The theory includes all the effects of rotary inertia, shear and axial deformations. Frobenius' scheme and the dynamic stiffness method are then applied to solve these equations. A computer program is coded in Mathematica according to the proposed method. As a special case, the dynamic stiffness and further the natural frequencies of a cylindrical helical spring under fixed-fixed boundary condition are carried out. Comparison of the present results with the FEM results using body elements in I-DEAS shows good accuracy in computation and validity of the model. Further, the present model is used for reciprocal spiral rods with different boundary conditions, and the comparison with FEM results shows that only a limited number of terms in the resultant provide a relatively accurate solution.

Keywords: curved beam; free vibration; variable curvature and torsion; dynamic stiffness; exact solution

1. Introduction

Curved beams have been found extensive use in modern world from traditional bridges and machinery structures to light rails for city transportation and scenic railways in the play grounds. Larger span with variable curvature and torsion and higher speed moving loads interest the researchers. Based on the static analysis, the studies began to focus on the dynamic response of spatial curved beams. Earliest work casts back to Michell's (1890), which obtained three equations of motion by using the first form of Lagrange's fundamental equation. Love (1899) obtained six equations of motion based on the same assumptions as Michell's. These equations were later modified by Yoshimura and Murata (1952) to include the torsional inertia, and then by Wittrick (1966) to add the rotary inertia and Timoshenko shear deformation effects. But because of the mathematical difficulties on solving the equations, various numerical methods were applied to attack such problems. Based on the Timoshenko beam theory, Kiral and Ertepinar (1974a, b) derived governing equations of the free and forced vibration of curved space rod in the canonical

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form and solved the free vibration problem by the transfer matrix method. Using the equations given by Wittrick (1966), Mottershead (1980) computed the natural frequencies of the free vibration of cylindrical helical rods by the finite element method and compared them with the results from experiments and Pearson (1982) studied the free vibration of the compressed circular cross-sectioned cylindrical helical springs by the transfer matrix method. Pearson and Wittrick (1986) obtained the dynamic stiffness matrix for the free vibration of a helical spring not considering the effect of shear deformation. Nagaya *et al*. (1986) gave the natural frequencies of noncircular helical springs with circular cross-section by the equivalence transfer matrix method and experimentally, where only the effect of axial deformation was considered. Tabarrok *et al*. (1988) examined free vibration of spatial curved and twisted rods with the aid of a finite element model and obtained displacement modes of a problem. Yildirim (1999, 2001, 2004) studied free vibration of cylindrical helical springs and unidirectional composite barrel and hyperboloidal springs with the help of the transfer matrix method and investigated the effect of the parameters to the natural frequencies. Lee and Thompson (2001) examined the free vibration and wave motion of helical springs and compared the natural frequencies of the dynamic stiffness matrix with those of the transfer matrix and the finite element methods. Temel and Calim (2003) presented a method for the analysis of the forced vibration of cylindrical helical rods under arbitrary time-dependent and impulsive loads in the Laplace domain. Taktak *et al*. (2008) introduced an efficiently finite element for dynamic analysis of a cylindrical isotropic helical spring. Calim (2009) investigated the dynamic behavior of composite barrel and hyperboloidal springs using the complementary functions method. Yu *et al*. (2008, 2010) and Hao and Yu (2011) carried out an analytical study for free vibration of naturally curved and twisted beams with uniform cross-sectional shapes using spatial curved beam theory based on the Washizu's static model (1964) and investigated the free vibrational behavior of cylindrical helical springs with different cross-section.

Of the vast literature in dynamic analysis of spatial curved beam, most of it aimed to the cylindrical helical rods which curvature and torsion are constant. Although there are some papers on the free and forced vibration analysis of composite barrel and hyperboloidal springs (Yildirim 2001, 2004, Calim 2009), research on the analysis of the universal spatial curved beams with variable curvature and torsion considering the effects of rotary inertia, shear and axial deformations is scarce. In this study, the governing equations for free vibration of spatial curved beams with variable curvature and torsion are set up, considering the effects of rotary inertia, shear and axial deformations. Frobenius' theory (Whittaker and Watson 1965) combined with the dynamic stiffness method (Tseng *et al*. 1997) is applied to solve these equations. Compared with the traditional finite element method and other approximate methods, the method allows an infinite number of natural frequencies and normal modes of a vibration structure to be computed through few degrees of freedom. A computer program is coded in Mathematica according to the proposed method. As a special case, the dynamic stiffness and further the natural frequencies of a cylindrical helical spring under fixed-fixed boundary condition are carried out. Comparison of the present results with the FEM results using body elements in I-DEAS shows good accuracy in computation of the model. Further, the validity of the present solution for free vibration is demonstrated through comparison with FEM results for a reciprocal spiral rod. This provides great convenience in the solution of the problems with general boundary conditions, which usually need to take high number of elements in FEM to get the satisfactory solutions.

2. Governing equations

Let the locus of the cross-sectional centroid of a spatial curved beam be a continuous spatial curve. The tangential, normal and binormal unit vectors of the curve are shown by $\boldsymbol{e}_s$, $\boldsymbol{e}_n$ and $\boldsymbol{e}_b$, respectively. Through the centroid O_1 of the cross-section, let ξ and η directions be in coincidence with the principal axes. $\boldsymbol{e}_\xi$ and $\boldsymbol{e}_\eta$ are unit vectors of ξ axis and η axis, respectively. s is the coordinate of the curve. Assuming the normal and binormal axes are the principle axes, the equations of motion obtained from dynamic equilibrium, if there is no external loading, are

$$\begin{aligned} &\frac{\partial}{\partial s}\boldsymbol{Q} - \boldsymbol{K}\boldsymbol{Q} = \boldsymbol{C}\ddot{\boldsymbol{u}} \\ &\frac{\partial}{\partial s}\boldsymbol{M} - \boldsymbol{K}\boldsymbol{M} - \boldsymbol{H}\boldsymbol{Q} = \boldsymbol{N}\ddot{\boldsymbol{\varphi}} \end{aligned} \tag{1}$$

where

$$\boldsymbol{Q} = \{Q_s \quad Q_\xi \quad Q_\eta\}^{\mathrm{T}}, \ \boldsymbol{M} = \{M_s \quad M_\xi \quad M_\eta\}^{\mathrm{T}},$$

$$\boldsymbol{u} = \{u_s \quad u_\xi \quad u_\eta\}^{\mathrm{T}}, \ \varphi = \{\varphi_{\mathrm{s}} \quad \varphi_\xi \quad \varphi_\eta\}^{\mathrm{T}},$$

$$\boldsymbol{K} = \begin{bmatrix} 0 & k & 0 \\ -k & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}, \ \boldsymbol{H} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix},$$

$$\boldsymbol{C} = \begin{bmatrix} \rho A_s & & \\ & \rho A_s & \\ & & \rho A_s \end{bmatrix}, \ \boldsymbol{N} = \begin{bmatrix} \rho J & & \\ & \rho I_\xi & \\ & & \rho I_\eta \end{bmatrix},$$

