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Vibrations of an axially accelerating, multiple supported flexible beam

S. Kural* and E. Özkaya[a]

Department of Mechanical Engineering, Celal Bayar University, 45140 Muradiye, Manisa, Turkey

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Abstract. In this study, the transverse vibrations of an axially moving flexible beams resting on multiple supports are investigated. The time-dependent velocity is assumed to vary harmonically about a constant mean velocity. Simple-simple, fixed-fixed, simple-simple-simple and fixed-simple-fixed boundary conditions are considered. The equation of motion becomes independent from geometry and material properties and boundary conditions, since equation is expressed in terms of dimensionless quantities. Then the equation is obtained by assuming small flexural rigidity. For this case, the fourth order spatial derivative multiplies a small parameter; the mathematical model converts to a boundary layer type of problem. Perturbation techniques (The Method of Multiple Scales and The Method of Matched Asymptotic Expansions) are applied to the equation of motion to obtain approximate analytical solutions. Outer expansion solution is obtained by using MMS (The Method of Multiple Scales) and it is observed that this solution does not satisfy the boundary conditions for moment and incline. In order to eliminate this problem, inner solutions are obtained by employing a second expansion near the both ends of the flexible beam. Then the outer and the inner expansion solutions are combined to obtain composite solution which approximately satisfying all the boundary conditions. Effects of axial speed and flexural rigidity on first and second natural frequency of system are investigated. And obtained results are compared with older studies.

Keywords: beam vibrations; supported end; flexible beam; multiple supports; axially accelerating; small flexural stiffness; transverse vibrations

1. Introduction

Axially moving continuous media have attracted great interest because of the importance in engineering applications such as ropes, high speed magnetic tapes, power transmission belts, band-saws, fiber textiles, paper sheets, aerial cable tramways, oil pipelines, etc. Especially, their vibration analyses are very important for optimum working conditions. Their usage fields are extended with developments of industrial process and technology from middle of last century to nowadays. A broad literature exists on the topic which is reviewed by Ulsoy *et al*. (1978), Wickert and Mote (1988). Wickert and Mote (1990) investigated the transverse vibrations of travelling strings and beams. Wickert (1992) investigated strained beam problem. First equation of motion for axially

*Corresponding author, Ph.D. Student, E-mail: saim.kural@cbu.edu.tr, saimkural@gmail.com
[a]E-mail: erdogan.ozkaya@cbu.edu.tr

moving continuous media is obtained by Miranker (1960). Mockenstrum *et al*. (1994) analyzed time-dependent stress force and stability state for systems with constant velocity. Pakdemirli *et al*. (1994) obtained the equation of motion for axially accelerating string by using Hamilton Principle and they investigated stability of vibrations numerically. Pakdemirli and Batan (1993) repeated this analysis for harmonically accelerating and decelerating systems with constant acceleration. Pakdemirli and Ulsoy (1997) investigated principal parametric resonances combination resonances for an axially accelerating string. They found that for velocity fluctuation frequencies near twice of natural frequency, an instability region occurs whereas for the frequencies close to zero, no instabilities were detected. Nayfeh *et al*. (1981) showed that, direct-perturbation method give better results for quadratic and cubic non-linearity. Öz and Pakdemirli (1999), Öz (2001), Özkaya and Öz (2002) investigated main parametric and combination resonance situation for axially moving beam with variable velocity by using perturbation and artificial neural networks method and they showed that these systems have sum type combination resonances and also they showed that difference type combination resonances not occurs. Pakdemirli *et al*. (1995) compared results that obtained from two methods for nonlinear cable vibrations. They showed that solutions obtained by direct - perturbation method better represent to real system behavior of the system than the common method of discretization-perturbation method for higher order expansions. Pakdemirli (1994), Pakdemirli and Boyacı (1995) showed that direct - perturbation method has much sensitive solutions by using general model with arbitrary quadratic and cubic nonlinearity. And again Pakdemirli and Boyacı (1994) showed that two methods have differences not only in nonlinear equations and also in linear equations. Chung *et al.* (2001) studied vibrations of an axially moving string with translating acceleration and they investigated the natural frequencies, the time histories of the deflections and the distributions of the deflection and stress. Chen and Zhao (2005) studied on a numerical method for simulating transverse vibrations of an axially moving string. In following studies researchers studied on non-ideal boundary conditions for beam vibrations Boyacı (2005), for stretched damped beam vibrations Boyacı (2006), for continuous systems Pakdemirli and Boyacı (2002), for simple-simple beam with a non-ideal support in between Pakdemirli and Boyacı (2003), for stretched beam vibrations Pakdemirli and Boyacı (2001). Linear and non-linear transverse vibrations are investigated by Özkaya (2001, 2002).

In recent studies, Ponomereva and van Horssen (2009) considered boundary value problem for a linear axially moving stretched beam. They assumed the velocity as time dependent. They examined higher order mods and they combined string and tensioned beam models to obtain flexible beam model. Pakdemirli and Özkaya (1998) obtained approximate boundary layer solution for an axially moving beam problem. Öz *et al*. (1998) considered axially moving beam with small transverse rigidity and investigated transition behavior from string to beam. An approximate analytical expression for natural frequency was given for the problem. For time-dependent velocity profiles, stability borders were determined analytically. This study gives approximately solutions for only simply supported beam with small transverse rigidity and has not possible usage for other conditions. The one close study to this article accomplished by Özkaya and Pakdemirli (2000). In this study transition behavior from string to beam is investigated by using different methods. But they obtained same natural frequency values for simply supported and fixed supported cases. This solution approximately corrects the simply supported beam but in fixed supported case it fails to satisfy exact solution. And one close study about article accomplished by Parker *et al*. (2004). Axially moving flexible beam with constant axial velocity was considered. In this study, they defined that obtained natural frequency values in studies of Öz *et al*. (1998) and Özkaya and

