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Free vibration of symmetric angle-ply layered conical shell frusta of variable thickness under shear deformation theory

K.K.Viswanathan*, Saira Javed and Zainal Abdul Aziz

Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia, 81310 Skudai, Johor Bahru, Johor, Malaysia

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Abstract. Free vibration of symmetric angle-ply layered conical shell frusta of variable thickness is analyzed under shear deformation theory with different boundary conditions by applying collocation with spline approximation. Linear and exponential variation in thickness of layers are assumed in axial direction. Displacements and rotational functions are approximated by Bickley-type splines of order three and obtained a generalized eigenvalue problem. This problem is solved numerically for an eigenfrequency parameter and an associated eigenvector of spline coefficients. The vibration of three and five–layered conical shells, made up of two different type of materials are considered. Parametric studies are made for analysing the frequencies of the shell with respect to the coefficients of thickness variations, length-to-radius ratio, length-to-thickness ratio and ply angles with different combination of the materials. The results are compared with the available data and new results are presented in terms of tables and graphs.

Keywords: free vibration; conical shells; shear deformation; spline; angle-ply[1]

1. Introduction

The study of free vibration of laminated composite shell structures are attracted to many researchers in the field of aerospace, shipbuilding and chemical industries. Since the composite structures have more desirable damping and shock absorbing characteristics than the homogeneous ones. The use of the lamination for the structures leads to design with the maximum reliability and minimum weight. It is also known that the laminated composite shells exhibit large thickness effects than the structures made of homogeneous materials. So, it tends to analyze the free vibration of laminated composite conical shells including shear deformation theory (Kayran and Vinson 1990). Sivadas and Ganesan (1991) presented a paper on vibration of laminated conical shells with variable thickness using FEM. Wu and Wu (2000) studied the 3-D elasticity solutions for the free vibration analysis of laminated conical shells by an asymptotic approach. The influence of orthotropic material and the frequency characteristics for a rotating thin truncated circular symmetrical cross-ply laminated composite conical shell with simply-supported boundary condition using the generalized differential quadrature method was studied by Hua and Lam (2001). Shu (1996) also analyzed vibration of composite laminated conical shells by generalized

*Corresponding author, Associate Professor, E-mail: visu20@yahoo.com, viswanathan@utm.my

differential quadrature technique. Wu and Lee (2001) presented the results on laminated conical shells with variable stiffness. Free vibration of laminated cross-ply and angle-ply plates and Axisymmetric vibration of layered cylindrical shells of variable thickness were analysed using spline method by Viswanathan *et al*. (2007, 2008, 2011). Spline strip method was used by Mizusawa and Kito (1995) to study the vibration of cross-ply laminated cylindrical shells.

In the present work, free vibration of symmetric angle-ply conical shell frusta of variable thickness including first order shear deformation theory (FSDT) is studied using spline function approximation. The thickness variations are assumed to be linear and exponential along the axial direction of the cone. The problem is formulated using FSDT to obtain the equilibrium equations of conical shell frusta. The system of coupled differential equations are obtained on a set of assumed displacement and rotational functions using stress-strain and strain-displacement relations in the equilibrium equations. These functions are assumed in the separable form to obtain the differential equations in terms of single variable. In preference to the many numerical methods to solve the problem, like those of Generalized differential quadrature (Shu 1996), Fourier series approach (Kabir *et al.* 2001) and FEM (Girgin 2006, Wang *et al.* 2006), spline method have been adopted to approximate the displacement and rotational functions. . These splines are simple and clear for analytical process and therefore have significant computational advantage.

The displacement and rotational functions are approximated using cubic spline and collocation procedure is applied to obtain a set of field equations. The field equations along with the equations of boundary conditions yield a system of homogeneous simultaneous algebraic equations on the assumed spline coefficients which resulting to generalized eigenvalue problem. This eigenvalue problem is solved using eigensolution technique to get many eigenfrequencies as we required. The effect of frequency parameters with respect to the coefficient of thickness variations, cone angle, aspect ratio, circumferential node number, boundary conditions, two types of layered materials with three- and five- layered conical shells are presented and discussed.

2. Theoretical formulation and method of solution

2.1 Formulation of the problem

Consider a laminated conical shell frusta of variable thickness along axial direction having arbitrary number of layers, which are perfectly bonded together is shown in Fig. 1. The orthogonal coordinate system (x, θ, z) is fixed at its reference surface, which is taken to be at the middle surface. The radius of the cone at any point along its length is $r = x\sin\alpha$. The radius at the small end of the cone is $r_a = a\sin\alpha$ and the other end is $r_b = b\sin\alpha$. α is the semi-vertical angle and ℓ is the length of the cone along its generator. The displacement components are assumed to be in the form (Toorani and Lakis 2000)

$$\begin{aligned} u(r,\theta,z,t) &= u_0(r,\theta,t) + z\,\psi_x(r,\theta,t) \\ v(r,\theta,z,t) &= v_0(r,\theta,t) + z\,\psi_\theta(r,\theta,t) \\ w(r,\theta,z,t) &= w_0(r,\theta,t) \end{aligned} \tag{1}$$

where u, v, and w are the displacement functions in x, θ and z directions respectively, u_0, v_0, and w_0 are the displacements of the middle surface of the cone and ψ_x, ψ_θ are shear rotations of any point on the middle surface of the cone.

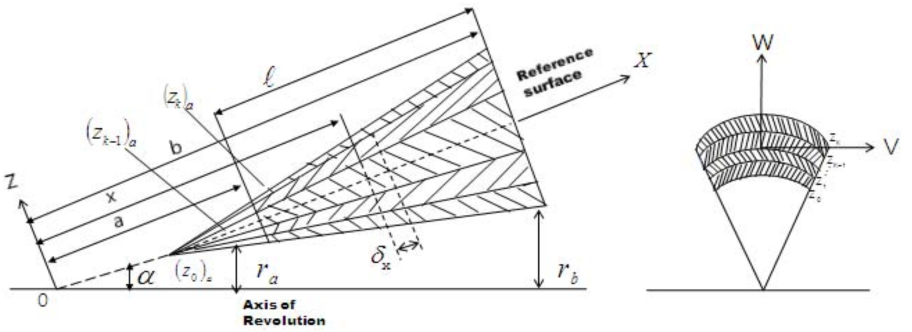

Fig. 1 Layered conical shell of variable thickness: geometry

The thickness variation of the k^{th} layer of the shell is assumed in the form as

$$h_k(x) = h_{0k}\, g(x) \tag{2}$$

where $g(x) = 1 + C_\ell(x - x_a/\ell) + C_e \exp(x - x_a/\ell)$, h_{0k} is a constant thickness of the k^{th} layer, $\ell = b - a$ is the length of the cone and x_a is the distance from the origin to $x = a$ (small end of the cone). The thickness of the shell becomes uniform when $g(x) = 1$.