$\boldsymbol{Q}$ and $\boldsymbol{M}$ is the principal vector and principal moment, respectively, when simplifying the cross-sectional stress vectors to the centroid O_1, of which components are respectively denoted by Q_s axial force, Q_ξ and Q_η shear forces in the ξ and η directions, M_s torque and M_ξ and M_η bending moments in $e_sO_1e_\eta$ and $e_sO_1e_\xi$ plane; k and τ, curvature and torsion of the curve; $\boldsymbol{u}$ and φ, the generalized displacements; ρ, the mass per unit volume; A_s, the cross-sectional area and J, torsional constant of cross-section; I_ξ and I_η, the area moments of inertia around ξ and η axes. The dots denote time derivatives.

Assuming that the effect of warping is ignored, and that the material of the beam is linear elastic, the geometrical equations and constitutive equations are written in matrix form as (Zhu and Zhao 2008, Zhu *et al*. 2010, Oz 2010)

$$\begin{aligned} &\frac{\partial}{\partial s}\varphi - \boldsymbol{K}\varphi - \boldsymbol{\omega} = 0 \\ &\frac{\partial}{\partial s}\boldsymbol{u} - \boldsymbol{K}\boldsymbol{u} - \boldsymbol{H}\varphi - \boldsymbol{\varepsilon} = 0 \end{aligned} \tag{2}$$

$$\begin{aligned} &\boldsymbol{Q} = \boldsymbol{B}\boldsymbol{\varepsilon} \\ &\boldsymbol{M} = \boldsymbol{D}\boldsymbol{\omega} \end{aligned} \tag{3}$$

where

$$\boldsymbol{B}=\begin{bmatrix} EA_s & & \\ & GA_s & \\ & & GA_s \end{bmatrix},\quad \boldsymbol{D}=\begin{bmatrix} GJ & & \\ & EI_\xi & \\ & & EI_\eta \end{bmatrix},$$

E and G are respectively Young's and shear moduli, ε and ω are the generalized strains.

Substituting Eq. (2) into Eq. (3), the relations between the displacement and rotation components and the stress resultants are given as

$$\begin{aligned} \boldsymbol{Q} &= \boldsymbol{B}\left(\frac{\partial}{\partial s}\boldsymbol{u}-\boldsymbol{K}\boldsymbol{u}-\boldsymbol{H}\varphi\right) \\ \boldsymbol{M} &= \boldsymbol{D}\left(\frac{\partial}{\partial s}\varphi-\boldsymbol{K}\varphi\right) \end{aligned} \tag{4}$$

And then substituting Eq. (4) into Eq. (1), the governing equations for free vibration of spatial curved beams are:

$$\begin{aligned} &\boldsymbol{B}\frac{\partial^2}{\partial s^2}\boldsymbol{u}-(\boldsymbol{BK}+\boldsymbol{KB})\frac{\partial}{\partial s}\boldsymbol{u}-\boldsymbol{BH}\frac{\partial}{\partial s}\varphi+\left(\boldsymbol{KBK}-\boldsymbol{B}\frac{\partial}{\partial s}\boldsymbol{K}\right)\boldsymbol{u}+\boldsymbol{KBH}\varphi=\boldsymbol{C}\ddot{\boldsymbol{u}} \\ &\boldsymbol{D}\frac{\partial^2}{\partial s^2}\varphi-(\boldsymbol{DK}+\boldsymbol{KD})\frac{\partial}{\partial s}\varphi-\boldsymbol{HB}\frac{\partial}{\partial s}\boldsymbol{u}+\boldsymbol{HBK}\boldsymbol{u}+\left(\boldsymbol{KDK}+\boldsymbol{HBH}-\boldsymbol{D}\frac{\partial}{\partial s}\boldsymbol{K}\right)\varphi=\boldsymbol{N}\ddot{\varphi} \end{aligned} \tag{5}$$

3. Method of solution

As is known, for a spatial curved beam such as a helix, it is simple to give its parametric equation as a function of the parameter ϕ. Therefore, in the following formulation of solution, the coordinate s is transformed to ϕ. Furthermore, introducing a representative length of the beam L, non-dimensional parameters are defined as

$$\overline{\boldsymbol{u}}=\frac{\boldsymbol{u}}{L},\quad \bar{s}=\frac{s}{L},\quad \overline{\boldsymbol{K}}=\begin{bmatrix} 0 & \bar{k} & 0 \\ -\bar{k} & 0 & \bar{\tau} \\ 0 & -\bar{\tau} & 0 \end{bmatrix}=L\boldsymbol{K} \tag{6}$$

And for the free vibration analysis, the solutions of Eq. (5) can be assumed to take the form as:

$$\overline{\boldsymbol{u}}(\phi,t)=\mathrm{e}^{\mathrm{i}\omega t}\overline{\boldsymbol{U}}(\phi),\varphi(\phi,t)=\mathrm{e}^{\mathrm{i}\omega t}\boldsymbol{\Phi}(\phi). \tag{7}$$