Pakdemirli (2000) does not valid for fixed supported case, and with their suggested method they obtained natural frequencies for simply and fixed supported cases. Simply supported case has same solutions with older studies but they obtained different frequency values for fixed supported case, but difference between these solutions and exact solutions are relatively high. Bağdatlı *et al*. (2008) investigated non-linear transverse vibrations and 3:1 internal resonances of a tensioned beam on multiple supports. Bağdatlı *et al*. (2009) stepped beam systems using artificial neural networks. Investigated nonlinear vibrations of curved Euler-Bernoulli beams carrying arbitrarily placed concentrated masses. Tekin *et al*. (2009) studied on three-to one resonance in multi stepped beam systems. Bağdatlı *et al*. (2011) dynamics of axially accelerating beams with an intermediate support. Ozkaya *et al*. (2011) nonlinear vibrations and 3:1 internal resonances on multiple supports were investigated and excitation frequency-frequency response curves drawn for different support numbers. Chen *et al*. (2009) summarized the latest progresses on nonlinear dynamics for transverse motion of axially moving strings. A uniform governing equation incorporated arbitrary forms of the constitutive law of the string material was presented. Ghayesh (2010) studied parametric vibrations and stability of an axially accelerating string guided by a non-linear elastic foundation, analytically. They presented some numerical simulations to highlight the effects of system parameters on vibration, natural frequencies, frequency-response curves, stability, and bifurcation points of the system. Nguyen and Hong (2010) investigated a robust adaptive boundary control for an axially moving string that showed nonlinear behavior resulting from spatially varying tension. Ghayesh (2011) investigated the forced dynamics of an axially moving viscoelastic beam. The governing equation of motion was obtained via Newton's second law of motion and constitutive relations. Huang *et al*. (2011) analyzed the nonlinear vibration of an axially moving beam subject to periodic lateral force excitations. Attention was paid to the fundamental and subharmonic resonances, since the excitation frequency was close to the first two natural frequencies of the system. Ghayesh *et al*. (2012) examined the sub- and super-critical dynamics of an axially moving beam subjected to a transverse harmonic excitation force for the cases where the system was tuned to a three-to-one internal resonance as well as for the case where it was not. Cetin and Simsek (2011) investigated free vibration of an axially functionally graded (AFG) pile embedded in Winkler-Pasternak elastic foundation within the framework of the Euler-Bernoulli beam theory, and they obtained results that will be a reference with which other researchers can compare their results.

Except a few studies, papers that mentioned above and their references, axial velocity accepted as constant and transverse vibrations are investigated.

In this paper, we investigated axially moving flexible beams with multiple supports. Axial velocity is accepted as harmonically varying about a mean velocity. The beam effects are assumed to be small. Since, in this case, the fourth order spatial derivative multiplies a small parameter; the mathematical model becomes a boundary layer type of problem. We suggested new solutions for inner expansion and eliminate errors in older studies. Obtained solutions are approximately satisfied all boundary conditions and natural frequency values approximately satisfied the exact values for small beam effects. Effects of harmonically varying velocity are analyzed and we showed that velocity can be modeled as constant. Some axially moving continuous media have multiple support like mass producing band. Effects of support number on the solutions and the natural frequency are investigated and discussed in detail.

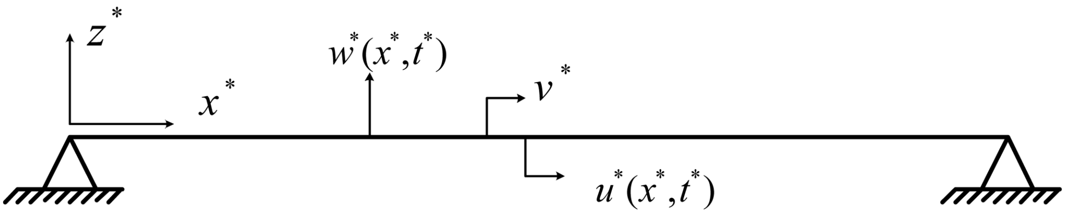

Fig. 1 Schematics of an axially moving beam

2. Equation of motion and approximately solutions

Linear dimensionless equation of motion for axially moving beam with two supports at both ends can be written as (1994).

$$\left(\frac{\partial^2 w}{\partial t^2} + 2\frac{\partial^2 w}{\partial x \partial t} + \frac{\partial w}{\partial x}\frac{dv}{dt}\right) + (v^2 - 1)\frac{\partial^2 w}{\partial x^2} + \bar{v}_f^2\frac{\partial^4 w}{\partial x^4} = 0 \tag{1}$$

where $\bar{v}_f^2$ dimensionless beam parameter, v axial velocity. The dimensionless quantities are defined from the corresponding dimensional ones as follows

$$w = \frac{w^*}{L}, \quad x = \frac{x^*}{L}, \quad t = t^*\sqrt{\frac{P}{\rho A L^2}}, \quad v = \frac{v^*}{\sqrt{P/\rho A}}, \quad \bar{v}_f^2 = \frac{EI}{PL^2} \tag{2}$$

where $(\ )^*$ dimensional parameters, ρ is the flexible beam density, A is the cross-sectional area of the beam, L is the length of the flexible beam, P is the axial tension force, E is the modulus of elasticity, I is the moment of inertia. If EI is small compared to PL^2, $\bar{v}_f^2$ can be chosen as.

$$\bar{v}_f^2 = \varepsilon^2 v_f^2 \tag{3}$$

The velocity is assumed as harmonically varying about a constant mean velocity

$$v = v_0 + \varepsilon v_1 \sin\Omega t \tag{4}$$

where ε is a small parameter. The dimensional velocity variation frequency (Ω^*) is related to the dimensionless one (Ω) through below relation.

$$\Omega = \Omega^*(1/L)\sqrt{P/\rho A} \tag{5}$$

3. Method of matched asymptotic expansions

For solution of the problem, the method of matched asymptotic expansions (MMAE) will be used to construct a uniform expansion valid for all ranges of the spatial variable (1981). Since the equation treated is a partial differential equation and elimination of secularities from the time variable is needed, this method is combined with MMS by introducing two time variables $T_0 = t$ and $T_1 = \varepsilon t$. To obtain valid solution; first, outer expansion solution will be found and then inner solutions are obtained by employing a second expansion near the both ends of the flexible beam. Then the outer and the inner expansion solutions are combined to construct composite solution which approximately satisfying all the boundary conditions. Two and Three supported beam systems are analyzed separately. In two supported case simple-simple and fixed-fixed, in three supported case simple-simple-simple and fixed-simple-fixed supported boundary conditions are investigated.