Since the thickness is assumed to be varying along the axial direction, one can define the elastic coefficients A_{ij}, B_{ij} and D_{ij} (extensional, bending-extensional coupling and bending stiffnesses) corresponding to layers of uniform thickness with superscript 'c' as

$$A_{ij} = A_{ij}^c\, g(x), \quad B_{ij} = B_{ij}^c\, g(x), \; D_{ij} = D_{ij}^c\, g(x) \tag{3}$$

$$A_{ij}^c = \sum_k \bar{Q}_{ij}^{(k)}(z_k - z_{k-1}), \; B_{ij}^c = \frac{1}{2}\sum_k \bar{Q}_{ij}^{(k)}(z_k^2 - z_{k-1}^2)$$

$$D_{ij}^c = \frac{1}{3}\sum_k \bar{Q}_{ij}^{(k)}(z_k^3 - z_{k-1}^3) \quad \text{for}\, i, j = 1, 2, 6 \tag{4}$$

$$\text{and} \;\; A_{ij}^c = K\sum_k \bar{Q}_{ij}^{(k)}(z_k - z_{k-1}) \quad \text{for} \quad i, j = 4, 5 \tag{5}$$

Here K is the shear correction factor, z_{k-1} and z_k are boundaries of the k-th layer and the quantities $\overline{Q}_{ij}^{(k)}$ are defined in the reference Viswanathan and Kim (2008). In the case of symmetric angle-ply lamination, the laminate stiffnesses A_{16}, A_{26}, D_{16}, D_{26}, A_{45} and all B_{ij} are identically zero.

The governing differential equations characterising the vibration of a conical shell frusta of variable thickness including first order shear deformation theory is derived in terms of displacement functions $u_0(x, \theta, t)$, $v_0(x, \theta, t)$, $w_0(x, \theta, t)$ and shear rotational functions $\psi_x(x, \theta, t)$, $\psi_\theta(x, \theta, t)$ using stress-strain and strain-displacement relations of the conical shell frusta (Reddy 1978).

The displacement components u_0, v_0, w and shear rotations ψ_x, ψ_θ are assumed in separable form given as

$$
\begin{aligned}
u_0(x,\theta,t) &= U(x)\cos n\theta\, e^{i\omega t} \\
v_0(x,\theta,t) &= V(x)\sin n\theta\, e^{i\omega t} \\
w(x,\theta,t) &= W(x)\cos n\theta\, e^{i\omega t} \\
\psi_x(x,\theta,t) &= \Psi_x(x)\cos n\theta\, e^{i\omega t} \\
\psi_\theta(x,\theta,t) &= \Psi_\theta(x)\sin n\theta\, e^{i\omega t}
\end{aligned}
\tag{6}
$$

where ω is the angular frequency of vibration, t is the time and n is the circumferential node number.

Using the Eq. (5) into the governing differential equations, the resulting equation becomes in the matrix form as

$$
\begin{bmatrix}
L_{11}\, L_{12}\, L_{13}\, L_{14}\, L_{15} \\
L_{21}\, L_{22}\, L_{23}\, L_{24}\, L_{25} \\
L_{31}\, L_{32}\, L_{33}\, L_{34}\, L_{35} \\
L_{41}\, L_{42}\, L_{43}\, L_{44}\, L_{45} \\
L_{51}\, L_{52}\, L_{53}\, L_{54}\, L_{55}
\end{bmatrix}
\begin{Bmatrix} U \\ V \\ W \\ \psi_X \\ \psi_\Theta \end{Bmatrix}
=
\begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}
\tag{7}
$$

where L_{ij} are the differential operators depends on the variable x only and are given in Appendix-A.

The non-dimensional parameters are introduced to modify the above equations as follows:

The non-dimensional parameters are introduced as follows:

$X = \dfrac{x-a}{l}$, $a \le x \le b$ and $X \in [0,1]$

$\lambda = \ell\lambda'$, a frequency parameter

$$\gamma = \frac{h_0}{r_a},\ \gamma' = \frac{h_0}{a}, \text{ ratios of thickness to radius and to a length} \tag{8}$$

$\beta = \dfrac{a}{b}$, a length ratio

$\delta_k = \dfrac{h_k}{h}$, relative layer thickness of the k-th layer.

The thickness $h_k(X)$ of the k-th layer at X distance from the smaller end of the cone, already explained, can expressed as

$$h_k(X) = h_{0k}\, g(X) \tag{9}$$

where $g(X) = 1 + C_\ell X + C_e \exp(X)$

The new set of differential equations are obtained using Eqs. (8) and (9) into Eq. (7), is given in the new set of matrix form as

$$\begin{bmatrix} L_{11}^* & L_{12}^* & L_{13}^* & L_{14}^* & L_{15}^* \\ L_{21}^* & L_{22}^* & L_{23}^* & L_{24}^* & L_{25}^* \\ L_{31}^* & L_{32}^* & L_{33}^* & L_{34}^* & L_{35}^* \\ L_{41}^* & L_{42}^* & L_{43}^* & L_{44}^* & L_{45}^* \\ L_{51}^* & L_{52}^* & L_{53}^* & L_{54}^* & L_{55}^* \end{bmatrix} \begin{Bmatrix} U \\ V \\ W \\ \ell\Psi_X \\ \ell\Psi_\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{10}$$

The differential operators L_{ij}^* of the matrix are given in the Appendix B.

2.2 Thickness variation

Case (i):

If $C_e = 0$, then the thickness variation becomes linear. In this case it can easily shown that

$$C_\ell = \frac{1}{\eta} - 1 \text{ , where } \eta \text{ is the taper ratio } h_k(0)/h_k(1) \tag{11}$$

Case (ii)

If $C_\ell = 0$, then the excess thickness over uniform thickness varies exponentially.

It may be noted that the thickness of any layer at the end $X = 0$ is h_{0k} for the cases (i), but is $h_{0k}(1 + C_e)$ for the case (ii).