Substituting Eq. (6) and Eq. (7) into Eq. (5) and after some calculations, Eq. (5) can be transformed into a set of second order ordinary differential equations with coefficients that are functions of only one independent variable ϕ:

$$\begin{aligned}
&\boldsymbol{B}\overline{\boldsymbol{U}}'' - \frac{1}{\xi}(\boldsymbol{B}\overline{\boldsymbol{K}} + \overline{\boldsymbol{K}}\boldsymbol{B} - \xi'\boldsymbol{B})\,\overline{\boldsymbol{U}}' - \frac{1}{\xi}\boldsymbol{B}\boldsymbol{H}\boldsymbol{\Phi}' + \frac{1}{\xi^2}\overline{\boldsymbol{K}}\boldsymbol{B}\overline{\boldsymbol{K}}\overline{\boldsymbol{U}} + \frac{1}{\xi^2}\overline{\boldsymbol{K}}\boldsymbol{B}\boldsymbol{H}\boldsymbol{\Phi} - \\
&\quad - \frac{1}{\xi}\boldsymbol{B}\overline{\boldsymbol{K}}'\overline{\boldsymbol{U}} = -\frac{L^2\omega^2}{\xi^2}\boldsymbol{C}\overline{\boldsymbol{U}} \\
&\boldsymbol{D}\boldsymbol{\Phi}'' - \frac{1}{\xi}(\boldsymbol{D}\overline{\boldsymbol{K}} + \overline{\boldsymbol{K}}\boldsymbol{D} - \xi'\boldsymbol{D})\,\boldsymbol{\Phi}' - \frac{L^2}{\xi}\boldsymbol{H}\boldsymbol{B}\overline{\boldsymbol{U}}' + \frac{L^2}{\xi^2}\boldsymbol{H}\boldsymbol{B}\overline{\boldsymbol{K}}\overline{\boldsymbol{U}} + \frac{1}{\xi^2}(\overline{\boldsymbol{K}}\boldsymbol{D}\overline{\boldsymbol{K}} + L^2\boldsymbol{H}\boldsymbol{B}\boldsymbol{H})\,\boldsymbol{\Phi} - \\
&\quad - \frac{1}{\xi}\boldsymbol{D}\overline{\boldsymbol{K}}'\boldsymbol{\Phi} = -\frac{L^2\omega^2}{\xi^2}\boldsymbol{N}\boldsymbol{\Phi}
\end{aligned} \tag{8}$$

where the primes denote derivatives with respect to ϕ, and $\xi = \dfrac{d\phi}{d\overline{s}} = L\dfrac{d\phi}{ds}$.

The Frobenius' theory (Whittaker and Watson 1965) and the dynamic stiffness method (Tseng *et al*. 1997) are applied now to solve Eq. (8). At first, variable coefficients of the equations should be expressed in Taylor expansion series about a point on the beam under consideration with the non-dimensional position coordinates, δ:

$$\begin{aligned}
&\frac{\xi'}{\xi} = \sum_{p=0}^{P} a_p(\phi-\delta)^p, \frac{\overline{k}^2}{\xi^2} = \sum_{p=0}^{P} b_p(\phi-\delta)^p, \frac{\overline{k}\,\overline{\tau}}{\xi^2} = \sum_{p=0}^{P} c_p(\phi-\delta)^p, \frac{\overline{k}}{\xi^2} = \sum_{p=0}^{P} d_p(\phi-\delta)^p, \\
&\frac{\overline{k}}{\xi} = \sum_{p=0}^{P} e_p(\phi-\delta)^p, \frac{\overline{k}'}{\xi} = \sum_{p=0}^{P} f_p(\phi-\delta)^p, \frac{1}{\xi^2} = \sum_{p=0}^{P} g_p(\phi-\delta)^p, \frac{\overline{\tau}}{\xi} = \sum_{p=0}^{P} h_p(\phi-\delta)^p, \\
&\frac{1}{\xi} = \sum_{p=0}^{P} i_p(\phi-\delta)^p, \frac{\overline{\tau}^2}{\xi^2} = \sum_{p=0}^{P} j_p(\phi-\delta)^p, \frac{\overline{\tau}}{\xi^2} = \sum_{p=0}^{P} k_p(\phi-\delta)^p, \frac{\overline{\tau}'}{\xi} = \sum_{p=0}^{P} l_p(\phi-\delta)^p.
\end{aligned} \tag{9}$$

Consequently, the solutions of Eq. (8) can be expressed in term of polynomials as

$$\begin{aligned}
&\overline{U}_s = \sum_{v=0}^{V} A_v(\phi-\delta)^v, \overline{U}_\xi = \sum_{v=0}^{V} B_v(\phi-\delta)^v, \overline{U}_\eta = \sum_{v=0}^{V} C_v(\phi-\delta)^v, \\
&\Phi_s = \sum_{v=0}^{V} D_v(\phi-\delta)^v, \Phi_\xi = \sum_{v=0}^{V} E_v(\phi-\delta)^v, \Phi_\eta = \sum_{v=0}^{V} F_v(\phi-\delta)^v.
\end{aligned} \tag{10}$$

Theoretically, V should approach infinity. Nevertheless, only a finite number of terms in Eq. (10) are needed to obtain very accurate results. Substituting Eq. (9) and Eq. (10) into Eq. (8) with some rearrangement, the relationships between the coefficients in Eq. (10) can be shown as follows

$$\begin{aligned}
A_{v+2} = \frac{-1}{(v+2)(v+1)}\{&\sum_{p=0}^{v}[(p+1)a_{v-p}A_{p+1} + (\frac{\rho L^2\omega^2}{E}g_{v-p} - \lambda^2 b_{v-p})A_p + \lambda^2 c_{v-p}C_p \\
&+ \lambda^2 d_{v-p}F_p - (1+\lambda^2)(p+1)e_{v-p}B_{p+1} - f_{v-p}B_p]\},
\end{aligned}$$