3.1 Two supported flexible beams

3.1.1 Outer solution

First, an outer solution valid for all ranges of spatial variable except near the both ends of the flexible beam will be constructed. For outer expansion solution we made this expansion

$$w^o = w_0^o + \varepsilon w_1^o + \varepsilon^2 w_2^o + \dots \tag{6}$$

Time derivatives are

$$\frac{d}{dt} = D_0 + \varepsilon D_1, \quad \frac{d^2}{dt^2} = D_0^2 + 2\varepsilon D_0 D_1, \quad t = T_0, \quad \varepsilon t = T_1, \quad \varepsilon^2 t = T_2 \tag{7}$$

Substituting Eqs. (6) and (7) into equation of motion and separating terms of different orders, one obtain

O(1):
$$D_0^2 w_0^o + 2v_0 D_0 w_1^{o\prime} + (v_0^2 - 1) w_2^{o\prime\prime} = 0 \tag{8}$$

O(ε):
$$D_0^2 w_1^o + 2v_0 D_0 w_1^{o\prime} + (v_0^2 - 1) w_1^{o\prime\prime} = -2D_0 D_1 w_0^o - 2v_0 D_1 w_0^{o\prime} \\ -2v_1 \sin\Omega t D_0 w_0^{o\prime} - 2v_0 v_1 \sin\Omega t w_0^{o\prime\prime} \tag{9}$$

O(ε^2):
$$D_0^2 w_2^o + 2v_0 D_0 w_2^{o\prime} + (v_0^2 - 1) w_2^{o\prime\prime} = -2D_0 D_1 w_1^o - 2v_0 D_1 w_1^{o\prime} - 2v_1 \sin\Omega t D_0 w_1^{o\prime} \\ -2v_1 \sin\Omega t D_0 w_0^{o\prime} - 2v_0 v_1 \sin\Omega t w_1^{o\prime\prime} - v_1^2 \sin^2\Omega t w_0^{o\prime\prime} - v_f^2 w_0^{oiv} \tag{10}$$

The solution of order 1 is

$$w_0^o(x, T_0, T_1, T_2) = A_n(T_1, T_2) e^{i\omega_0 T_0} Y_n(x) + c.c. \tag{11}$$

Substituting Eq. (11) into Eq. (8) yields

$$(v_0^2 - 1) Y_n'' + 2iv_0 \omega_n Y_n' - \omega_n^2 Y_n = 0 \tag{12}$$

From Eq. (12) the order 1 solution obtains as

$$w_0^o = A_n e^{i\omega_0 T_0} \left(c_1 e^{-\frac{i\omega_n}{1+v_0} x} + c_2 e^{\frac{i\omega_n}{1-v_0} x} \right) + c.c. \tag{13}$$

Equations that obtained from order 1 represents string problem. Because of this the outer expansion solution can only satisfies two boundary conditions (displacements are zero at support points). But moment and incline conditions for beam problem must satisfy at support points, too. Other boundary conditions will be satisfied by combining the inner expansion solutions and the outer solution. Outer solution conflicts with composite solution far from support regions. Outer solution satisfies string conditions at its valid region and thus we can provide string conditions to outer solution to find solvability conditions.

Boundary conditions for string

$$w(0,t) = 0, \quad w(1,t) = 0 \tag{14}$$

By substituting (14) into (13) order 1 solution becomes

$$w_0^o = A_n e^{i\omega_n^* T_0} e^{\frac{i\omega_n^* v_0}{1-v_0^2}x} \sin\frac{\omega_n^*}{1-v_0^2}x + c.c. \tag{15}$$

where

$$\omega_n^* = n\pi(1-v_0^2) \tag{16}$$

ω_n^* is the natural frequency value that obtained from providing string conditions to outer expansion. Natural frequency of system will be obtained from composite solution that contains inner expansion solutions, too.

Substituting Eq. (15) into order ε, Eq. (9) one has

$$D_0^2 w_1^o + 2v_0 D_0 w_1^{o\prime} + (v_0^2 - 1)w_1^{o\prime\prime} = -2[D_0 D_1 A_n Y_{n_0} + v_0 D_1 A_n Y'_{n_0}]e^{i\omega_n^* T_0} + c.c. + NST \tag{17}$$

Eliminating secular terms, solvability condition obtains

$$D_1 A_n = 0 \tag{18}$$

Hence obtains that A_n parameter has no T_1 dependency.

$$A_n = A_n(T_2) \tag{19}$$

At order ε^2 we'll search only solvability condition to describe A_n parameter and rest terms in Eq. (17) do not bring secular terms to order ε^2. Thus we can take order ε solution as

$$w_1^o = A_n e^{i\omega_n^* T_0} e^{\frac{i\omega_n^* v_0}{1-v_0^2}x} \sin\frac{\omega_n^*}{1-v_0^2}x + c.c. \tag{20}$$

Substituting Eq. (15) and (20) into Eq. (10) and eliminating secular terms, solvability condition obtains for order ε^2

$$D_1 A_n - i\frac{\omega_n^{*3}}{2(1-v_0^2)^3} v_f^2 (v_0^4 + 6v_0^2 + 1) A_n = 0 \tag{21}$$

Solution of Eq. (21) yields

$$A_n = A_n e^{ik_0 T_2} \tag{22}$$

where

$$k_0 = \frac{n^3 \pi^3}{2} v_f^2 (v_0^4 + 6v_0^2 + 1) \tag{23}$$

k_0 parameter in outer expansion solution shows effects of beam characteristic on natural frequency of system. Also beam characteristic will appear in inner expansion solutions and effect on calculating natural frequency. k_0 parameter will be added to natural frequency that obtained from composite solution.

3.1.2 Inner solutions

For each ends of the beam, separate inner solutions should be constructed.

Inner solution at the $x \approx 0$ side

Assuming now an inner expansion of the form

$$w^I(x, t) = y_1(\zeta) y_2(x) e^{i\omega_n t} + c.c. \tag{24}$$

Where, the spatial variable ζ as follows

$$\zeta = \frac{x}{\varepsilon} \tag{25}$$

Substituting Eqs. (24) and (25) into equation of motion and separating terms of different orders, one obtains

$$O(1): \quad \frac{\partial^4 y_1}{\partial \zeta^4} y_2 + \frac{(v_0^2 - 1)}{v_f^2} \frac{\partial^2 y_1}{\partial \zeta^2} y_2 = 0 \tag{26}$$

$$O(\varepsilon): \quad 4v_f^2 \frac{\partial^3 y_1}{\partial \zeta^3} \frac{\partial y_2}{\partial x} + 2(v_0^2 - 1) \frac{\partial y_1}{\partial \zeta} \frac{\partial y_2}{\partial x} + 2v_0 v_1 \sin\Omega t \frac{\partial^2 y_1}{\partial \zeta^2} y_2 + 2iv_0 \omega_n \frac{\partial y_1}{\partial \zeta} y_2 = 0 \tag{27}$$