The following range of values of the thickness coefficients are considered

$$0.5 \le \eta \le 2.1, \quad -0.2 \le C_e \le 0.2 \tag{12}$$

Spline collocation procedure

The displacement functions *U, V, W* and rotational functions Ψ_x, Ψ_Θ are approximated by cubic spline functions in the range of $X \in [0, 1]$ as

$$U^*(X) = \sum_{i=0}^{2} a_i X^i + \sum_{j=0}^{N-1} b_j (X - X_j)^3 H(X - X_j)$$

$$V^*(X) = \sum_{i=0}^{2} c_i X^i + \sum_{j=0}^{N-1} d_j (X - X_j)^3 H(X - X_j)$$

$$W(X) = \sum_{i=0}^{2} e_i X^i + \sum_{j=0}^{N-1} f_j (X - X_j)^3 H(X - X_j)$$

$$\Psi_X^*(X) = \sum_{i=0}^{2} g_i X^i + \sum_{j=0}^{N-1} p_j (X - X_j)^3 H(X - X_j)$$

$$\Psi_\Theta^*(X) = \sum_{i=0}^{2} l_i X^i + \sum_{j=0}^{N-1} q_j (X - X_j)^3 H(X - X_j) \quad 13)$$

Here, $H(X - X_j)$ is the Heaviside step functions and defined as

$$H(X - X_j) = \begin{cases} 1, & X \geq X_j \\ 0, & X < X_j \end{cases} \tag{14}$$

The range of X is divided in to N subintervals, at the points $X = X_s$, $s = 1, 2, 3, \ldots, N-1$. The width of each subinterval is $1/N$ and $X_s = s/N$ ($s = 0, 1, 2, \ldots, N$), since the knots X_s are chosen equally spaced.

The assumed spline functions given in Eq. (13) are approximated at the nodes (coincide with the knots) and these splines satisfy the differential equations given in Eq. (10), at all X_s and resulting into the homogeneous system of $(5N + 5)$ equations in the $(5N + 15)$ unknown spline coefficients, $a_i, c_i, e_i, g_i, l_i, b_j, d_j, f_j, k_j, m_j$ ($i = 0, 1, 2$; $j = 0, 1, 2, \ldots, N-1$). To obtain 10 more equations, the following boundary conditions are considered in this problem.

(i) Clamped-Clamped (C-C) (both the ends are clamped)

(ii) Simply-supported (S-S) (both ends are simply supported)

Combining these 10 equations with the earlier $(5N + 5)$ equations, one can get $(5N + 15)$ homogeneous equations in the same number unknowns. Thus, we get a generalized eigenvalue problem in the form

$$[M]\{q\} = \lambda^2 [P]\{q\} \tag{15}$$

Where $[M]$ and $[P]$ are the square matrices, $\{q\}$ is the column matrix of the spline coefficients and λ is the eigenfrequency parameter. This eigenvalue problem is solved using FORTRAN programme by applying numerical technique (power method) to get the eigenvalues and eigenvectors as many as we required.

3. Results and discussions

Convergence study carried out for frequency parameter λ by fixing other parameters, material combinations, number of layers and ply angles under C-C and S-S boundary conditions with the number of subintervals N of the range $X \in [0, 1]$. The value of N started from 4 and finally it is fixed for $N = 14$, since for the next value of N, the percent changes in the values of λ are very low, the maximum being 3%. Comparison results are made for frequency parameter $\omega_{pi} = \omega_i R_2 \sqrt{\rho h / A_{11}}$, (where $R_2 = r_b$) with cone angle $\alpha = 30^\circ$ for axisymmetric vibration of two layered antisymmetric cross-ply truncated conical shells with coupling under S-S boundary conditions using FSDT and Classical shell theory (CST) shown in Table 1. The material properties of the individual layers are considered as: $E_x / E_\theta = 15$, $G_{x\theta} / E_\theta = 0.5$, $\upsilon_{x\theta} = 0.25$, $\upsilon_{xz} = \upsilon_{\theta z} = 0.3$, $G_{x\theta} = E_\theta / 2(1 + \upsilon_{xz})$ and $G_{\theta z} = E_\theta / 2(1 + \upsilon_{\theta z})$. It is seen, from the Table 1, that the maximum percentage changes between present value and available result is 5.7%. The agreement of the current result is quite good.

In the present work, frequency parameter for symmetric angle-ply layers of truncated conical shells is presented using two kinds of materials with respect to the circumferential mode number, aspect ratio, ratio of thickness to radius and ratio of thickness to length parameters. The materials AS4/3501-6 Graphite/epoxy (GE) and E-glass/epoxy (EGE) are considered to analyze the problem. The shear correction factor is fixed as 5/6 throughout the problem (Reddy 1978, Ghosh and Dey 1994). The effect of cone angle, aspect ratio, circumferential mode number and variation

Table 1 Comparison of natural frequencies for axisymmetric cross-ply laminated conical shell frusta with simply supported boundary conditions ($\alpha = 30^\circ$, $\ell/r_b = 0.5$)

h/r_b	Present	Tong (1994)	Wu and Lee (2001)	Shu (1996) (Classical Theory)
0.01	0.1753	0.1768	0.1759	0.1799
0.02	0.2088	0.2091	0.2093	0.2153
0.03	0.2316	0.2304	0.2320	0.2397
0.04	0.2516	0.2495	0.2520	0.2620
0.05	0.2706	0.2681	0.2710	0.2841
0.06	0.2894	0.2862	0.2892	0.3061
0.07	0.3078	0.3033	0.3061	0.3277
0.08	0.3252	0.3193	0.3217	0.3484
0.09	0.3418	0.3338	0.3358	0.3680
0.10	0.3568	0.3469	0.3484	0.3863

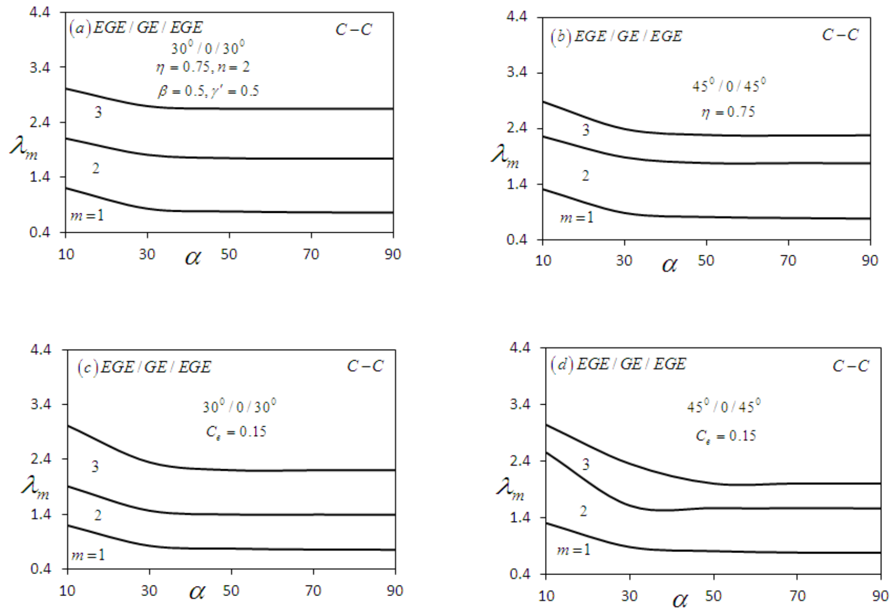

Fig. 2 Variation of frequency parameter with respect to cone angle under C − C boundary conditions: linear and exponential thickness variations

of thickness coefficients on the frequency parameter values are investigated using three and five layered conical shells. Linear and exponential variation in thickness of layers are considered.