$$B_{v+2}=\frac{-1}{(v+2)(v+1)}\{\sum_{p=0}^{v}[(p+1)a_{v-p}B_{p+1}+(1+\frac{1}{\lambda^2})(p+1)e_{v-p}A_{p+1}-2(p+1)h_{v-p}C_{p+1}$$
$$-(p+1)i_{v-p}F_{p+1}+(\frac{\rho L^2\omega^2}{E\lambda^2}g_{v-p}-\frac{1}{\lambda^2}b_{v-p}-j_{v-p})B_p-k_{v-p}E_p+f_{v-p}A_p-l_{v-p}C_p]\},$$
$$C_{v+2}=\frac{-1}{(v+2)(v+1)}\{\sum_{p=0}^{v}[(p+1)a_{v-p}C_{p+1}+2(p+1)h_{v-p}B_{p+1}+(p+1)i_{v-p}E_{p+1}+c_{v-p}A_p$$
$$-k_{v-p}F_p+l_{v-p}B_p+(\frac{\rho L^2\omega^2}{E\lambda^2}g_{v-p}-j_{v-p})C_p]\},$$
$$D_{v+2}=\frac{-1}{(v+2)(v+1)}\{\sum_{p=0}^{v}[(p+1)a_{v-p}D_{p+1}-(p+1)(1+\frac{I_\xi}{\lambda^2 J})e_{v-p}E_{p+1}+(\frac{\rho L^2\omega^2}{E\lambda^2}g_{v-p}-$$
$$\frac{I_\xi}{\lambda^2 J}b_{v-p})D_p+\frac{I_\xi}{\lambda^2 J}c_{v-p}F_p-f_{v-p}E_p]\},$$
$$E_{v+2}=\frac{-1}{(v+2)(v+1)}\{\sum_{p=0}^{v}[(p+1)a_{v-p}E_{p+1}+(p+1)(1+\frac{\lambda^2 J}{I_\xi})e_{v-p}D_{p+1}-(p+1)(1+\frac{I_\eta}{I_\xi})$$
$$h_{v-p}F_{p+1}+(\frac{\rho L^2\omega^2}{E}g_{v-p}-\frac{\lambda^2 J}{I_\xi}b_{v-p}-\frac{I_\eta}{I_\xi}j_{v-p}-\frac{L^2\lambda^2 A_s}{I_\xi}g_{v-p})E_p-(p+1)\frac{L^2\lambda^2 A_s}{I_\xi}$$
$$i_{v-p}C_{p+1}-\frac{L^2\lambda^2 A_s}{I_\xi}k_{v-p}B_p+f_{v-p}D_p-l_{v-p}F_p]\},$$
$$F_{v+2}=\frac{-1}{(v+2)(v+1)}\{\sum_{p=0}^{v}[(p+1)a_{v-p}F_{p+1}+(p+1)(1+\frac{I_\xi}{I_\eta})h_{v-p}E_{p+1}+\frac{I_\xi}{I_\eta}c_{v-p}D_p+(\frac{\rho L^2\omega^2}{E}\tag{11}$$
$$g_{v-p}-\frac{I_\xi}{I_\eta}j_{v-p}-\frac{L^2\lambda^2 A_s}{I_\eta}g_{v-p})F_p+(p+1)\frac{L^2\lambda^2 A_s}{I_\eta}i_{v-p}B_{p+1}+\frac{L^2\lambda^2 A_s}{I_\eta}d_{v-p}A_p-\frac{L^2\lambda^2 A_s}{I_\eta}$$
$$k_{v-p}C_p+l_{v-p}E_p]\}.$$

where v = 0, 1, 2, … From Eq. (11), A_{v+2}, B_{v+2}, C_{v+2}, D_{v+2}, E_{v+2}, F_{v+2} can be determined if A_0, A_1, B_0, B_1, C_0, C_1, D_0, D_1, E_0, E_1, F_0 and F_1 are known. As in the finite-element approach, the beam can be decomposed into several elements. Substituting Eq. (11) into Eq. (10), the end displacements for each element can be determined from Eq. (10) and expressed as

$$\boldsymbol{w}_n=\boldsymbol{\alpha}_n\boldsymbol{\chi}_n \tag{12}$$

where the subscript n for vectors and the matrix represents the results for the nth element. For the nth element, $\delta=\dfrac{\phi_n+\phi_{n+1}}{2}$.

$$\boldsymbol{w} = \{\overline{U}_{s0} \quad \overline{U}_{\xi 0} \quad \overline{U}_{\eta 0} \quad \varPhi_{s0} \quad \varPhi_{\xi 0} \quad \varPhi_{\eta 0} \quad \overline{U}_{s1} \quad \overline{U}_{\xi 1} \quad \overline{U}_{\eta 1} \quad \varPhi_{s1} \quad \varPhi_{\xi 1} \quad \varPhi_{\eta 1}\}^{\mathrm{T}}$$

are the nodal displacements and rotations of the nth element, and

$$\boldsymbol{\chi} = \{A_0 \quad A_1 \quad B_0 \quad B_1 \quad C_0 \quad C_1 \quad D_0 \quad D_1 \quad E_0 \quad E_1 \quad F_0 \quad F_1\}^{\mathrm{T}},$$

$$\boldsymbol{\alpha}_n = \begin{bmatrix} \overline{u}_{s0}(\phi_n) & \overline{u}_{s1}(\phi_n) & \overline{u}_{s2}(\phi_n) & \overline{u}_{s3}(\phi_n) & \overline{u}_{s4}(\phi_n) & \cdots & \overline{u}_{s11}(\phi_n) \\ \overline{u}_{\xi 0}(\phi_n) & \overline{u}_{\xi 1}(\phi_n) & \overline{u}_{\xi 2}(\phi_n) & \overline{u}_{\xi 3}(\phi_n) & \overline{u}_{\xi 4}(\phi_n) & \cdots & \overline{u}_{\xi 11}(\phi_n) \\ \overline{u}_{\eta 0}(\phi_n) & \overline{u}_{\eta 1}(\phi_n) & \overline{u}_{\eta 2}(\phi_n) & \overline{u}_{\eta 3}(\phi_n) & \overline{u}_{\eta 4}(\phi_n) & \cdots & \overline{u}_{\eta 11}(\phi_n) \\ \varphi_{s0}(\phi_n) & \varphi_{s1}(\phi_n) & \varphi_{s2}(\phi_n) & \varphi_{s3}(\phi_n) & \varphi_{s4}(\phi_n) & \cdots & \varphi_{s11}(\phi_n) \\ \varphi_{\xi 0}(\phi_n) & \varphi_{\xi 1}(\phi_n) & \varphi_{\xi 2}(\phi_n) & \varphi_{\xi 3}(\phi_n) & \varphi_{\xi 4}(\phi_n) & \cdots & \varphi_{\xi 11}(\phi_n) \\ \varphi_{\eta 0}(\phi_n) & \varphi_{\eta 1}(\phi_n) & \varphi_{\eta 2}(\phi_n) & \varphi_{\eta 3}(\phi_n) & \varphi_{\eta 4}(\phi_n) & \cdots & \varphi_{\eta 11}(\phi_n) \\ \overline{u}_{s0}(\phi_{n+1}) & \overline{u}_{s1}(\phi_{n+1}) & \overline{u}_{s2}(\phi_{n+1}) & \overline{u}_{s3}(\phi_{n+1}) & \overline{u}_{s4}(\phi_{n+1}) & \cdots & \overline{u}_{s11}(\phi_{n+1}) \\ \overline{u}_{\xi 0}(\phi_{n+1}) & \overline{u}_{\xi 1}(\phi_{n+1}) & \overline{u}_{\xi 2}(\phi_{n+1}) & \overline{u}_{\xi 3}(\phi_{n+1}) & \overline{u}_{\xi 4}(\phi_{n+1}) & \cdots & \overline{u}_{\xi 11}(\phi_{n+1}) \\ \overline{u}_{\eta 0}(\phi_{n+1}) & \overline{u}_{\eta 1}(\phi_{n+1}) & \overline{u}_{\eta 2}(\phi_{n+1}) & \overline{u}_{\eta 3}(\phi_{n+1}) & \overline{u}_{\eta 4}(\phi_{n+1}) & \cdots & \overline{u}_{\eta 11}(\phi_{n+1}) \\ \varphi_{s0}(\phi_{n+1}) & \varphi_{s1}(\phi_{n+1}) & \varphi_{s2}(\phi_{n+1}) & \varphi_{s3}(\phi_{n+1}) & \varphi_{s4}(\phi_{n+1}) & \cdots & \varphi_{s11}(\phi_{n+1}) \\ \varphi_{\xi 0}(\phi_{n+1}) & \varphi_{\xi 1}(\phi_{n+1}) & \varphi_{\xi 2}(\phi_{n+1}) & \varphi_{\xi 3}(\phi_{n+1}) & \varphi_{\xi 4}(\phi_{n+1}) & \cdots & \varphi_{\xi 11}(\phi_{n+1}) \\ \varphi_{\eta 0}(\phi_{n+1}) & \varphi_{\eta 1}(\phi_{n+1}) & \varphi_{\eta 2}(\phi_{n+1}) & \varphi_{\eta 3}(\phi_{n+1}) & \varphi_{\eta 4}(\phi_{n+1}) & \cdots & \varphi_{\eta 11}(\phi_{n+1}) \end{bmatrix}_n,$$