By solving equations of order1 and order ε, y_1 and y_2 solutions yields

$$y_1 = a_1 e^{-\sqrt{1-v_0^2}\zeta} = a_1 e^{\sqrt{\frac{1-v_0^2}{\varepsilon v_f}} x} \tag{28}$$

$$y_2 = a_2 e^{\frac{-i\omega_n v_0 + v_0 v_0 \sqrt{1-v_0^2} \sin\Omega t}{1-v_0^2} x} \tag{29}$$

Inner solution at the $x \approx 0$ side obtains as

$$w_1^i = c_3 e^{-\frac{\sqrt{1-v_0^2}x}{\varepsilon v_f}} e^{\frac{-i\omega_n v_0 + v_0 v_1 \sqrt{1-v_0^2}\sin\Omega t}{1-v_0^2}} e^{i\omega_n t} \tag{30}$$

Inner solution at the $x \approx 1$ side;
The spatial variable ζ as follows

$$\zeta = \frac{1-x}{\varepsilon} \tag{31}$$

With using similar process for $x \approx 1$ side, solution for $x \approx 1$ side obtains

$$w_2^i = c_4 e^{-\frac{\sqrt{1-v_0^2}x}{\varepsilon v_f}} e^{\frac{-i\omega_n v_0 - v_0 v_1 \sqrt{1-v_0^2}\sin\Omega t}{1-v_0^2}} e^{i\omega_n t} \tag{32}$$

Combining all solutions, the composite solution, that valid for all ranges of x is

$$Y^c = c_1 e^{-\frac{i\omega_n}{1+v_0}x} + c_2 e^{-\frac{i\omega_n}{1-v_0}x} + c_3 e^{\left(-\frac{\sqrt{1-v_0^2}}{\varepsilon v_f} + \frac{-i\omega_n v_0 + v_0 v_1 \sqrt{1-v_0^2}\sin\Omega t}{1-v_0^2}\right)x} + c_4 e^{\left(\frac{\sqrt{1-v_0^2}}{\varepsilon v_f} + \frac{-i\omega_n v_0 - v_0 v_1 \sqrt{1-v_0^2}\sin\Omega t}{1-v_0^2}\right)x} \tag{33}$$

Natural frequency values can be calculated by considering composite solution and boundary conditions. By adding k_0 natural frequency values obtains for axially moving flexible beam. If inner expansion solutions are not made near supports, natural frequency values will be same for each boundary conditions. This situation is not realistic.

Figs. 2-5 shows natural frequencies for flexible beams that studied. Frequency values for different

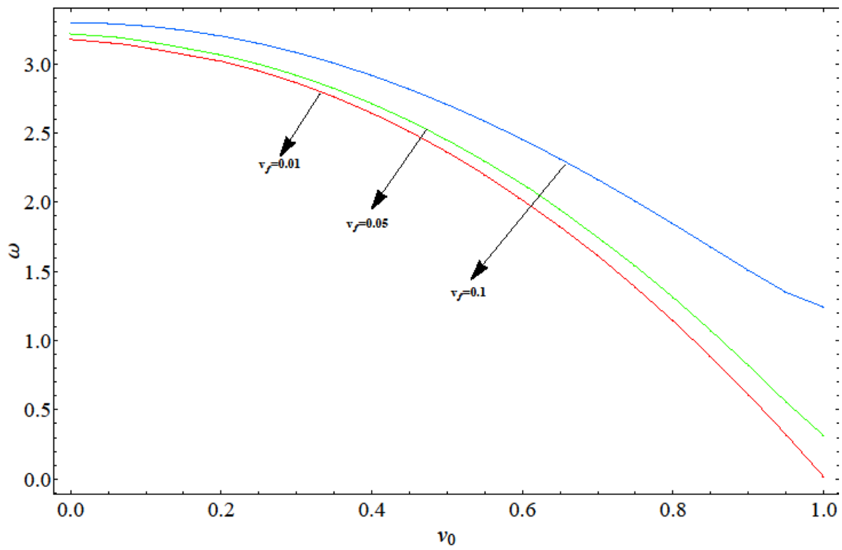

Fig. 2 Comparison of first mode mean velocity-dependent natural frequency changes for simple-simple supported flexible beam

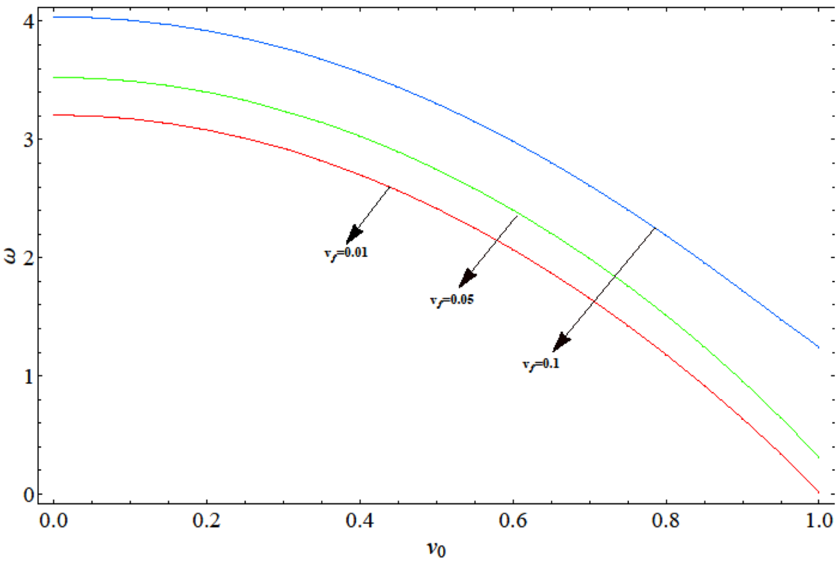

Fig. 3 Comparison of first mode mean velocity-dependent natural frequency changes for fixed-fixed supported flexible beam

