Vibration analysis of high nonlinear oscillators using accurate approximate methods

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Abstract. In this paper, two new methods called Improved Amplitude-Frequency Formulation (IAFF) and Energy Balance Method (EBM) are applied to solve high nonlinear oscillators. Two cases are given to illustrate the effectiveness and the convenience of these methods. The results of Improved Amplitude-Frequency Formulation are compared with those of EBM. The comparison of the results obtained using these methods reveal that IAFF and EBM are very accurate and can therefore be found widely applicable in engineering and other science. Finally, to demonstrate the validity of the proposed methods, the response of the oscillators, which were obtained from analytical solutions, have been shown graphically and compared with each other.

Keywords: Improved Amplitude-Frequency Formulation (IAFF); Energy Balance Method (EBM); Nonlinear Oscillators

1. Introduction

Nonlinear vibration is an interesting filed in the mechanical and civil engineering. Nonlinear oscillator models have been widely used in many areas of physics and engineering and are of significant importance in mechanical and structural dynamics for the comprehensive understanding and accurate prediction of motion. In the last few decades applied mathematics and innovative methods to solve and obtain an accurate solution for nonlinear dynamic problems has been an interesting area in the field of mechanical vibration and dynamical systems. Generally, for many nonlinear problems, it is very difficult to obtain exact solutions of nonlinear differential equations; it is possible to prepare approximate solutions in many cases. Bayat *et al*. (2012a) did a complete review on the recent analytical methods for nonlinear vibrations equations in their review paper. Many new techniques have appeared in the open literature to overcome the shortcomings of traditional analytical methods such as: Variational Approach (He 2006, Liu 2009, Bayat 2013a, Pakar 2011, 2012a), Variational iteration method (Pakar 2012b), differential transform method (Thongmoon 2010), Homotopy analysis method (Ganji 2009), Max-min approach (Pakar 2013, Bayat 2011, Shen 2009), Homotopy perturbation method (Baki 2011, Jalaal *et al*. 2010), Hamiltonian approach (He 2010a, Xu and He 2010, He *et al*. 2010b, Bayat 2012b, 2013b), energy

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balance method (Fu 2011, He 2002, Zhang 2009) other numerical and analytical methods (Geng 2007, He 2004, 2008, Zhang 2008, Konuralp 2009).

Zhang (2008) presented an application of He's amplitude-frequency formulation in order to show the capability of this method to nonlinear oscillations even to some of equations which have fractional term. Finally the Improved Amplitude-Frequency Formulation (IAFF) is proposed by He using a modification and discussion on the crucial point of the equations to improve the Amplitude-Frequency Formulation (He 2008).

Two practical cases have been considered to show the efficiency of Amplitude-Frequency Formulation (IAFF) and Energy Balance Method (EBM). We have solved the governing equations of circular cylindrical shell and nonlinear vibration of simply support edges beam. Many researchers have been studied on the vibrations of the shells and beams in the recent years (Amiro and Zarutsky 1981, Zarutsky 1993, Koiter 1966, Manevitch 1972, Andrianov *et al.* 2004, Ba datl 2009, Piccardo 2012, Kural 2012).

This paper has been collocated as follows: The first section contains the basic idea of improved amplitude frequency formulation and energy balance method. The second section is related to the application of these methods to the governing equations of circular cylindrical shell and simply support edges beam. In the third section, comparisons of the obtained results are also presented to demonstrate the accuracy and applicability of these methods. Finally, we show that the IAFF and EBM could be very useful mathematical tools to prepare a precise cyclic solution for nonlinear systems.

2. Basic idea of Improved Amplitude-Frequency Formulation

We consider a generalized nonlinear oscillator in the form (He 2008)

$$\ddot{\xi}+F(\xi)=0, \qquad \xi(0)=a, \quad \dot{\xi}(0)=0, \tag{1}$$

We use two following trial functions

$$\xi_1(t)=a\cos(\omega_1 t), \tag{2}$$

And

$$\xi_2(t)=a\cos(\omega_2 t), \tag{3}$$

The residuals are

$$R_1(\omega t)=-a\,\omega_1^{\,2}\cos(\omega_1 t)+F\left(a\cos(\omega_1 t)\right), \tag{4}$$

And

$$R_2(\omega t)=-a\,\omega_2^2\cos(\omega_2 t)+F\left(a\cos(\omega_2 t)\right), \tag{5}$$

The original Frequency-amplitude formulation reads (He 2004, 2006)

$$\omega^2=\frac{\omega_1^2R_2-\omega_2^2R_1}{R_2-R_1}, \tag{6}$$

He used the following formulation and Geng and Cai improved the formulation by choosing another location point (Geng 2007).

$$\omega^2 = \frac{\omega_1^2 R_2\ (\omega_2 t = 0) - \omega_2{}^2 R_1 (\omega_1 t = 0)}{R_2 - R_1}, \tag{7}$$

This is the improved form by Geng and Cai (2007)

$$\omega^2 = \frac{\omega_1^2 R_2 (\omega_2 t = \pi/3) - \omega_2^2 R_1 (\omega_1 t = \pi/3)}{R_2 - R_1}, \tag{8}$$

The point is: $\cos(\omega_1 t) = \cos(\omega_2 t) = k$

Substituting the obtained ω into $\xi(t) = a\cos(\omega t)$, we can obtain the constant k in ω^2 equation in order to have the frequency without irrelevant parameter.