in which $\overline{u}_{sj}$, $\overline{u}_{\xi j}$, $\overline{u}_{\eta j}$, φ_{sj}, $\varphi_{\xi j}$ and $\varphi_{\eta j}$ (j = 0, 1, 2, …, 11) are polynomials the coefficients of which are determined when substituting Eq. (11) into Eq. (10) through the aid of some commercial softwares such as Mathematica and so on.

According to Eq. (7), the solutions for the stress resultants can be expressed as

$$\overline{\boldsymbol{Q}}(\phi,t) = \mathrm{e}^{\mathrm{i}\omega t}\overline{\boldsymbol{q}}(\phi), \overline{\boldsymbol{M}}(\phi,t) = \mathrm{e}^{\mathrm{i}\omega t}\overline{\boldsymbol{m}}(\phi).$$

After transforming the coordinate s to ϕ and using Eq. (6), Eq. (4) can be expressed as follows

$$\begin{aligned} \overline{\boldsymbol{q}} &= \boldsymbol{B}\,(\overline{\boldsymbol{U}}'\xi - \overline{\boldsymbol{K}}\overline{\boldsymbol{U}} - \boldsymbol{H\Phi}) \\ \overline{\boldsymbol{m}} &= \frac{1}{L}\boldsymbol{D}\,(\boldsymbol{\Phi}'\xi - \overline{\boldsymbol{K}}\boldsymbol{\Phi}) \end{aligned} \tag{13}$$

Substituting Eq. (12) into Eq. (13), the end stress resultants for the nth element of the beam can be shown as follows

$$\boldsymbol{f}_n = \boldsymbol{\beta}_n\boldsymbol{\chi}_n = \boldsymbol{\beta}_n\boldsymbol{\alpha}_n^{-1}\boldsymbol{w}_n = \boldsymbol{G}_n\boldsymbol{w}_n \tag{14}$$

where

$$\boldsymbol{f} = \{\overline{q}_{s0} \quad \overline{q}_{\xi 0} \quad \overline{q}_{\eta 0} \quad \overline{m}_{s0} \quad \overline{m}_{\xi 0} \quad \overline{m}_{\eta 0} \quad \overline{q}_{s1} \quad \overline{q}_{\xi 1} \quad \overline{q}_{\eta 1} \quad \overline{m}_{s1} \quad \overline{m}_{\xi 1} \quad \overline{m}_{\eta 1}\}^{\mathrm{T}},$$

$\boldsymbol{G}_n$ is the local dynamic stiffness matrix for the nth element.

From the continuity conditions between adjacent elements, the local dynamic stiffness matrices of all elements can be assembled to obtain the global dynamic stiffness matrix $\widetilde{\boldsymbol{G}}$, such that:

$$\tilde{\boldsymbol{G}}\tilde{\boldsymbol{U}} = \tilde{\boldsymbol{F}} \tag{15}$$

where $\tilde{\boldsymbol{U}}$ is the nodal displacement vector for the curved beam system under consideration while $\tilde{\boldsymbol{F}}$ is the equivalent external loading vector applied at the ends of each element.

After leaving out the rows and columns of $\tilde{\boldsymbol{G}}$ associated with the geometry boundary conditions, the resultant dynamic stiffness matrix can be denoted by $\boldsymbol{G}_{sub}$. The natural frequencies are the roots making the determinant of $\boldsymbol{G}_{sub}$ equal to zero.

4. Computation procedure

With the help of symbolic computing package Mathematica, the proposed method can be organized as a computation procedure as illustrated in Fig. 1. It consists of the following steps:

Step 1 Input the control data, which includes the geometric parameters, material characteristics and boundary conditions of the spatial curved beam.

Step 2 Express the geometric relationship of the spatial curved beam using the Frobenius' theory.

Step 3 Compute the local dynamic stiffness matrix, the procedure of which is illustrated in Fig. 2.

Step 4 Substitute the geometric boundary conditions into the global dynamic stiffness matrix, which is obtained by assembling the local dynamic stiffness matrices of all elements.

Step 5 Find the natural frequencies of the spatial curved beam.