Fig. 2 depicts the variation of frequency parameter λ_m (m = 1, 2, 3) with respect to the cone angle α for three layered shells of the angles 30° / 0° / 30° and 45° / 0° / 45° using GE and EGE materials arranged in the order of EGE-GE-EGE under C-C boundary conditions. The other parameters β, γ', n, η and C_e are fixed. Figs. 2(a) and (b) are the effect of cone angle with the variation of frequency parameter for linear variation in thickness in two different ply-angles. The value of λ_m (m = 1, 2, 3) decreases when α increases. The decrease of λ is up to α = 30° and then the frequency values are almost constant for all $\alpha \geq 30^\circ$. The nature of variations of λ_m (m = 1, 2, 3) are the same for both the ply-angles 30° / 0° / 30° and 45° / 0° / 45°, but the values are higher for

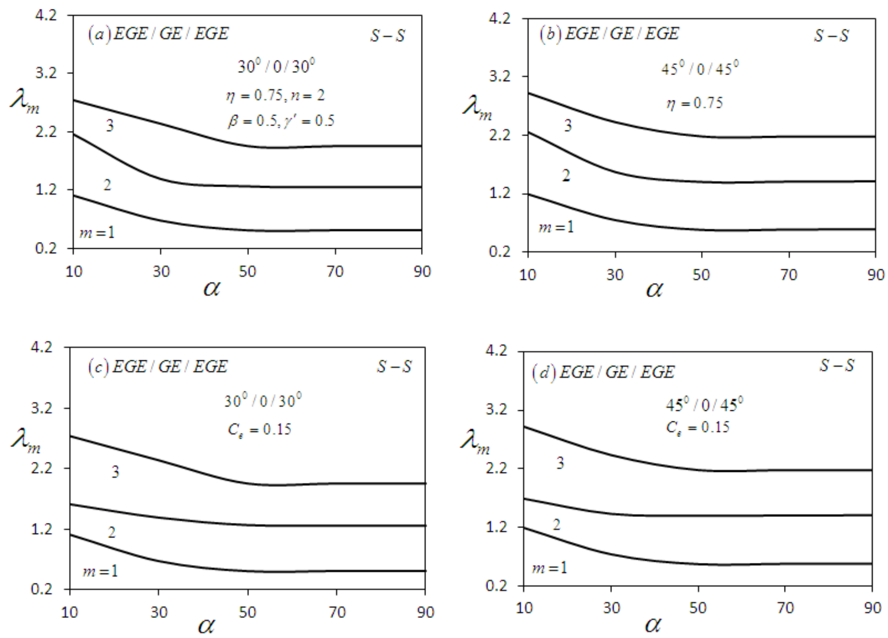

Fig. 3 Variation of frequency parameter with respect to cone angle under S – S boundary conditions: linear and exponential thickness variations

higher angles. Figs. 2(c) and (d) shows the frequency parameter λ_m (m = 1, 2, 3) with respect to cone angle under the exponential variation in thickness. This also shows the same trend as shown in Figs. 2(a) and (b). Fig. 3 shows the effect of cone angle with frequency parameter λ_m (m = 1, 2, 3) under S-S boundary conditions for three layered symmetric angle-ply shells. The thickness variation coefficients η and C_e are held fixed. Analyzing with reference to the boundary conditions, the values of the frequency parameter λ_m (m = 1, 2, 3) of S-S boundary conditions are least when compared to the corresponding values of C-C boundary conditions.

Figs. 4(a)-(d) shows the manner of variation of the frequency parameter with reference to the circumferential mode number n. The value of n ranges from 1 to 10. A shell of linear thickness variation with three layered symmetric angle-ply under C-C boundary conditions are considered in Figs. 4 (a) and (b). Figs. 4 (c) and (d) relates to a shell of three layered symmetric angle-ply of exponential variation in thickness under C-C conditions are considered. The layered materials are in the order of EGE-GE-EGE. All the shells have semi cone angle $\alpha = 30^\circ$, $\beta = 0.5$ and $\gamma = 0.05$. It is seen for the figures that all the frequency parameter values decreases upto n = 3 or 4 and then increases. The curvature at turning points seem to be greater for lower modes (m). Figs. 5 (a)-(d) shows the manner of variation of the frequency parameter with reference to the circumferential mode number n under S-S boundary conditions. All other parameters are fixed. The nature of variation of frequency parameters for linear and exponential thickness variations with circumferential mode number n is the same as defined in Fig. 4.

Fig. 6 describes the variation of angular frequencies ω (not λ) with respect to the length ratio β under C-C boundary conditions. Since λ is a function of the length ℓ of the conical shell, so it is not meaningful to study the variation of λ with β. In this case, the thickness parameter to be fixed

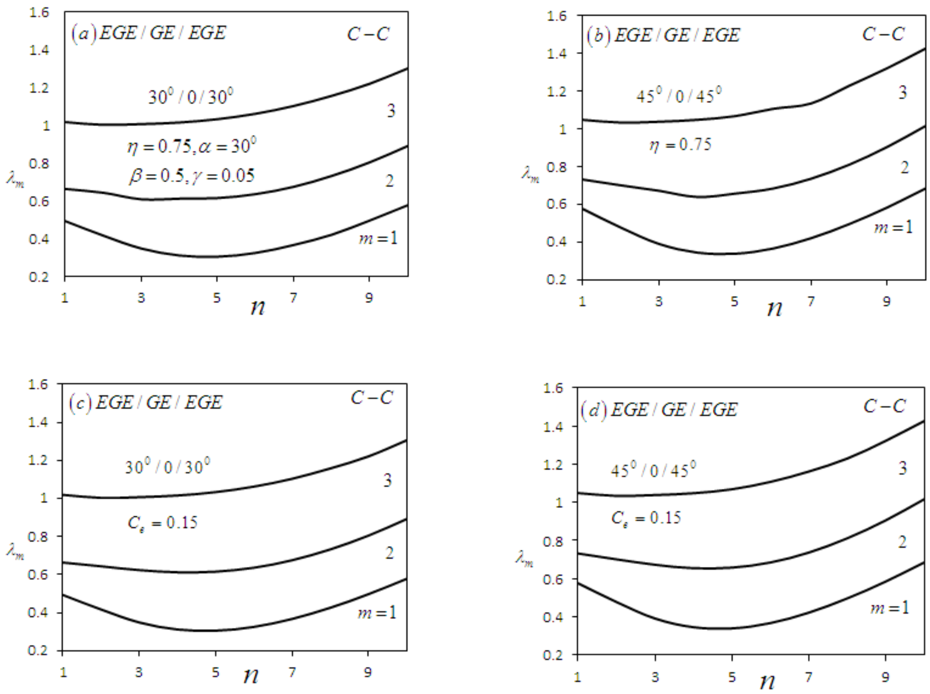

Fig. 4 Variation of frequency parameter with respect to circumferential mode number under C – C boundary conditions: linear and exponential thickness variations