To improve its accuracy, we can use the following trial function when they are required.

$$\xi_1(t) = \sum_{i=1}^{m} a_i \cos(\omega_i t), \quad \text{and} \quad \xi_2(t) = \sum_{i=1}^{m} a_i \cos(\Omega_i t), \tag{9}$$

Or

$$\xi_1(t) = \frac{\sum_{i=1}^{m} a_i \cos(\omega_1 t)}{\sum_{j=1}^{m} b_j \cos(\omega_j t)}, \quad \text{and} \quad \xi_2(t) = \frac{\sum_{i=1}^{m} a_i \cos(\Omega_i t)}{\sum_{j=1}^{m} b_j \cos(\Omega_j t)}, \tag{10}$$

But in most cases because of the sufficient accuracy, trial functions are as follow and just the first term

$$\xi_1(t) = a\cos t, \quad \text{and} \quad \xi_2(t) = b\cos(\omega t) + (a-b)\cos(\omega t), \tag{11}$$

and

$$\xi_1(t) = a\cos t, \quad \text{and} \quad \xi_2(t) = \frac{a(1+c)\cos(\omega t)}{1 + c\cos(2\omega t)}, \tag{12}$$

Where a and c are unknown constants. In addition we can set:

$\cos t = k$ in ξ_1, and $\cos(\omega t) = k$ in ξ_2

3. Basic idea of Energy Balance Method

In the present paper, we consider a general nonlinear oscillator in the form (He 2002)

$$\ddot{\xi} + f\ (\xi(t)) = 0, \tag{13}$$

In which ξ and t are generalized dimensionless displacement and time variables, respectively, and $f = f\ (\xi, \dot{\xi}, \ddot{\xi}, t)$. Its variational principle can be easily obtained

$$J(\xi)=\int_0^t \left(-\tfrac{1}{2}\dot{\xi}^2+F(\xi)\right)dt, \tag{14}$$

Where $T = 2\pi/\omega$ is period of the nonlinear oscillator**,** $F(\xi)=\int f(\xi)d\xi$.

Its Hamiltonian, therefore, can be written in the form

$$\Delta H=\frac{1}{2}\dot{\xi}^2+F(\xi)=F(a), \tag{15}$$

or

$$R(t)=\frac{1}{2}\dot{\xi}^2+F(\xi)-F(a)=0, \tag{16}$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation, a. So let us consider such initial conditions

$$\xi(0)=a \quad , \quad \dot{\xi}(0)=0, \tag{17}$$

We use the following trial function to determine the angular frequency ω

$$\xi(t)=a\cos(\omega t), \tag{18}$$

Substituting Eq. (18) into Eq. (16), we obtain the following residual equation

$$R(t)=\frac{1}{2}a^2\omega^2\sin^2(\omega t)+F(a\cos(\omega t))-F(a)=0, \tag{19}$$

If, by chance, the exact solution had been chosen as the trial function, then it would be possible to make R zero for all values of t by appropriate choice of ω. Since Eq. (18) is only an approximation to the exact solution, R, can not be made zero everywhere. Collocation at $\omega t = \pi/4$ gives

$$\omega=\sqrt{\frac{2F(a)-F(a\cos(\omega t))}{a^2\sin^2(\omega t)}} \quad , \quad \text{rad/sec}, \tag{20}$$

Its period can be written in the form

$$T=\frac{2\pi}{\sqrt{\frac{2F(a)-F(a\cos(\omega t))}{a^2\sin^2(\omega t)}}}, \tag{21}$$

4. Applications

In order to achieve the accuracy and the applicability of Improved Amplitude-frequency Formulation and Energy Balance Method for solving nonlinear vibration equations, we will consider the following cases.

4.1 Case 1

The linear theory of the shells can be found in many papers and discussed completely (Manevitch 1972, Amiro 1981).The non linear dynamic boundary value problems of theory of closed circular cylindrical shell eccentrically reinforced in the two principal directions are investigated within the framework of the structurally orthotropic scheme. The middle surface of the shell is chosen as the main one. Therefore we consider the final results. Fig. 1 shows the schematic of closed eccentrically circular cylindrical shell.

In dimensionless form, the governing differential equation of closed eccentrically circular cylindrical shell with constant coefficient for the time function is given by Awrejcewicz *et al.* (1998)

$$\frac{d^2\xi}{dt^2}+\alpha\xi\left[\left(\frac{d\xi}{dt}\right)^2+\xi\left(\frac{d^2\xi}{dt^2}\right)\right]+A_1\xi+A_2\xi^3+A_3\xi^5=0, \tag{22}$$

With the initial conditions of

$$\xi(0)=a\ ,\quad \frac{d\xi}{dt}(0)=0\ , \tag{23}$$

The expressions of the governing coefficients are presented in Appendix A. The complete formulation of Eq. (22) can be referred to Awrejcewicz *et al.* (1998) for details.

4.1.1 Solution of case 1 using Improved Amplitude-Frequency Formulation

We use trial functions for solving Eq. (22), as follows

$$\xi_1(t)=a\cos(t), \tag{24}$$

And

$$\xi_2(t)=a\cos(2t), \tag{25}$$

Respectively, the residual equations are

$$R_1(t)=a\cos(t)\left(A_1+A_2a^2\cos^2(t)+A_3a^4\cos^4(t)-2\alpha a^2\cos^2(t)+\alpha a^2-1\right), \tag{26}$$