5. Numerical examples

In this section, two sample problems are presented. First, in order to validate the developed computer program, the free vibration frequencies of a cylindrical helical spring that is fixed at two ends are compared with the FEM results. Second, a reciprocal spiral arch having uniform circular cross-section is considered. The analysis is done by using the present computer program and I-DEAS, respectively. The free vibration frequencies with different boundary conditions are compared in the tables.

Example 1. The parametric equation of a cylindrical helix is (Fig. 3):

$$\boldsymbol{r} = \{a\cos\phi, a\sin\phi, h\phi\}$$

where ϕ is the horizontal angle of the helix.

According to

$$\boldsymbol{r}' = \{-a\sin\phi, a\cos\phi, h\},$$
$$\boldsymbol{r}'' = \{-a\cos\phi, -a\sin\phi, 0\},$$
$$\boldsymbol{r}''' = \{a\sin\phi, -a\cos\phi, 0\},$$

then

$$|\boldsymbol{r}'| = \sqrt{a^2 + h^2}\ ,$$

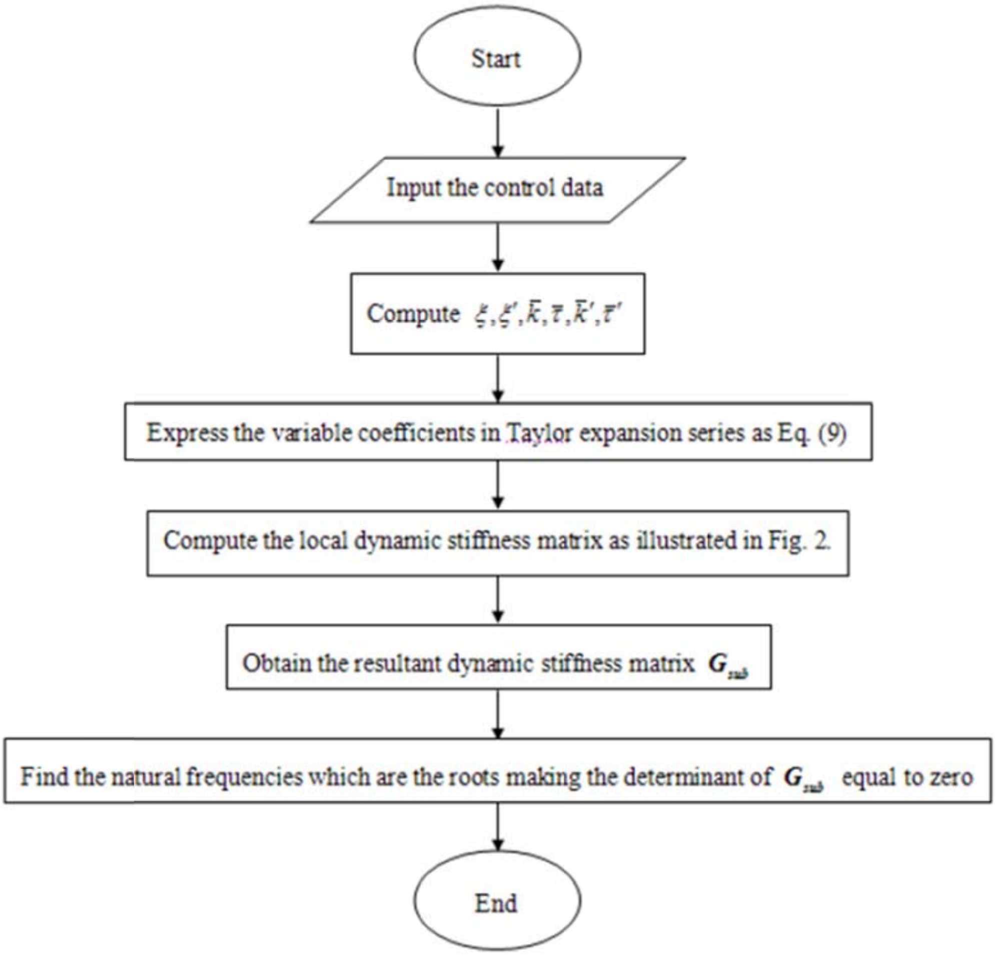

Fig. 1 Flowchart of computation procedure

$$\boldsymbol{r}' \times \boldsymbol{r}'' = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ -a\sin\phi & a\cos\phi & h \\ -a\cos\phi & -a\sin\phi & 0 \end{vmatrix} = \left\{ah\sin\phi, -ah\cos\phi, a^2\right\},$$

$$\left|\boldsymbol{r}' \times \boldsymbol{r}''\right| = a\sqrt{a^2 + h^2} \ ,$$

Because the curvature and torsion of a cylindrical helical spring are:

$$k = \frac{\left|\boldsymbol{r}' \times \boldsymbol{r}''\right|}{\left|\boldsymbol{r}'\right|^3} = \frac{a}{a^2 + h^2} = \text{constant} \ ,$$

$$\tau = \frac{\left(\boldsymbol{r}', \boldsymbol{r}'', \boldsymbol{r}'''\right)}{\left(\boldsymbol{r}' \times \boldsymbol{r}''\right)^2} = \frac{h}{a^2 + h^2} = \text{constant} \ ,$$

the f_p and l_p in Eq. (9) equal to zero.