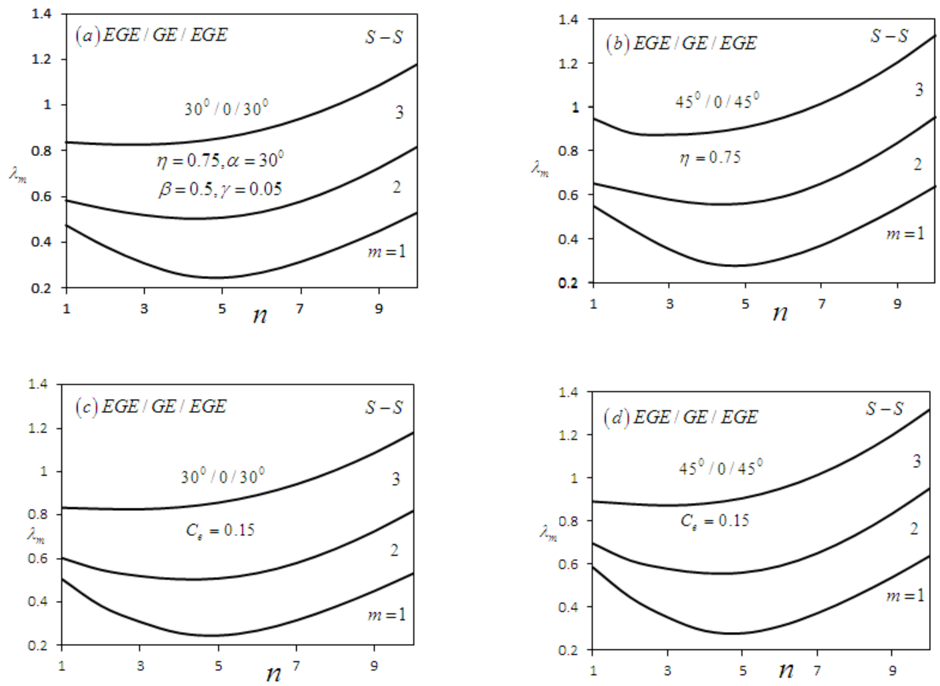

Fig. 5 Variation of frequency parameter with respect to circumferential mode number under S – S boundary conditions: Linear and exponential thickness variations

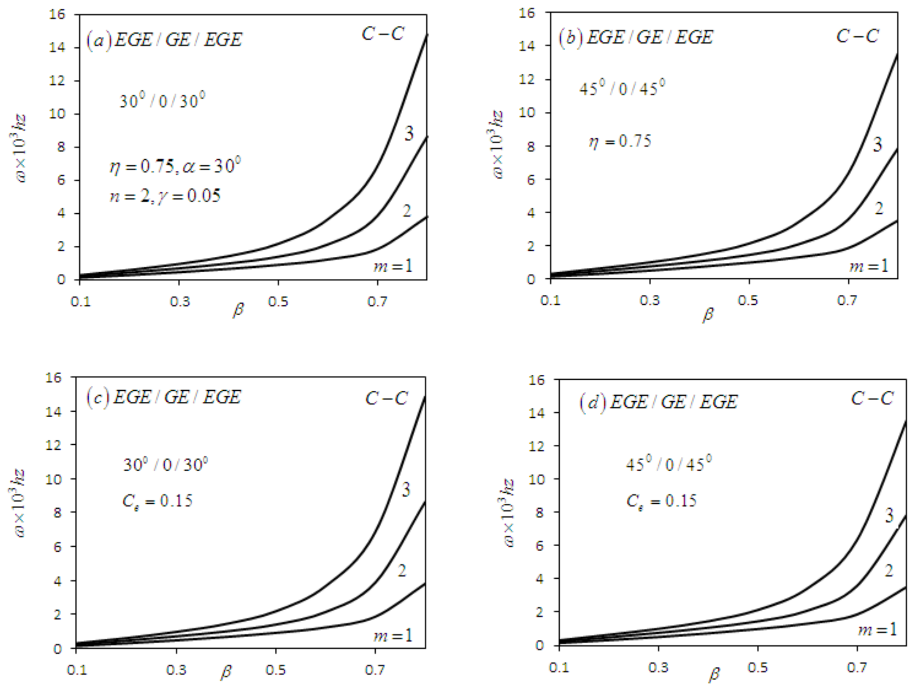

Fig. 6 Variation of frequency with respect to aspect ratio β under C – C boundary conditions: linear and exponential thickness variations

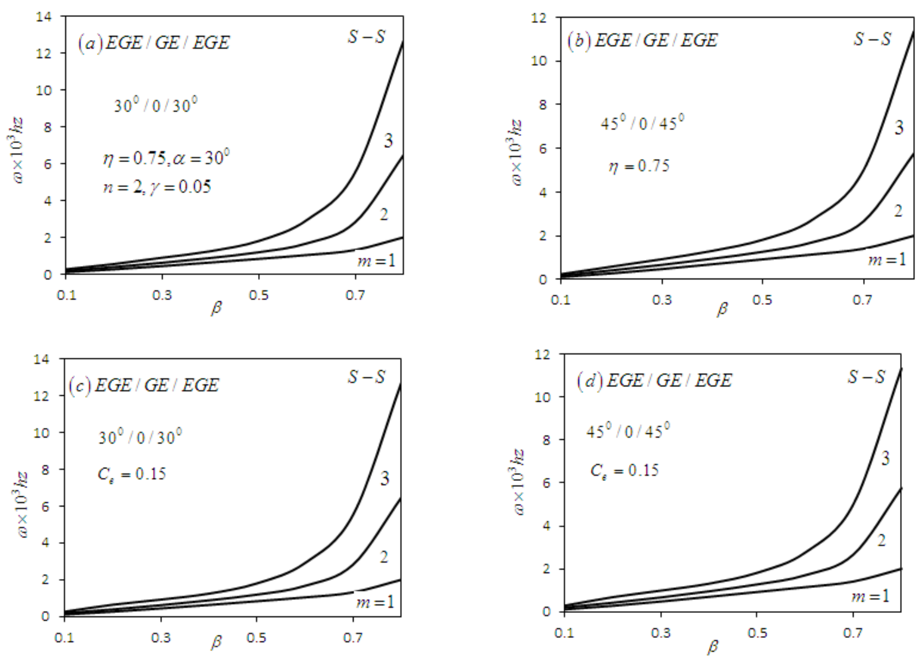

Fig. 7 Variation of frequency with respect to aspect ratio β under S – S boundary conditions: linear and exponential thickness variations