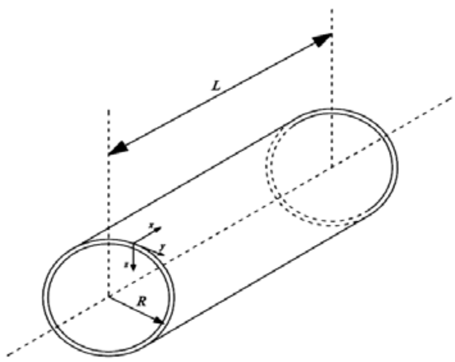


Fig. 1 schematic of closed eccentrically circular cylindrical shell

And

$$R_2(t) = a\cos(2t)\left(A_1 + A_2 a^2\cos^2(2t) + A_3 a^4\cos^4(2t) - 8\alpha a^2\cos^2(2t) + 4\alpha a^2 - 4\right), \tag{27}$$

Considering $\cos(\omega_1 t) = \cos(\omega_2 t) = k$ we have

$$\omega^2 = \frac{\omega_1^2 R_2 - \omega_2^2 R_1}{R_2 - R_1} = \frac{A_1 + A_2 a^2 k^2 + A_3 a^4 k^4}{2\alpha a^2 k^2 - \alpha a^2 + 1}, \tag{28}$$

We can rewrite $\xi(t) = a\cos(\omega t)$ in the form

$$\xi(t) = a\cos\left(\sqrt{\frac{A_1 + A_2 a^2 k^2 + A_3 a^4 k^4}{2\alpha a^2 k^2 - \alpha a^2 + 1}}\, t\right), \tag{29}$$

In view of the approximate solution, we can rewrite the main equation in the form

$$\frac{d^2\xi}{dt^2} + \left(\frac{A_1 + A_2 a^2 k^2 + A_3 a^4 k^4}{2\alpha a^2 k^2 - \alpha a^2 + 1}\right)\xi = \left(\frac{A_1 + A_2 a^2 k^2 + A_3 a^4 k^4}{2\alpha a^2 k^2 - \alpha a^2 + 1}\right)\xi - \alpha\xi\left[\left(\frac{d\xi}{dt}\right)^2 + \xi\left(\frac{d^2\xi}{dt^2}\right)\right] - A_1\xi - A_2\xi^3 - A_3\xi^5, \tag{30}$$

If by any chance Eq. (29) is the exact solution, then the right side of Eq. (30) vanishes completely. Considering our approach which is just an approximation one, we set

$$\int_0^T \left[\left(\frac{A_1 + A_2 a^2 k^2 + A_3 a^4 k^4}{2\alpha a^2 k^2 - \alpha a^2 + 1}\right)\xi - \alpha\xi\left[\left(\frac{d\xi}{dt}\right)^2 + \xi\left(\frac{d^2\xi}{dt^2}\right)\right] - A_1\xi - A_2\xi^3 - A_3\xi^5\right]\cos(\omega t)\,dt = 0\ ,\quad T = \frac{2\pi}{\omega}\ , \tag{31}$$

Considering the term $\xi(t) = a\cos(\omega t)$ and substituting the term to Eq. (31) and solving the integral term, we have

$$k^4 = \frac{1}{16}\frac{1}{a^4 A_3^2(\alpha a^2 + 2)^2}\left(5a^4 A_3\alpha + 8A_1\alpha + 4a^2 A_2\alpha - 4A_2 + \begin{pmatrix} 5a^8 A_3^2\alpha^2 + 32a^4 A_3\alpha^2 A_1 + 16a^6 A_3\alpha^2 A_2 - 64a^4\alpha A_3 A_2 \\ +64A_1^2\alpha^2 + 64A_1\alpha^2 a^2 A_2 - 64\alpha A_1 A_2 + 16a^4 A_2^2\alpha^2 - 32a^2 A_2^2\alpha \\ +16A_2^2 - 20a^6\alpha A_3^2 + 48a^2 A_3 A_2 + 40a^4 A_3^2 - 96a^2\alpha A_3 A_1 \end{pmatrix}^{\frac{1}{2}}\right)^2 \tag{32}$$

So, substituting Eq. (32) into Eq. (28), we have

$$\omega = \frac{1}{2}\sqrt{\frac{5a^4 A_3 + 6a^2 A_2 + 8A_1}{\alpha a^2 + 2}}\ , \tag{33}$$

We can obtain the following approximate solution

$$\xi(t) = a\cos\left(\frac{1}{2}\sqrt{\frac{5a^4 A_3 + 6a^2 A_2 + 8A_1}{\alpha a^2 + 2}}\, t\right), \tag{34}$$

4.1.2 Solution of case 1 using Energy Balance Method

Variational and Hamiltonian formulations of Eq. (22) can be readily obtained as

$$J(\xi)=\int_0^t\left(-\frac{1}{2}\left(\frac{d^2\xi}{dt^2}\right)^2+\frac{1}{2}\alpha\xi^2\left(\frac{d\xi}{dt}\right)^2+\frac{1}{2}A_1\xi^2+\frac{1}{4}A_2\xi^4+\frac{1}{6}A_3\xi^6\right)dt$$

$$\begin{aligned}H&=\frac{1}{2}\left(\frac{d^2\xi}{dt^2}\right)^2+\frac{1}{2}\alpha\xi^2\left(\frac{d\xi}{dt}\right)^2+\frac{1}{2}A_1\xi^2+\frac{1}{4}A_2\xi^4+\frac{1}{6}A_3\xi^6\\&=\frac{1}{2}A_1a^2+\frac{1}{4}A_2a^4+\frac{1}{6}A_3a^6\end{aligned}\tag{35}$$