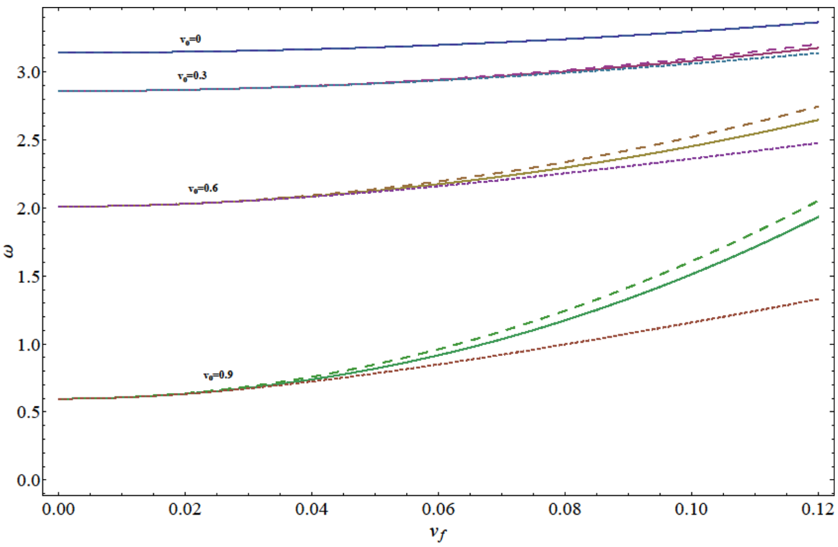

Fig. 4 Comparison of our solutions with solutions in Parker *et al*. (2004). (---) Solutions of Kong and Parker; (—) our solution; (…) exact solution (first mode natural frequency and constant velocity for simple-simple supported flexible beam)

transverse rigidity versus mean velocity values are investigated in Figs. 2 and 3. Frequency value; inversely with mean velocity and directly proportion with beam parameter. This situation is projected. Flexible beam approaches beam behavior with increasing transversal rigidity. Natural

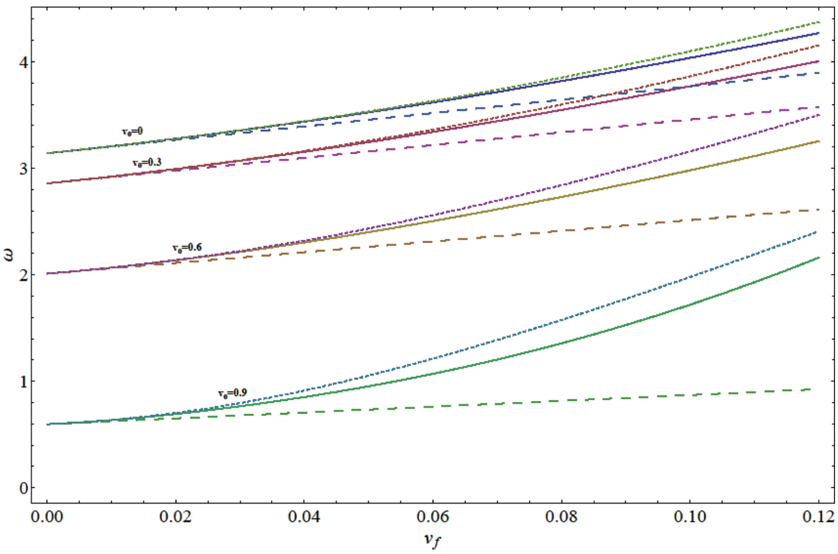

Fig. 5 Comparison of our solutions with solutions in Parker *et al*. (2004). (---) Solutions of Kong and Parker; (—) our solution; (…) exact solution (first mode natural frequency and constant velocity for fixed-fixed supported flexible beam)

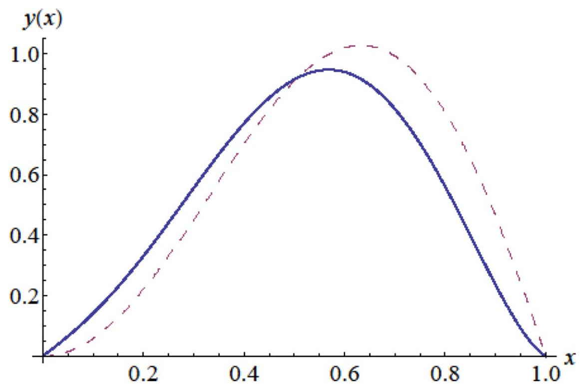

Fig. 6 Comparison of first mode, deflection curves for outer expansion solution and composite solution for simple-simple supported flexible beam. (---) outer expansion solution; (—) composite solution ($v_0 = 0.3$, $\Omega = 5$, $t = 0.3$, $v_1 = 0.1$, $v_f = 0.1$)

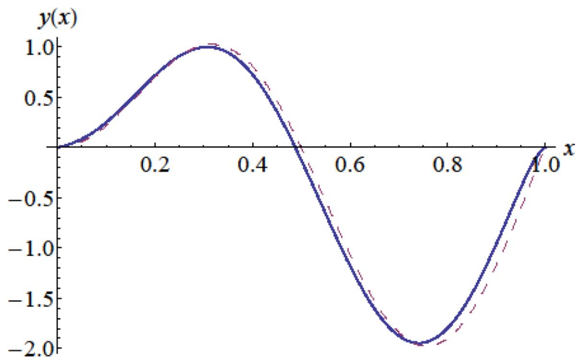

Fig. 7 Comparison of second mode, deflection curves for outer expansion solution and composite solution for simple-simple supported flexible beam. (---) outer expansion solution; (—) composite solution ($v_f = 0.025$, $v_0 = 0.3$, $\Omega = 5$, $t = 0.3$, $v_1 = 0.1$)

frequency values increased by using fixed supported beam instead of simply supported. Our solutions compared with Kong and Parker's (2004) solutions for constant velocity in Figs. 4 and 5. From this graphics we can see that our solutions have better approximation than other solution. Especially our solution has high achievement for fixed-fixed supported system.

The first mode shape graphic compares outer expansion solution and composite solution at simple-simple supported system in Fig. 6. Both solutions satisfy conditions at boundary points. But

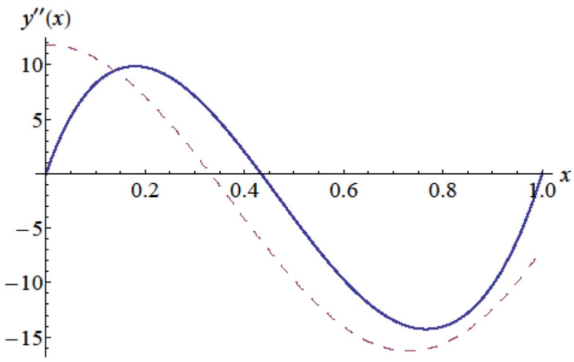

Fig. 8 Comparison of first mode, moment curves for outer expansion solution and composite solution for simple-simple supported flexible beam. (---) outer expansion solution; (—) composite solution ($v_0 = 0.3$, $\Omega = 5$, $t = 0.3$, $v_1 = 0.1$, $v_f = 0.1$)