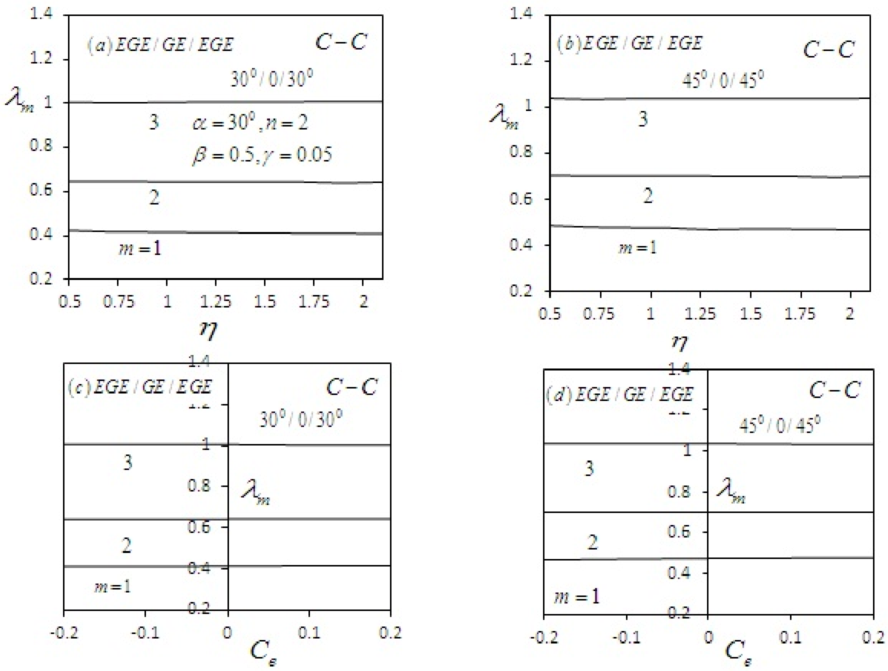

Fig. 8 Variation of frequency parameter with respect to taper ratio and coefficient of exponential variation of thickness of layers under C – C boundary conditions

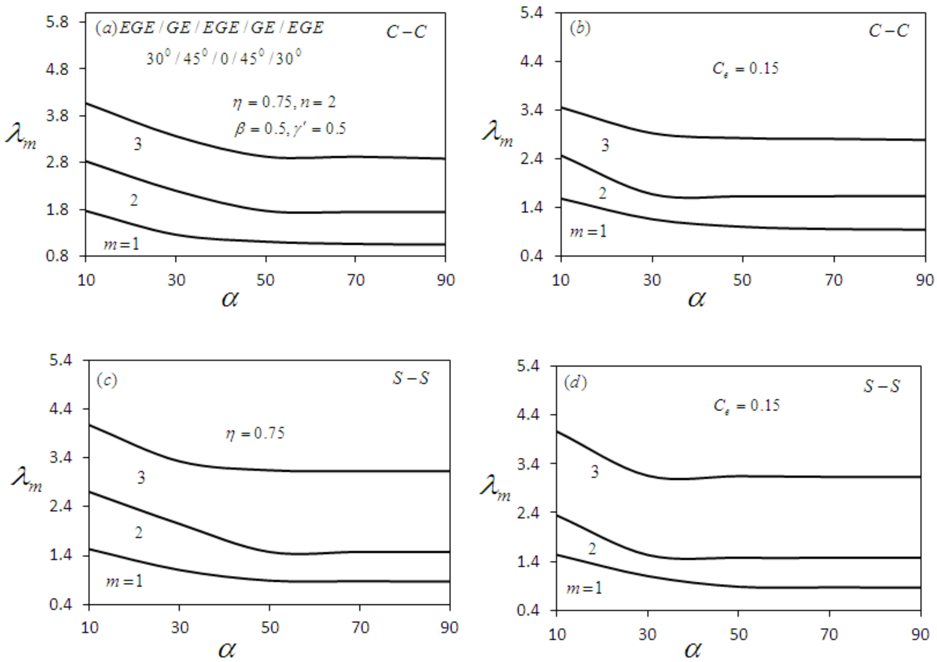

Fig. 9 Variation of frequency parameter with respect to cone angle under C – C and S – S boundary conditions: linear and exponential thickness variations

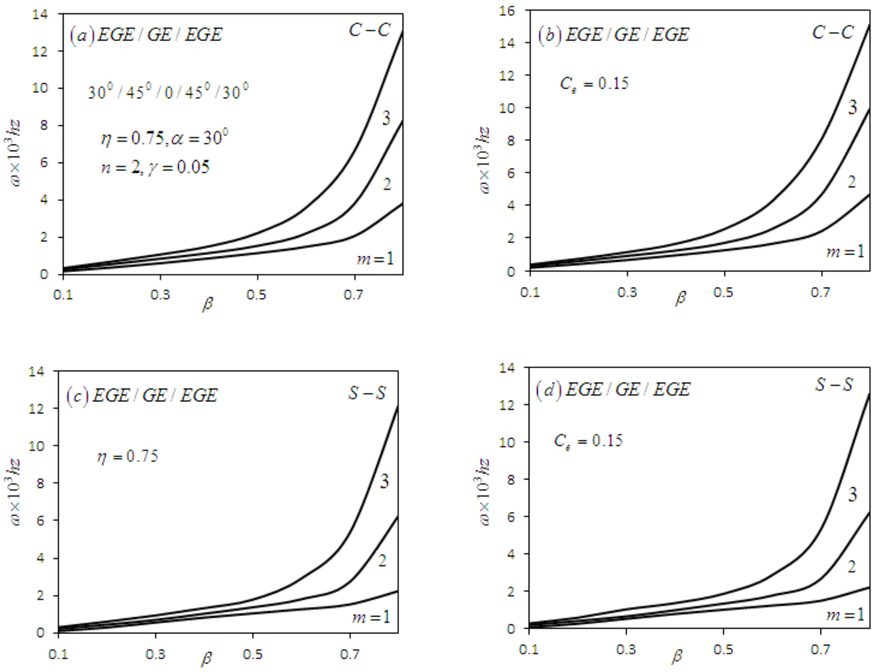

Fig. 10 Variation of frequency with respect to aspect ratio β under C – C and S – S boundary conditions: Linear and exponential thickness variations

with some value and it is given that $h(a)=1$cm for all cases considered. Figs. 6(a) and (b) relates to the linear thickness variation where as Figs. 6(c) and (d) for exponential thickness variation. It is seen from the figure that the frequencies monotonically increase with increase in β i.e., with decreasing cone-length. The increase of $\omega_m(m = 1, 2, 3)$ is gradual and steady up to some value of β, and rapid increase after wards. The rate of gradual change is higher, and rapid increase in ω starts earlier, for higher modes. As expected, for very short shells ($\beta > 0.8$), frequencies are very high. The variation of angular frequency with length ratio β for three layered conical shell under S-S boundary conditions are shown in Fig. 7. Both, linear and exponential thickness variations are considered.