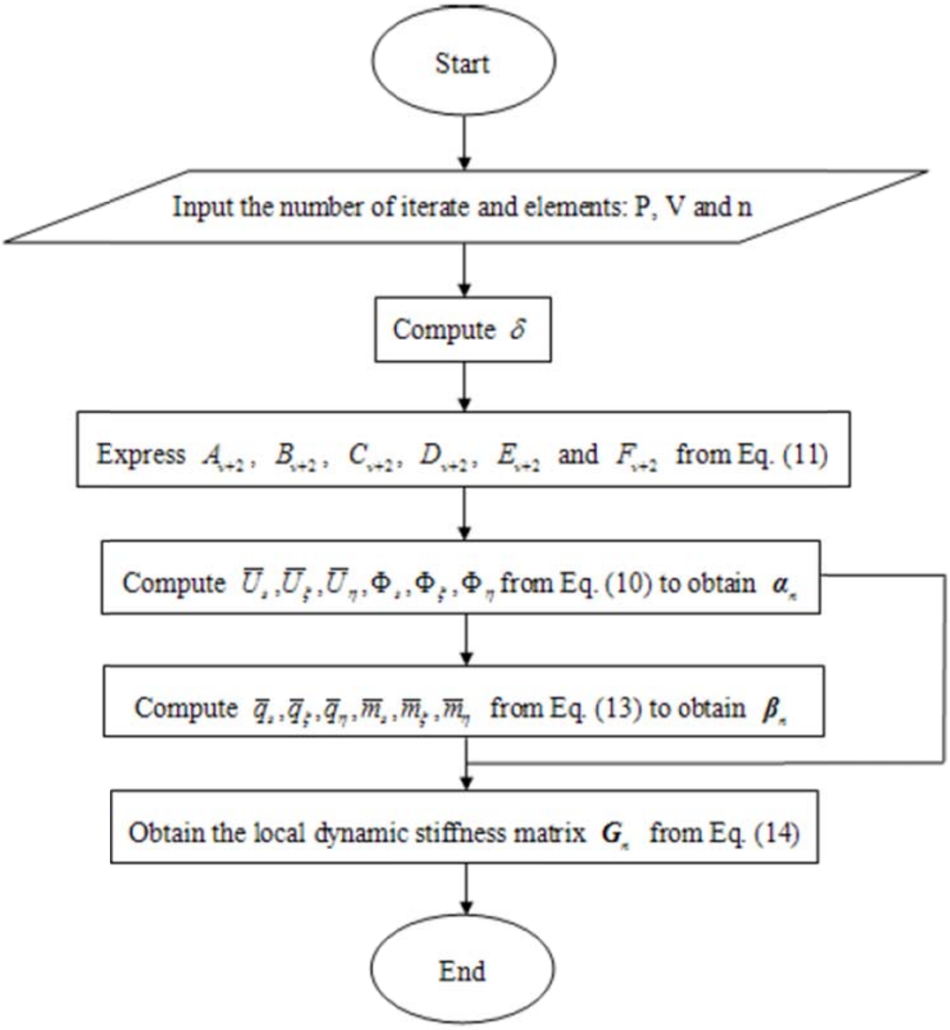

Fig. 2 Flowchart of stiffness matrix subroutine of element

From length of curve, it can be found that:

$$s = \int_0^\phi |\boldsymbol{r}'(\phi)| d\phi = \int_0^\phi \sqrt{a^2 + h^2}\, d\phi = \sqrt{a^2 + h^2}\,\phi \,.$$

The cylindrical helical spring fixed at the both ends shown in Fig. 4 is considered. The spring is made of steel and has a circular cross-section with the diameter d = 1 mm. The pitch angle, horizontal angle and radius of the helix circle are chosen as $\varphi_0 = 8.5744^\circ$, $\phi = [0, \pi]$, and a = 5 mm, respectively. The material properties are $E = 2.06 \times 10^{11}$ N/m^2, ρ = 7900 kg/m^3, and ν = 0.3. The representative length of the arch, L, in the solution is set equal to $2a$. The fundamental natural frequencies calculated by using the present computer program are given in Table 1 with the FEM results obtained from I-DEAS (Fig. 5) for a comparison. In the present method, 16 elements with V = 20 and P = 15 are used to achieve the 4.57 percent maximum error as opposed to 4559 body elements needed in I-DEAS.

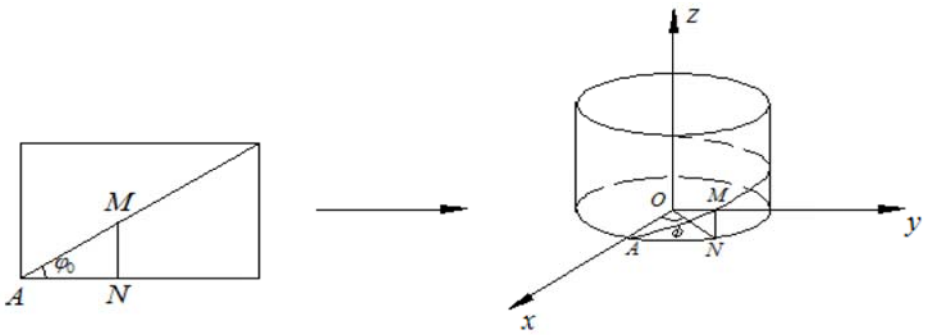

Fig. 3 Geometry of a cylindrical helix

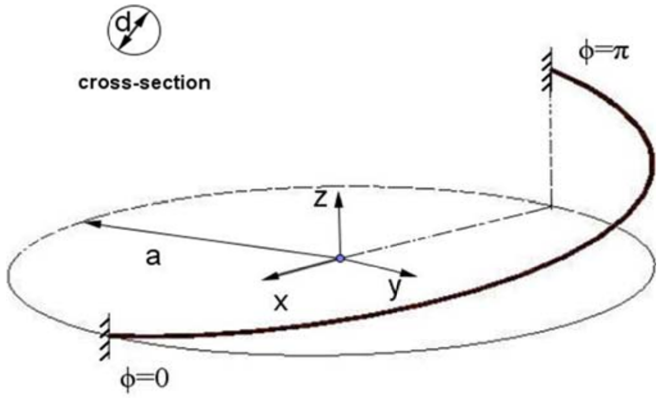

Fig. 4 Sketch of the cylindrical helical rod

Table 1 Frequencies ω for free vibration of a fixed-fixed hyperbolic helical rod (Unit: Hertz)

	1	2	3	4	5	6
Present study	14754.60	33960.50	40124.80	69905.30	81857.10	127835.00
I-DEAS results	14128.49	33143.90	39350.77	67235.97	79473.70	121992.21

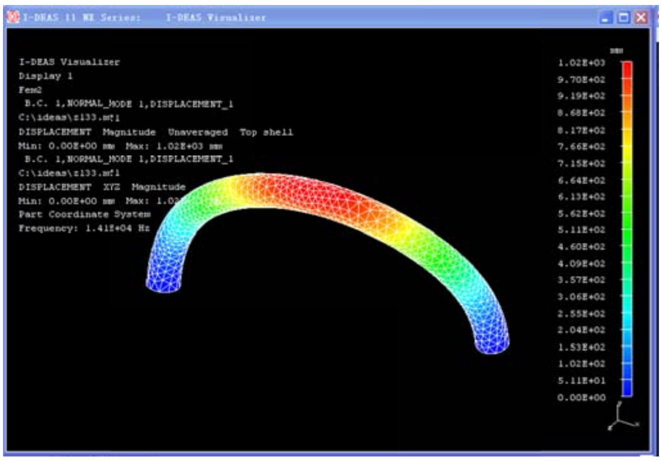

Fig. 5 Cylindrical helical rod in I-DEAS program

Example 2. Using the method presented above, a general-purpose computer program is used to analyze free vibration of spatial curved beams with variable curvature and torsion. In order to validate the developed computer program, the free-vibration frequencies of a fixed-fixed reciprocal spiral arch having uniform circular cross-section (Fig. 6) are compared with the FEM results.