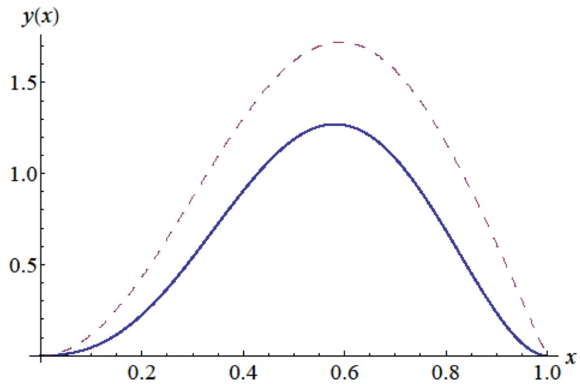

Fig. 9 Comparison of first mode, deflection curves for outer expansion solution and composite solution for fixed-fixed supported flexible beam. (---) outer expansion solution; (—) composite solution ($v_0 = 0.6$, $\Omega = 5$, $t = 0.3$, $v_1 = 0.1$, $v_f = 0.1$)

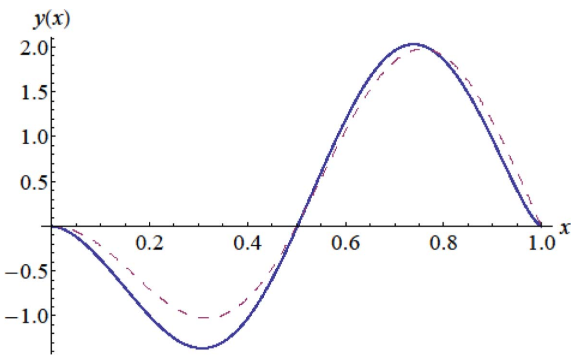

Fig. 10 Comparison of second mode, deflection curves for outer expansion solution and composite solution for fixed-fixed supported flexible beam. (---) outer expansion solution; (—) composite solution ($v_f = 0.025$, $v_0 = 0.3$, $\Omega = 5$, $t = 0.3$, $v_1 = 0.1$)

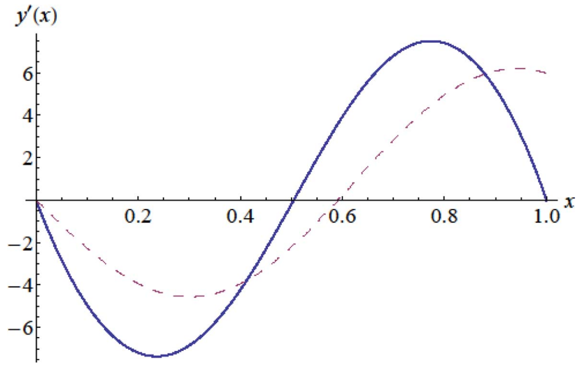

Fig. 11 Comparison of first mode, incline curves for outer expansion solution and composite solution for simple-simple supported flexible beam. (---) outer expansion solution; (—) composite solution ($v_f = 0.095$, $v_0 = 0.6$, $\Omega = 5$, $t = 0.3$, $v_1 = 0.1$)

there are differences at other regions of system. The second mode shape graphic showed in Fig. 7. Moment variation graphic for outer expansion solution and composite solution at same supported system given in Fig. 8. While composite solutions satisfy moment conditions at the support points, outer expansion solution is not.

Displacement variation graphic for outer expansion solution and composite solution at fixed-fixed supported systems plotted in Figs. 9 (for first mode) and 10 (for second mode). And again both solutions satisfy conditions at boundary points and there are differences at other regions of system. These differences are larger than simple-simple supported case. Incline variation graphic for outer expansion solution and composite solution at same supported system given in Fig. 11. Composite solutions satisfy incline conditions at the each support points. Outer expansion solution satisfies incline conditions at left hand side, but at right hand has big difference.

3.2 Three supported flexible beams

Our solutions will be investigated separately for left and right regions of the middle support. Boundary conditions for outer expansion assume as follows

$$w_1(0,t) = 0, \quad w_1(\eta,t) = 0, \quad w_2(1,t) = 0, \quad w_2(\eta,t) = 0 \tag{34}$$

η is dimensionless distance of middle support from left support. If we repeat same process with two supported systems under these conditions, we can obtain the correction term for frequency. Correction term for left hand side found as

$$k_{0_I} = \frac{n^3\pi^3}{2\eta^3} v_f^2 (v_0^4 + 6v_0^2 + 1) \tag{35}$$

And for right hand side obtains as

$$k_{0_{II}} = \frac{n^3\pi^3}{2(1-\eta)^3} v_f^2 (v_0^4 + 6v_0^2 + 1) \tag{36}$$

Eqs. (35) and (36) are same at the middle support where $x = 0.5$. We have two correction terms. Both correction terms can be used at $x = 0.5$ point. While middle support is near the right hand side k_{0_I} and while middle support is near left hand side $k_{0_{II}}$ will used for calculating the first natural frequency. We must determine occurring region of mode to calculate the second and higher modes of natural frequency values and consider k_0 value for that region. Outer expansion solutions obtain as

$$w_1^0 = c_1 e^{-\frac{i\omega_n}{1+v_0}x} + c_2 e^{\frac{i\omega_n}{1-v_0}x} \quad ve \quad w_2^0 = c_5 e^{-\frac{i\omega_n}{1+v_0}x} + c_6 e^{\frac{i\omega_n}{1-v_0}x} \tag{37}$$

Inner solutions are also investigated separately for left and right parts of the system and the spatial variable ζ described for each parts as follows;

For left hand side

$$x \approx 0 \Rightarrow \zeta = \frac{x}{\varepsilon} \tag{38}$$

$$x \approx \eta \Rightarrow \zeta = \frac{\eta - x}{\varepsilon} \tag{39}$$

for right hand side

$$x \approx \eta \Rightarrow \zeta = \frac{x - \eta}{\varepsilon} \tag{40}$$

$$x \approx 1 \Rightarrow \zeta = \frac{1 - x}{\varepsilon} \tag{41}$$

Thus inner expansion solutions obtains as

$$Y_1^i = c_3 e^{\left(-\frac{\sqrt{1-v_0^2}}{\varepsilon v_f}+\frac{-i\omega_n v_0+v_0 v_1\sqrt{1-v_0^2}\sin\Omega t}{1-v_0^2}\right)x} + c_4 e^{\left(\frac{\sqrt{1-v_0^2}}{\varepsilon v_f}+\frac{-i\omega_n v_0-v_0 v_1\sqrt{1-v_0^2}\sin\Omega t}{1-v_0^2}\right)x} \tag{42}$$

$$Y_2^i = c_5 e^{\left(-\frac{\sqrt{1-v_0^2}}{\varepsilon v_f}+\frac{-i\omega_n v_0+v_0 v_1\sqrt{1-v_0^2}\sin\Omega t}{1-v_0^2}\right)x} + c_6 e^{\left(\frac{\sqrt{1-v_0^2}}{\varepsilon v_f}+\frac{-i\omega_n v_0-v_0 v_1\sqrt{1-v_0^2}\sin\Omega t}{1-v_0^2}\right)x} \tag{43}$$