In Fig. 8, the influence of the nature of variation of thickness of the three layered symmetric angle-ply conical shell with ply-angles 30˚ / 0˚ / 30˚ and 45˚ / 0˚ / 45˚ on its vibrational behaviour is studied. The order of the layers are arranged in the form of EGE-GE-EGE materials under C-C boundary conditions and the other parameters β, γ, α and n are fixed. Figs. 8 (a) and (b) relates to liner variation in thickness of layers. The variation of $\lambda_m(m = 1, 2, 3)$ with respect to η for $0.5 \le \eta \le 2.1$ is studied. The thickness becomes constant when $\eta = 1$ and the thickness at the larger end of the cone is larger or smaller than the thickness at the smaller end according as $\eta \lessgtr 1$. It is seen that the variation of λ_m is so small (almost constant) for all the values of η and the values are higher for higher modes. Figs. 8(c) and (d) relates to exponential variation in thickness of layers. The variation of $\lambda_m(m = 1, 2, 3)$ with respect to C_e for $-0.2 \le C_e \le 0.2$ is studied. The same nature is applicable for exponential variation as we discussed earlier in linear thickness variation.

Five layered symmetric angle-ply conical shells with arranging the EGE and GE materials in the order of EGE-GE-EGE-GE-EGE and also arranging the ply angles in the form of 30˚ / 45˚ / 0˚

/ 45˚ / 30˚ shown in Figs. 9-11. Fig. 9 depicts the influence of the cone angle on its frequency parameter under linear and exponential thickness variation under C-C and S-S boundary conditions. It seems from the figure that the frequency values are higher for five layered shells when it is compared with the corresponding values of three layered shells given in Figs. 2 and 3. This may be due to the order of the angles and order of the material properties. In Fig. 10, the influence of length ratio on its angular frequencies of variation with linear and exponential variation in thickness under C-C and S-S boundary conditions are shown by fixing the other parameters. The effect of circumferential mode number on frequency parameter is studied in Fig. 11. Both C-C and S-S conditions are applied to analyse the five layered angle-ply shells with η = 0.75 and C_e = 0.15 by fixing cone angle, length ratio. The effect of frequency parameter is the same as defined in the three layered angle-ply shells. The values may be lower or higher, depending on the ply-angles, order of the material properties and other parameters.

4. Conclusions

The values of natural frequencies of the layered conical shells with different material properties are being different with those of homogeneous shells of any one of the layered materials. The inclusion of shear deformation theory is more significant to analyse the vibration of layered shell structure since which yields lower values on the frequency parameters when we compared to the values predicted by classical shell theory. The results presented in this paper may be fruitful for designers to choose the materials, ply-angle, cone angle, length ratio and circumferential mode number for making the conical shell structure according to their needs in designing appropriately. Also, these results shows the elegance and usefulness of spline function approximations.

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Appendix A

The differential operators L_{ij} of the matrix are

$$L_{11} = \frac{d^2}{dx^2} + \left(\frac{g'}{g} + \frac{1}{x}\right)\frac{d}{dx} - s_{10}\frac{1}{x^2}n^2 \cos ec^2\alpha + s_2\frac{g'}{g}\frac{1}{x} - s_3\frac{1}{x^2} + \lambda'^2 \tag{A1}$$

$$L_{12} = \left(s_2 + s_{10}\right)\frac{n\cos ec\alpha}{x}\frac{d}{dx} + \left(s_2\frac{g'}{g}\frac{1}{x} - s_3\frac{1}{x^2} - s_{10}\frac{1}{x^2}\right)n\cos ec\alpha \tag{A2}$$

$$L_{13} = s_2\frac{1}{x}\cot\alpha\frac{d}{dx} + \left(s_2\frac{g'}{g}\frac{1}{x} - s_3\frac{1}{x^2}\right)\cot\alpha, L_{14} = L_{15} = 0 \tag{A3}$$

$$L_{21} = -\left(s_2 + s_{10}\right)\frac{n\cos ec\alpha}{x}\frac{d}{dx} - \left(s_{10}\frac{g'}{g}\frac{1}{x} + s_{10}\frac{1}{x^2} + s_3\frac{1}{x^2}\right)n\cos ec\alpha \tag{A4}$$

$$L_{22} = s_{10}\frac{d^2}{dx^2} + \left(s_{10}\frac{g'}{g} + s_{10}\frac{1}{x}\right)\frac{d}{dx} - s_3\frac{1}{x^2}n^2\cos ec^2\alpha - s_{10}\frac{g'}{g}\frac{1}{x}$$
$$-s_{10}\frac{1}{x^2} - Ks_{13}\frac{1}{x^2}\cot^2\alpha + \lambda'^2 \tag{A5}$$

$$L_{23} = -\left(s_3 + Ks_{13}\right)\frac{n\cos ec\alpha\cot\alpha}{x^2}, L_{24} = 0, L_{25} = Ks_{13}\frac{1}{x}\cot\alpha \tag{A6}$$

$$L_{31} = -s_2\frac{1}{x}\cot\alpha\frac{d}{dx} - s_3\frac{1}{x^2}\cot\alpha, L_{32} = -\left(s_3 + Ks_{13}\right)\frac{n\cos ec\alpha\cot\alpha}{x^2} \tag{A7}$$

$$L_{33} = Ks_{14}\frac{d^2}{dx^2} + Ks_{14}\left(\frac{g'}{g} + \frac{1}{x}\right)\frac{d}{dx} - Ks_{13}\frac{1}{x^2}n^2\cos ec^2\alpha - s_3\frac{1}{x^2}\cot^2\alpha + \lambda'^2 \tag{A8}$$

$$L_{34} = Ks_{14}\frac{d}{dx} + Ks_{14}\left(\frac{g'}{g} + \frac{1}{x}\right), L_{35} = Ks_{13}\frac{n\cos ec\alpha}{x} \tag{A9}$$