The parametric equation of a reciprocal spiral arch is

$$\boldsymbol{r} = \{a\cosh\phi, a\sinh\phi, a\phi\},$$

According to

$$\boldsymbol{r}' = \{a\sinh\phi, a\cosh\phi, a\},$$
$$\boldsymbol{r}'' = \{a\cosh\phi, a\sinh\phi, 0\},$$
$$\boldsymbol{r}''' = \{a\sinh\phi, a\cosh\phi, 0\},$$

and

$$|\boldsymbol{r}'| = \sqrt{2}a\cosh\phi,$$

$$\boldsymbol{r}'\times\boldsymbol{r}'' = \begin{vmatrix} \boldsymbol{e_1} & \boldsymbol{e_2} & \boldsymbol{e_3} \\ a\sinh\phi & a\cosh\phi & a \\ a\cosh\phi & a\sinh\phi & 0 \end{vmatrix} = \{-a^2\sinh\phi, a^2\cosh\phi, -a^2\},$$

$$|\boldsymbol{r}'\times\boldsymbol{r}''| = \sqrt{2}a^2\cosh\phi,$$

So the variable curvature and torsion of it are:

$$k = \frac{|\boldsymbol{r}'\times\boldsymbol{r}''|}{|\boldsymbol{r}'|^3} = \frac{1}{2a\cosh^2\phi},$$

$$\tau = \frac{(\boldsymbol{r}', \boldsymbol{r}'', \boldsymbol{r}''')}{(\boldsymbol{r}'\times\boldsymbol{r}'')^2} = \frac{1}{2a\cosh^2\phi}.$$

From length of curve, it can be found that:

$$s = \int_0^\phi |\boldsymbol{r}'(\phi)| d\phi = \int_0^\phi \sqrt{2a^2\cosh^2\phi}\, d\phi = \sqrt{2}a\sinh\phi.$$

The diameter of the cross-section is d = 1 mm, a = 5 mm, ϕ = 5 mm, $\phi = [0, \pi]$. The material properties are $E = 2.06 \times 10^{11}$ N/m^2, G = 0.79×10^{11} N/m^2 and ρ = 7900 kg/m^3. The representative length of the arch, L, in the solution is set equal to $\sqrt{2}a\sinh\pi$. A comparison of the free-vibration frequencies calculated by using the present computer program and obtained from I-DEAS (Fig. 7) are shown in Table 2. It can be seen from Table 2 that the results of the present model demonstrate a good agreement with the FEM results. It should be noted that, in the present method, only 16 elements with V = 20 and P = 15 are used to achieve the desired accuracy as opposed to 26765 body elements needed in I-DEAS.

After having tested the validity of the present model, the free-vibration frequencies of the reciprocal spiral arch with other types of boundary conditions are presented in Table 3; namely, fixed-hinged and hinged-hinged. As expected, the natural frequencies increase as the constraints of the boundary conditions increase, from hinged-hinged to fixed-hinged to fixed-fixed if the geometry parameters remain constant.

Fig. 6 Sketch of the fixed-fixed reciprocal spiral arch

Table 2 Frequencies ω for free vibration of a fixed-fixed reciprocal spiral rod (Unit: Hertz)

	1	2	3	4	5	6	7	8
Present study	678.18	1211.73	1860.59	2657.25	3640.14	4830.93	6007.65	7334.49
I-DEAS results	675.04	1210.33	1852.76	2661.20	3627.15	4841.90	5990.87	7338.88

Table 3 Frequencies ω for free vibration of fixed-hinged and hinged-hinged reciprocal spiral rods (Unit: Hertz)

	1	2	3	4	5	6	7	8
Fixed-hinged	467.42	891.51	1508.99	2235.36	3141.09	4245.54	5362.83	6672.49
Hinged-hinged	282.66	750.17	1108.01	2050.75	2551.82	4017.00	4607.81	6498.52

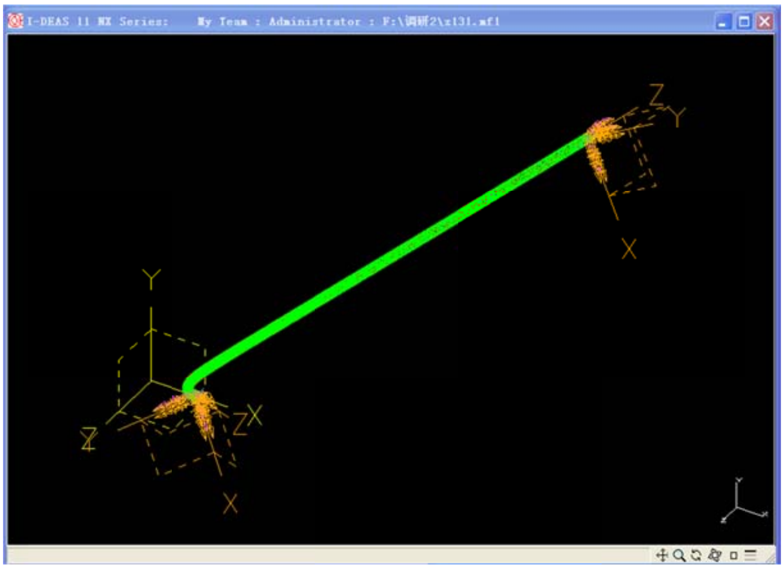

Fig. 7 Reciprocal spiral rod in I-DEAS program

6. Conclusions

•The governing equations for free vibration of spatial curved beams with variable curvature and torsion are derived. The effects of rotary inertia, shear and axial deformations are taken into account.

•Frobenius' theory and the dynamic stiffness method are applied to solve the governing equations.

•The numerical results show that only a limited number of terms are needed in series solutions and in Taylor expansion series to ensure an accurate solution.

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