Only difference between left and right parts of the system is "c" parameters as seen from Eqs. (42) and (43). Composite solutions obtain with considering of this information as

$$Y_1^c = c_1 e^{r_1 x} + c_2 e^{r_2 x} + c_3 e^{r_3 x} + c_4 e^{r_4 x} \tag{44}$$

$$Y_2^c = c_5 e^{r_1 x} + c_6 e^{r_2 x} + c_7 e^{r_3 x} + c_8 e^{r_4 x} \tag{45}$$

where

$$r_1 = -\frac{i\omega_n}{1+v_0}, \quad r_2 = \frac{i\omega_n}{1-v_0}, \quad r_3 = -\frac{\sqrt{1-v_0^2}}{\varepsilon v_f}+\frac{-i\omega_n v_0 - v_0 v_1\sqrt{1-v_0^2}\sin\Omega t}{1-v_0^2}$$

$$r_4 = \frac{\sqrt{1-v_0^2}}{\varepsilon v_f}+\frac{-i\omega_n v_0 - v_0 v_1\sqrt{1-v_0^2}\sin\Omega t}{1-v_0^2} \tag{46}$$

If v_1 is taken as "0", obtained solution yields axially moving flexible beam with constant velocity. In Fig. 12 time-dependent natural frequency changes are investigated. Amplitude of frequency directly proportion with v_1 value. Amplitude of fluctuation is small for small values of v_1. v_1 selected as small ($v_1 = 0{,}1$) in handled study. Thus amplitude of frequency changes is negligible size

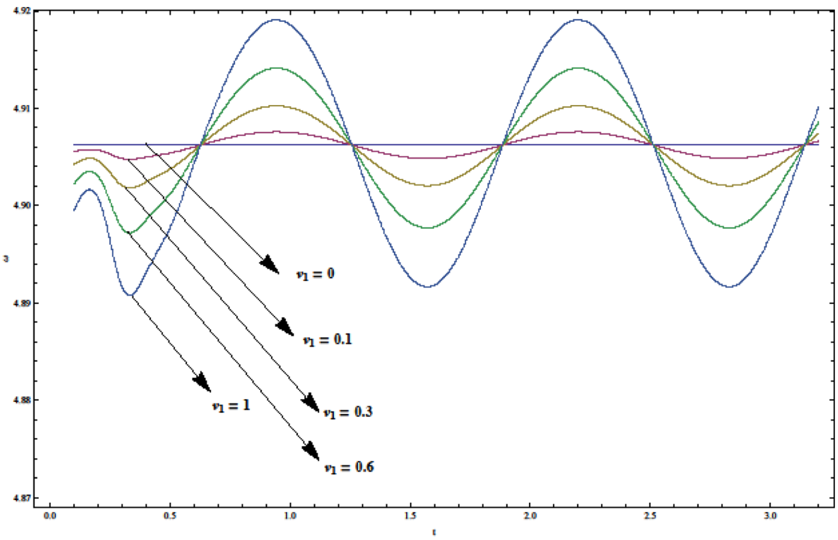

Fig. 12 Comparison of first mode time-dependent natural frequency changes for simple-simple-simple supported flexible beam ($v_0 = 0.6$, $\Omega = 5$, $v_f = 0.05$, $\eta = 0.5$)

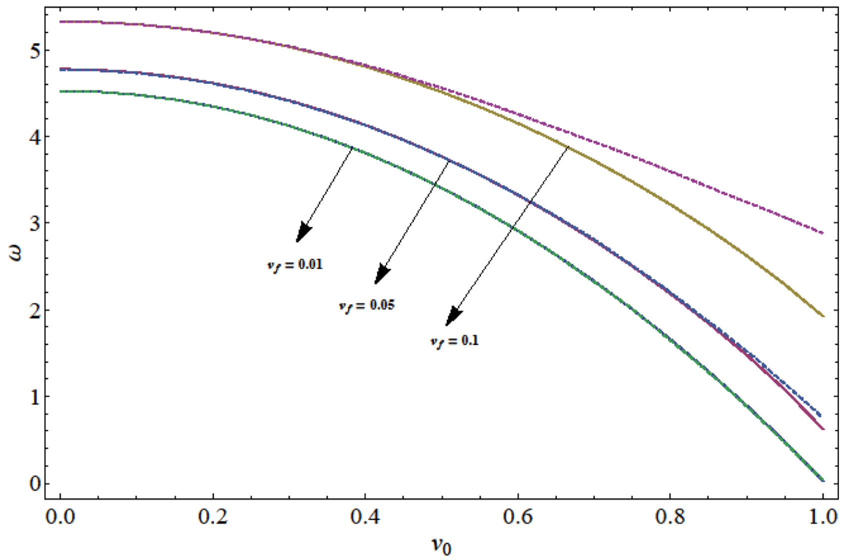

Fig. 13 Comparison of first mode mean velocity-dependent natural frequency changes in composite solution and exact solution for fixed-simple-fixed supported flexible beam (---) composite solution, (—) exact solution