$$L_{41} = L_{42} = 0, L_{43} = -Ks_{14}\frac{d}{dx} \tag{A10}$$

$$L_{44} = s_7\frac{d^2}{dx^2} + s_7\left(\frac{g'}{g} + \frac{1}{x}\right)\frac{d}{dx} + s_8\frac{g'}{g}\frac{1}{x} - s_9\frac{1}{x^2} - Ks_{14} - s_{12}\frac{1}{x^2}n^2\cos ec^2\alpha + \lambda'^2\left(\frac{I_3}{4I_1}\right) \tag{A11}$$

$$L_{45} = \left(s_8 + s_{12}\right)\frac{n\cos ec\alpha}{x}\frac{d}{dx} + \left(s_8\frac{g'}{g}\frac{1}{x} - s_9\frac{1}{x^2} - s_{12}\frac{1}{x^2}\right)n\cos ec\alpha \tag{A12}$$

$$L_{51} = 0, L_{52} = Ks_{13}\frac{1}{x}\cot\alpha, L_{53} = Ks_{13}\frac{1}{x}n\cos ec\alpha \tag{A13}$$

$$L_{54} = -\left(s_8 + s_{12}\right)\frac{n\cos ec\alpha}{x}\frac{d}{dx} - s_{12}\frac{g'}{g}\frac{n\cos ec\alpha}{x} - \left(s_9 + s_{12}\right)n\frac{1}{x^2}\cos ec\alpha \tag{A14}$$

$$L_{55} = s_{12}\frac{d^2}{dx^2} + s_{12}\left(\frac{g'}{g} + \frac{1}{x}\right)\frac{d}{dx} - s_9 n^2\cos ec^2\alpha\frac{1}{x^2} - s_{12}\frac{1}{x^2} - s_{12}\frac{g'}{g}\frac{1}{x} - Ks_{13} + \lambda'^2\left(\frac{I_3}{4I_1}\right) \tag{A15}$$

where

$$\lambda' = \omega\sqrt{\frac{I_1}{A_{11}}}, \quad g = g(x), \quad g' = \frac{dg}{dx}\ , \ (I_1, I_3) = \int \rho^{(k)}(1, z^2)\,dz \quad \text{and} \tag{A16}$$

$$s_2 = \frac{A_{12}^c}{A_{11}^c}, s_3 = \frac{A_{22}^c}{A_{11}^c}, s_8 = \frac{D_{12}^c}{l^2 A_{11}^c}, s_9 = \frac{D_{22}^c}{l^2 A_{11}^c}$$

$$s_{10} = \frac{A_{66}^c}{A_{11}^c}, s_{12} = \frac{D_{66}^c}{l^2 A_{11}^c}, s_{13} = \frac{A_{44}^c}{A_{11}^c}, s_{14} = \frac{A_{55}^c}{A_{11}^c} \tag{A17}$$

Appendix B

The differential operators L_{ij}^* of the matrix are

$$L_{11} = \frac{d^2}{dX^2} + \left(\frac{g'}{g} + p\right)\frac{d}{dX} - S_{10}p^2n^2\cos ec^2\alpha + S_2\frac{g'}{g}p - S_3p^2 + \lambda^2 \tag{B1}$$

$$L_{12} = \left(S_2 + S_{10}\right)pn\cos ec\alpha\frac{d}{dX} + \left(S_2\frac{g'}{g}p - S_3p^2 - S_{10}p^2\right)n\cos ec\alpha \tag{B2}$$

$$L_{13} = S_2 p\cot\alpha\frac{d}{dX} + \left(S_2\frac{g'}{g}p - S_3p^2\right)\cot\alpha, L_{14} = L_{15} = 0 \tag{B3}$$

$$L_{21} = -\left(S_2 + S_{10}\right)pn\cos ec\alpha\frac{d}{dX} - \left(S_{10}\frac{g'}{g}p + S_{10}p^2 + S_3p^2\right)n\cos ec\alpha \tag{B4}$$

$$L_{22} = S_{10}\frac{d^2}{dX^2} + \left(S_{10}\frac{g'}{g} + S_{10}p\right)\frac{d}{dX} - S_3p^2n^2\cos ec^2\alpha - S_{10}\frac{g'}{g}p - S_{10}p^2 - KS_{13}p^2\cot^2\alpha + \lambda^2 \tag{B5}$$

$$L_{23} = -\left(S_3 + KS_{13}\right)p^2n\cos ec\alpha\cot\alpha, L_{24} = 0, L_{25} = KS_{13}p\cot\alpha \tag{B6}$$

$$L_{31} = -S_2p\cot\alpha\frac{d}{dX} - S_3p^2\cot\alpha, L_{32} = -\left(S_3 + KS_{13}\right)p^2n\cos ec\alpha\cot\alpha \tag{B7}$$

$$L_{33} = KS_{14}\frac{d^2}{dX^2} + KS_{14}\left(\frac{g'}{g} + p\right)\frac{d}{dX} - KS_{13}p^2n^2\cos ec^2\alpha - S_3p^2\cot^2\alpha + \lambda^2 \tag{B8}$$

$$L_{34} = KS_{14}\frac{d}{dX} + KS_{14}\left(\frac{g'}{g} + p\right), L_{35} = KS_{13}np\cos ec\alpha \tag{B9}$$

$$L_{41} = L_{42} = 0, L_{43} = -KS_{14}\frac{d}{dx} \tag{B10}$$

$$L_{44} = S_7 \frac{d^2}{dX^2} + S_7\left(\frac{g'}{g} + p\right)\frac{d}{dX} + S_8 \frac{g'}{g} p - S_9 p^2 - KS_{14} - S_{12} p^2 n^2 \cos ec^2\alpha + \lambda^2 \left(\frac{I_3}{4\ell^2 I_1}\right) \tag{B11}$$

$$L_{45} = \left(S_8 + S_{12}\right) pn \cos ec\alpha \frac{d}{dX} + \left(S_8 \frac{g'}{g} p - S_9 p^2 - S_{12} p^2\right) n \cos ec\alpha \tag{B12}$$

$$L_{51} = 0, L_{52} = KS_{13} p \cot\alpha, L_{53} = KS_{13} pn \cos ec\alpha \tag{B13}$$

$$L_{54} = -\left(S_8 + S_{12}\right) pn \cos ec\alpha \frac{d}{dX} - S_{12} \frac{g'}{g} pn \cos ec\alpha - \left(S_9 + S_{12}\right) np^2 \cos ec\alpha \tag{B14}$$

$$L_{55} = S_{12} \frac{d^2}{dX^2} + S_{12}\left(\frac{g'}{g} + p\right)\frac{d}{dX} - S_9 n^2 \cos ec^2\alpha p^2 - S_{12} p^2 - S_{12} \frac{g'}{g} p - KS_{13} + \lambda^2 \left(\frac{I_3}{4\ell^2 I_1}\right) \tag{B15}$$

where

$$g = g(X), g' = \frac{dg(X)}{dX} \tag{B16}$$