Choosing the trial function $\xi(t) = a\cos(\omega t)$, we obtain the following residual equation

$$\begin{aligned}R(t)&=\frac{1}{2}a^2\sin^2(\omega t)\omega^2+\frac{1}{2}\alpha\ \mathrm{a}^4\cos^2(\omega t)\sin^2(\omega t)\omega^2+\frac{1}{2}A_1a^2\cos^2(\omega t)\\&+\frac{1}{4}A_2a^4\cos^4(\omega t)+\frac{1}{6}A_3a^6\cos^6(\omega t)-\frac{1}{2}A_1a^2-\frac{1}{4}A_2a^4-\frac{1}{6}A_3a^6\end{aligned}\tag{36}$$

If we collocate at $\omega t = \pi/4$, we obtain the following result

$$\omega=\frac{\sqrt{6}}{6}\frac{\sqrt{\left(2+\alpha a^2\right)\left(12A_1+9A_2a^2+7A_3a^4\right)}}{2+\alpha a^2},\tag{37}$$

We can obtain the following approximate solution

$$\xi(t)=a\cos\left(\frac{\sqrt{6}}{6}\frac{\sqrt{\left(2+\alpha a^2\right)\left(12A_1+9A_2a^2+7A_3a^4\right)}}{2+\alpha a^2}t\right),\tag{38}$$

4.2 Case 2

For the second case, the governing equation of the nonlinear beam vibration is considered (Andrianov *et al*. 2004)

$$EI\frac{\partial^4 w}{\partial x^4}-\frac{EA}{2l\left(1-\nu^2\right)}\frac{\partial^2 w}{\partial x^2}\int_0^1\left(\frac{\partial w}{\partial x}\right)^2dx+\rho A\frac{\partial^2 w}{\partial t^2}=0\tag{39}$$

In this equation E is the Young's modulus of the beam, I is the second moment of area of the cross section, w is the beam deflection, t is the time, A is the cross sectional area of the beam and L is the length of the beam in x direction.

By a process which is applied on the governing equation in reference (Andrianov *et al.* 2004), we have the following Duffing equation for simply support edges of beam

$$\frac{d^2\xi}{dt^2}+\Omega^2\left(1+\gamma\xi^2\right)\xi=0,\tag{40}$$

With initial condition

$$\xi(0)=a\ ,\quad \frac{d\xi(0)}{dt}=0\ ,\tag{41}$$

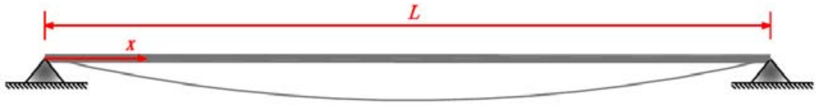


Fig. 2 Schematic of simply support beam

Where coefficients of Eq. (40) are given in Appendix B. the complete formulation of Eq. (40) can be referred to (Andrianov *et al*. 2004) for details.

4.2.1 Solution of case 2 using Improved Amplitude-Frequency Formulation

We use trial functions for solving Eq. (40), as follow

$$\xi_1(t) = a\cos t, \tag{42}$$

And

$$\xi_2(t) = a\cos(2t), \tag{43}$$

Respectively the residual equations are

$$R_1(t) = a\cos(t)\left(-1+\Omega^2+\Omega^2\gamma a^2\cos^2(t)\right), \tag{44}$$

And

$$R_2(t) = a\cos(2t)\left(-4+\Omega^2+\Omega^2\gamma\, a^2\cos^2(2t)\right), \tag{45}$$

Considering $\cos(\omega_1 t) = \cos(\omega_2 t) = k$ we have

$$\omega^2 = \frac{\omega_1^2 R_2 - \omega_2^2 R_1}{R_2 - R_1} = \Omega^2\left(1+\gamma\, a^2k^2\right), \tag{46}$$

We can rewrite $\xi(t) = a\cos(\omega t)$ in the form

$$\xi(t) = a\cos(\Omega\sqrt{1+\gamma\, a^2k^2}\ t), \tag{47}$$

In view of the approximate solution, we can rewrite the main equation in the form

$$\frac{d^2\xi}{dt^2} + \Omega^2(1+\gamma\ a^2k^2)\xi = \Omega^2\gamma\, a^2k^2\xi - \Omega^2\gamma\,\xi^3, \tag{48}$$

If by any chance Eq. (47) is the exact solution, then the right side of Eq. (48) vanishes completely. Considering our approach which is just an approximation one, we set

$$\int_0^T [\Omega^2\gamma a^2k^2\xi - \Omega^2\gamma\xi^3]\cos(\omega t)dt = 0\ , \qquad T = \frac{2\pi}{\omega}\ , \tag{49}$$

Considering the term $\xi(t) = a\cos(\omega t)$ and substituting the term to Eq. (49) and solving the integral term, we have

$$k^2 = 3/4, \tag{50}$$

So, substituting Eq. (50) into Eq. (46), we have

$$\omega^2 = \Omega^2 + \frac{3}{4}\Omega^2 \gamma a^2, \tag{51}$$

We can obtain the following approximate solution

$$\xi(t) = a\cos(\Omega\sqrt{1+\frac{3}{4}\gamma a^2}\ t)\ . \tag{52}$$

4.2.2 Solution of case 2 using Energy Balance Method

Variational and Hamiltonian formulations of Eq. (40) can be readily obtained as

$$\begin{aligned} J(\xi) &= \int_0^t \left(-\frac{1}{2}\left(\frac{d\xi}{dt}\right)^2 + \frac{1}{2}\Omega^2\xi^2 + \frac{1}{4}\Omega^2\gamma\xi^4 \right) dt \\ H &= \frac{1}{2}\left(\frac{d\xi}{dt}\right)^2 + \frac{1}{2}\Omega^2\xi^2 + \frac{1}{4}\Omega^2\gamma\xi^4 = \frac{1}{2}\Omega^2 a^2 + \frac{1}{4}\Omega^2\gamma a^4 \end{aligned} \tag{53}$$

Choosing the trial function $\xi(t) = a\cos(\omega t)$, we obtain the following residual equation

$$R(t) = \frac{1}{2}a^2\omega^2\sin^2(\omega t) + \frac{1}{2}\Omega^2 a^2\cos^2(\omega t) + \frac{1}{4}\Omega^2\gamma a^4\cos^4(\omega t) - \frac{1}{2}\Omega^2 a^2 - \frac{1}{4}\Omega^2\gamma a^4 = 0. \tag{54}$$

If we collocate at $\omega t = \pi/4$, we obtain the following result

$$\omega = \Omega\left(1 + \tfrac{3}{4}\gamma a^2\right)^{1/2}, \tag{55}$$

We can obtain the following approximate solution

$$\xi(t) = a\ \cos\ (\Omega\sqrt{1+\frac{3}{4}\gamma a^2}\ t). \tag{56}$$

5. Results and discussions

In this section, to illustrate and verify the accuracy of these new approximate analytical methods, some comparisons are shown in Table 1 and Figs. 3 to 5 for case 1 and Table 2 and Figs. 6 to 8 for case 2.

From Table 1 it can be observed that the maximum relative error between the IAFF results and EBM results is 3.5094% for large amplitude of the system (A = 100) in closed eccentrically circular cylindrical shell case. An excellent agreement can be seen in this table.

The Table 2 shows the same comparison between IAFF and EBM for the simply support beam, the results are identical completely as it can seen form Eqs. (51) and (55). Figs. 3 and 6 show comparisons of the IAFF solution of $\xi(t)$ based on time with the EBM solution. The behaviors of the systems are periodic. Figs. 4 and 7 represent the phase plan diagram comparisons of IAFF and EBM solution of ($\xi(t)/dt$ versus $\xi(t)$ curve) of the Eqs. (22) and (40) to show the effect of small parameters of the closed eccentrically circular cylindrical shell and simply support edges of beam.

The amplitudes of the shell and beam vibration are a function of the initial conditions. The best accuracies can be seen at extreme points. To have a better understanding from the behavior of the system and the effects of other small parameters on the frequency of the system, we consider the three parameters simultaneously as sensitivities analysis of frequency in Figs. 5 and 8 for both cases.

Fig. 5 shows the analysis of IAFF solution for these parameters:

(a): $0<a<2,\ 0<\alpha<2,\ A_1=0.5,\ A_2=0.5\ ,A_3=0.5$

(b): $0<a<2,\ \alpha=0.5,\ 0<A_1<3,\ A_2=0.5,\ A_3=0.5$

(c): $0<a<2,\ \alpha=0.5,\ A_1=0.5,\ 0<A_2<3,\ A_3=0.5$

(d): $0<a<2,\ \alpha=0.5,\ A_1=0.5, A_2=0.5,\ 0<A_3<3$

And Fig. 8 shows the same analysis of IAFF and EBM solutions for these parameters:

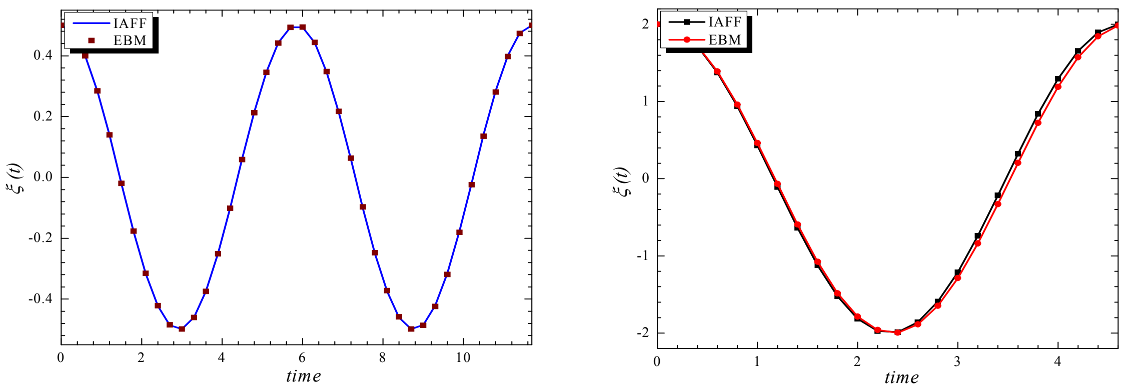
(a): $\gamma=1,\quad 0<a<5,\quad 1<\Omega<4$

(b): $\Omega=0.5,\quad 0<a<5,\quad 1<\gamma<4$

It is completely evident that IAFF and EBM show an excellent agreement with each other and quickly convergent and valid for a wide range of vibration amplitudes and initial conditions as it is indicated in the tables. The accuracy of the results shows that the IAFF and EBM can be potentiality used for the analysis of strongly nonlinear oscillation problems accurately.

Table1 Comparisons of IAFF frequencies with EBM frequencies at $\alpha = 0.5$, $A_1 = 1$, $A_2 = 1$, $A_3 = 1$ for case1

Amplitude a	Improved Amplitude-frequency Formulation ω_{IAFF}	Energy Balance Method ω_{EBM}	Error %
0.1	1.0025	1.0025	0.0002
0.5	1.0744	1.0733	0.1063
1	1.3784	1.3663	0.8889
5	7.5235	7.2809	3.3324
10	15.5983	15.0759	3.4654
20	31.5132	30.4479	3.4987
50	79.0127	76.3349	3.5081
100	158.0918	152.7318	3.5094



(I) (II)

Fig. 3 (Case 1) Comparison of time history response of the IAFF solution with the EBM solution for circular cylindrical shell (I): $a = 0.5$, $\alpha = 0.5$, $A_1 = 1$, $A_2 = 1$, $A_3 = 1$, (II): $a = 2$, $\alpha = 1$, $A_1 = 2$, $A_2 = 0.5$, $A_3 = 0.2$

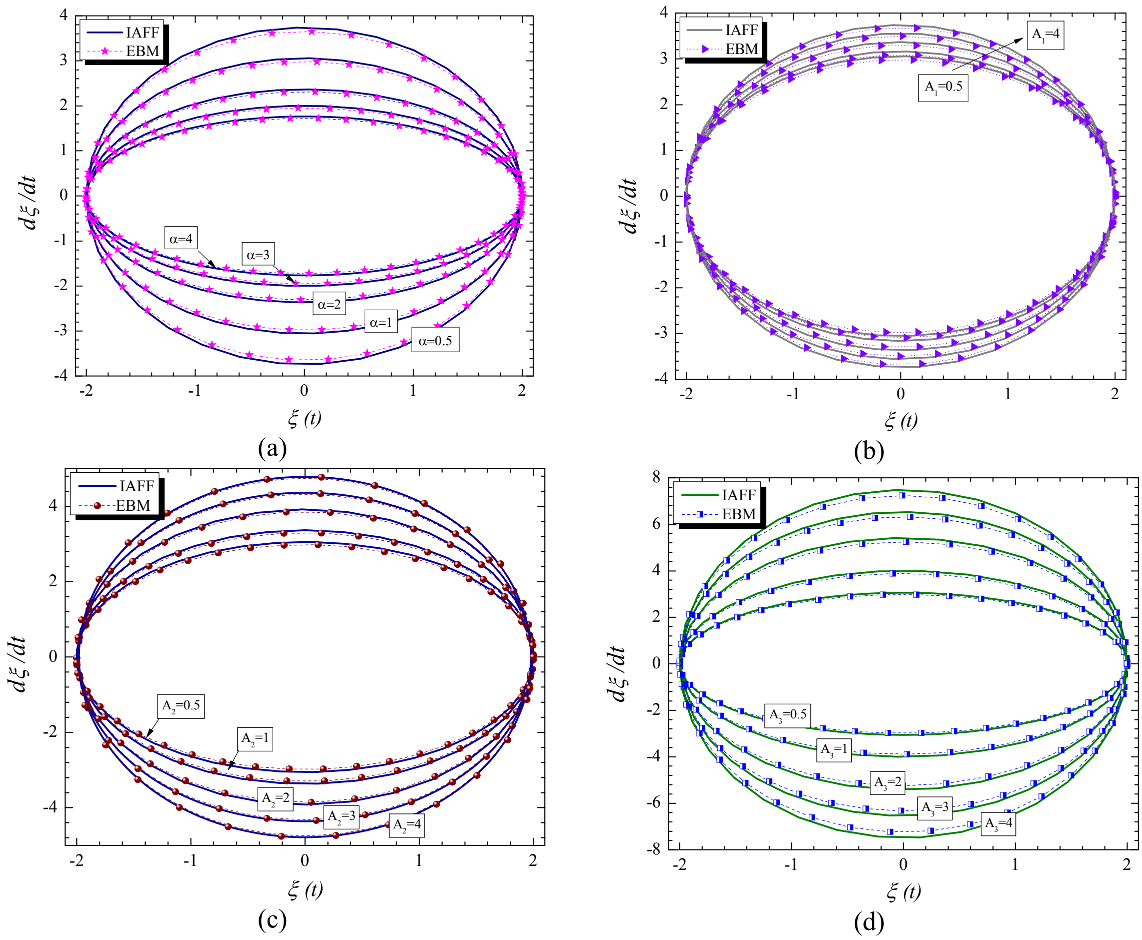
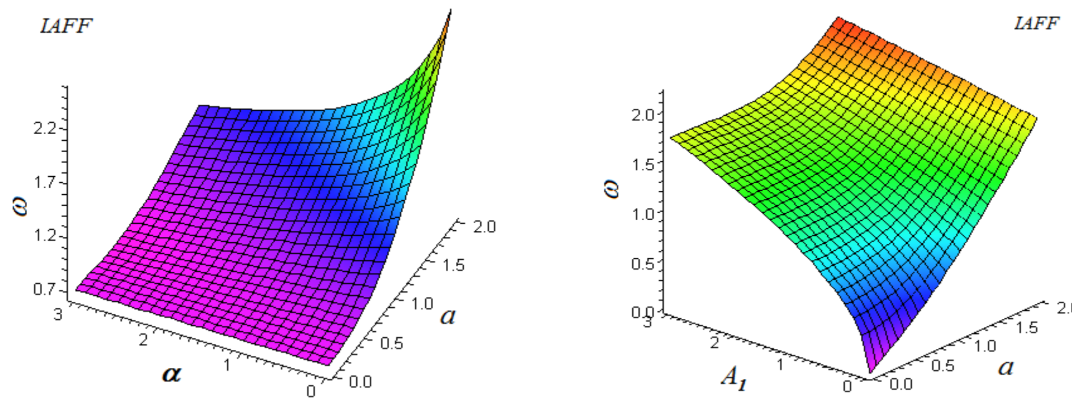
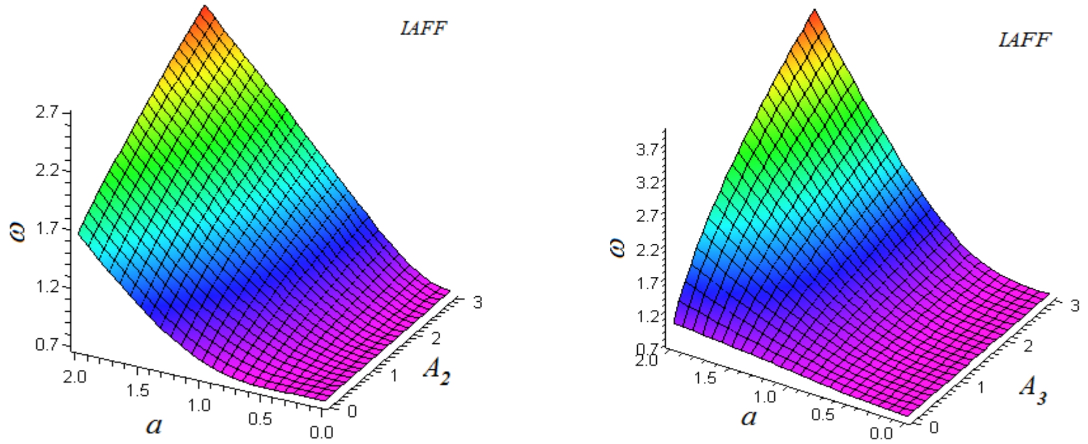


Fig. 4 (Case 1) Effect of various parameters on phase-plan diagram for the following parameter (a): $A_1 = 0.5$, $A_2 = 0.5$, $A_3 = 0.5$, (b): $\alpha = 0.5$, $A_2 = 0.5$, $A_3 = 0.5$, (c): $\alpha = 0.5$, $A_1 = 0.5$, $A_3 = 0.5$, (d): $\alpha = 0.5$, $A_1 = 0.5$, $A_2 = 0.5$



(a) $0 < a < 2$, $0 < \alpha < 2$, $A_1 = 0.5$, $A_2 = 0.5$, $A_3 = 0.5$ (b) $0 < a < 2$, $\alpha = 0.5$, $0 < A_1 < 3$, $A_2 = 0.5$, $A_3 = 0.5$

Fig. 5 (Case 1) Sensitivity analysis of frequency of IAFF solution for the above parameter cases

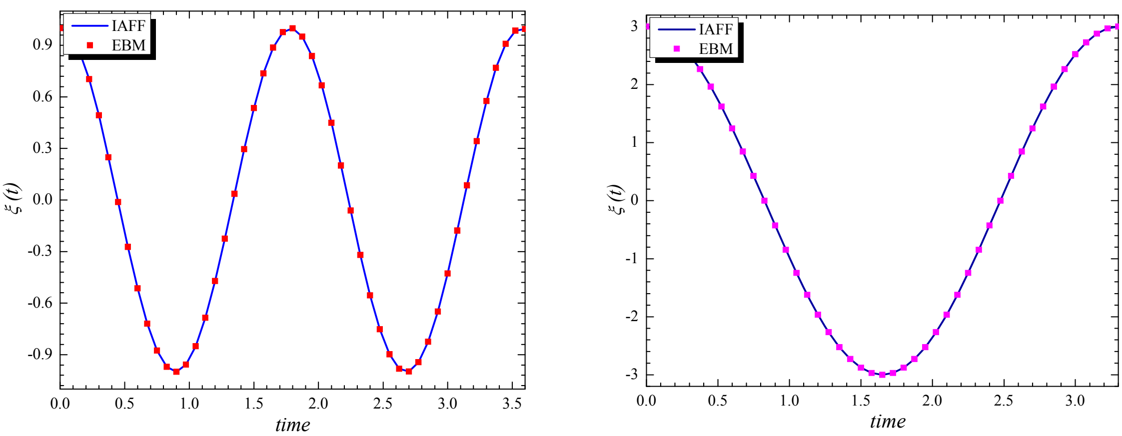


(c) $0 < a < 2$, $\alpha = 0.5$, $A_1 = 0.5$, $0 < A_2 < 3$, $A_3 = 0.5$ (d) $0 < a < 2$, $\alpha = 0.5$, $A_1 = 0.5$, $A_2 = 0.5$, $0 < A_3 < 3$

Fig. 5 Continued

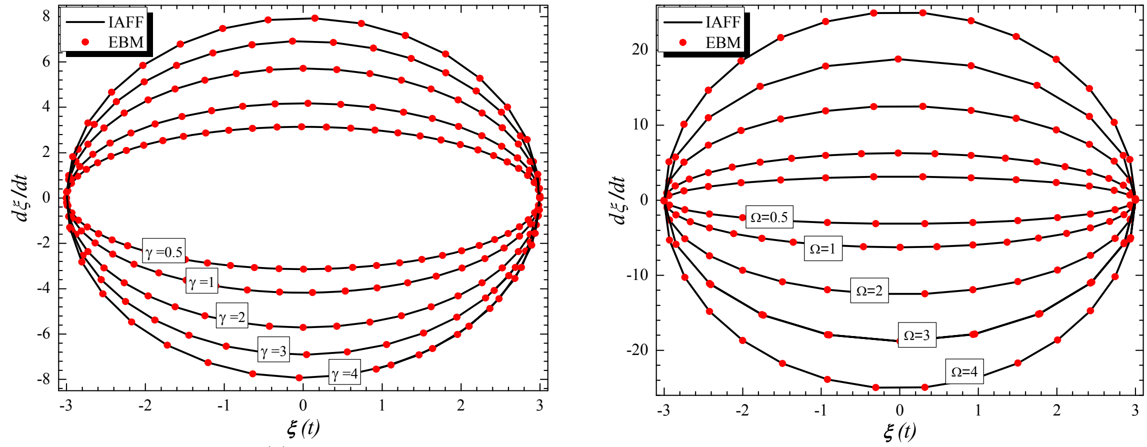
Table 2 Comparison of IAFF frequency with EBM frequency at $\Omega = 3$, $\gamma = 0.5$ for case 2

Amplitude a	Improved Amplitude-frequency Formulation ω_{IAFF}	Energy Balance Method ω_{EBM}
0.1	3.0056	3.0056
0.5	3.1375	3.1375
1	3.5178	3.5178
5	9.6631	9.6631
10	18.6145	18.6145
20	36.8646	36.8646
50	91.9048	91.9048
100	183.7362	183.7362



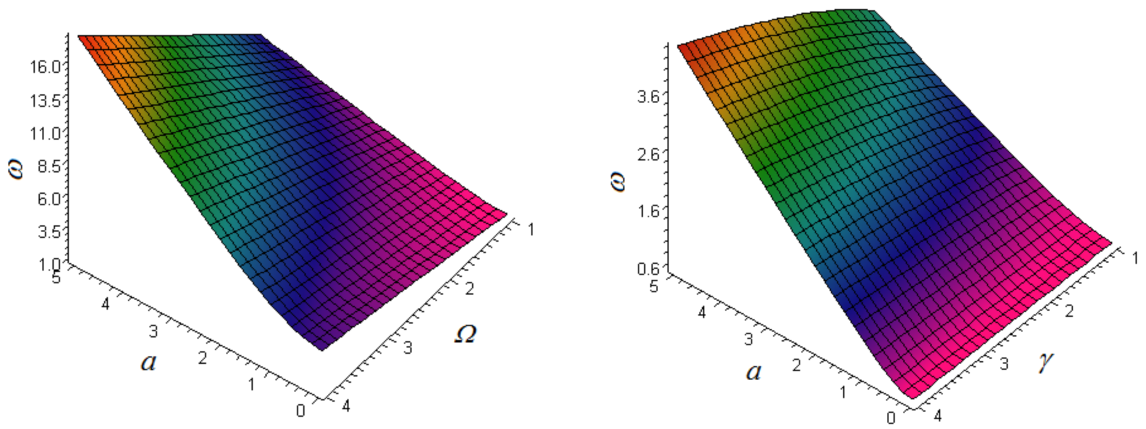
(I) (II)

Fig. 6 (Case 2) Comparison of time history response of the IAFF solution with the EBM solution for simply support beam: (I) $a = 1$, $\Omega = 3$, $\gamma = 0.5$, (II) $a = 3$, $\Omega = 0.5$, $\gamma = 2$



(a) (b)

Fig. 7 (Case 2) Effect of various parameters on phase-plan diagram for the following cases (a): a = 3, Ω = 0.5, (b): a = 3, γ = 0.5



(a): $\gamma = 1$, $0 < a < 5$, $1 < \Omega < 4$ (b): $\Omega = 0.5$, $0 < a < 5$, $1 < \gamma < 4$

Fig. 8 (Case 2) Sensitivity analysis of frequency of IAFF and EBM solution for the above parameter cases

6. Conclusions

In this work, we used applications of Improved Amplitude-Frequency Formulation (IAFF) and Energy Balance Method (EBM) for solving the two nonlinear oscillatory systems. The methods, which are proved to be powerful mathematical tools to study of nonlinear oscillators, can be found widely applicable in engineering and science. The results obtained from these methods have been compared with each other. These cases have shown that the approximate analytical solutions are in excellent agreement. Improved Amplitude-Frequency Formulation and Energy Balance Method are easy and direct procedures for determining approximations to the periodic solutions.

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Appendix A.

$$t=\sqrt{B_1\left(\rho R^2\right)^{-1}}\tau;\quad A_1=\varepsilon_1^2\varepsilon_4+2\varepsilon_1^2\varepsilon_3\varepsilon_4p^{-2}+\varepsilon_1^2\varepsilon_3\varepsilon_4p^{-4}+s_2^{-4}\left(1-\varepsilon_6^2s_2^2\right)^2;$$
$$A_2=\frac{1}{16}+\frac{1}{2}s_2^4\varepsilon_1^2\varepsilon_4-\frac{3}{4}\left(1-\varepsilon_6^2s_2^2\right);\ A_3=\frac{1}{4}s_2^4;\ \alpha=\frac{3}{32}s_2^4$$

R is shell radius; ρ_1 is densities of rib

The ε_1 parameter characterizes the relative thickness of shell; the ε_2 characterizes the ratio of the bending rigidities; the ε_3 characterizes the ratio of the torsion and bending rigidity in longitudinal direction; the ε_4 characterizes the ratio of the membrane rigidities; $\varepsilon_6$related to the eccentricities of stringer and rings.

Here $s_1 = \pi m l^{-1}$, $s_2 = n$ and m and n are the wave numbers in the axial direction.

Appendix B.

$$\Omega^2=\frac{\pi^4}{\rho_1\lambda^4},\quad \gamma=\left(1+\delta\right)\left(\frac{\Lambda}{r}\right)^2,\quad \delta=\frac{\lambda}{2\pi l}\left[\sin\frac{2\pi(1-x_0)}{\lambda}+\sin\frac{2\pi x_0}{\lambda}\right],\quad r=\sqrt{I/A},\quad \rho_1=\frac{\rho A}{EI}$$

Where x_0 is the phase shift and λ is the length of oscillation wave.