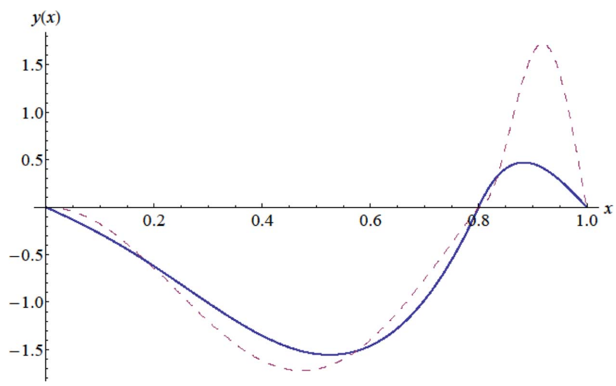

Fig. 14 Comparison of first mode deflection curves for outer expansion solution and composite solution for simple-simple-simple supported flexible beam. (---) outer expansion solution; (—) composite solution ($v_f = 0.12$, $v_0 = 0.6$, $\eta = 0.8$, $\Omega = 5$, $t = 0.3$, $v_1 = 0.1$)

and we can assume that frequency changes independent from time for our system. Similar graphics are obtained for three supported cases in Figs. 13-18. The first mode mean velocity-dependent on natural frequency changes in composite solution versus exact solution compared in Fig. 13 for S-S-S (simple-simple-simple) supported beam. These solutions coincide with exact solution for small values of beam parameter but they separated with increasing beam parameter. This case is natural, because we obtained solutions for small beam parameter. Additionally our solutions approximate to exact values for small axial velocity in high beam parameter situation. First and second mode deflection curves plotted for outer expansion solution and composite solution for S-S-S supported

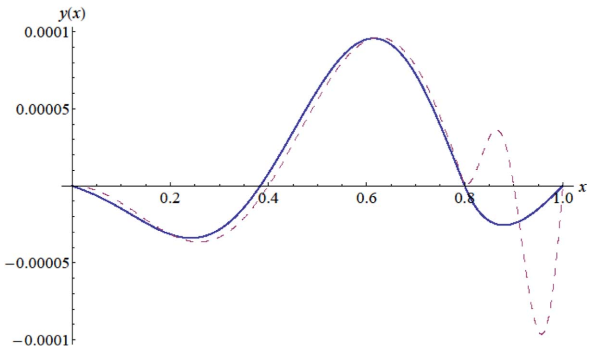

Fig. 15 Comparison of second mode deflection curves for outer expansion solution and composite solution for simple-simple-simple supported flexible beam. (---) outer expansion solution; (—) composite solution ($v_f = 0.055$, $v_0 = 0$, $\eta = 0.8$, $\Omega = 5$, $t = 0.3$, $v_1 = 0.1$)

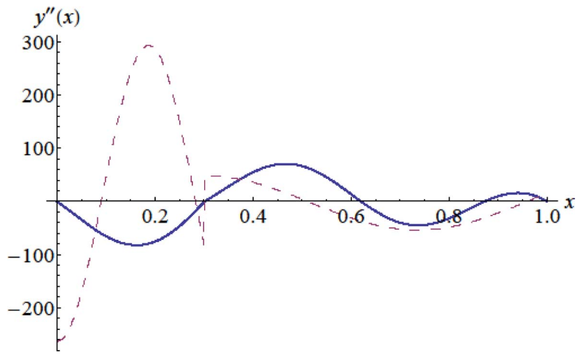

Fig. 16 Comparison of first mode moment curves for outer expansion solution and composite solution for simple-simple-simple supported flexible beam. (---) outer expansion solution; (—) composite solution ($v_f = 0.12$, $v_0 = 0.6$, $\eta = 0.3$, $\Omega = 5$, $t = 0.3$, $v_1 = 0.1$)

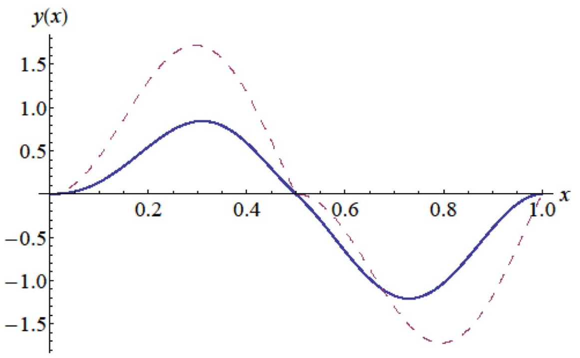

Fig. 17 Comparison of first mode deflection curves for outer expansion solution and composite solution for fixed-simple-fixed supported flexible beam. (---) outer expansion solution; (—) composite solution ($v_f = 0.1$, $v_0 = 0.6$, $\eta = 0.5$, $\Omega = 5$, $t = 0.3$, $v_1 = 0.1$)

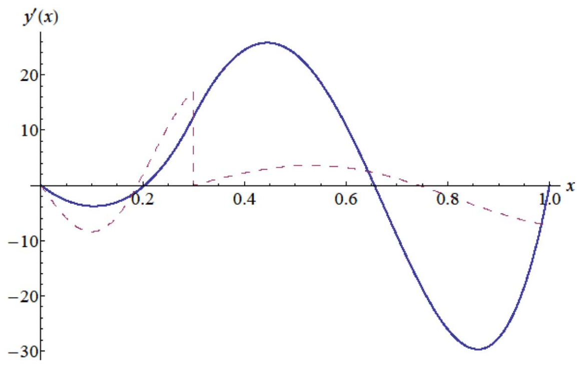

Fig. 18 Comparisons of first mode incline curves for outer expansion solution and composite solution for fixed-simple-fixed supported flexible beam. (---) outer expansion solution; (—) composite solution ($v_f = 0.1$, $v_0 = 0.3$, $\eta = 0.3$, $\Omega = 5$, $t = 0.3$, $v_1 = 0.1$)

flexible beam in Fig. 14 and Fig. 15. Both solutions satisfy boundary conditions but there are big constructive differences between these solutions. Composite solution has a shape that much compatible with physical situation. Moment variations showed in Fig. 16 for same supported system. While composite solutions satisfy moment conditions at the support points, outer expansion solution is not. Deflection and incline variations for fixed-simple-fixed supported flexible beam plotted in Fig. 17 and Fig. 18. While deflection conditions satisfied by all solutions, incline conditions only satisfied by composite solution.

4. Conclusions

Axially moving beam equation becomes fair enough to flexible beam by assuming small flexural rigidity. Outer expansion solution is obtained by the method of multiple scales. It is observed that this outer expansion solution does not satisfy the boundary conditions for moment at simple-simple and incline at fixed-fixed supported cases. In order to eliminate this problem, inner expansion solution is obtained by employing a second expansion near the both ends of the flexible beam. Then outer and inner expansion solutions are combined to obtain composite solutions approximately satisfying all the boundary conditions. At first dealt with axially moving two supported systems with constant velocity and obtained solutions for simple-simple and fixed-fixed supported cases. And then three supported systems are investigated and solutions produce for simple-simple-simple and fixed-simple-fixed supported cases. Effects of axial speed and flexural rigidity on first and second natural frequency of system for every condition are discussed.

It's clearly seen that an outer expansion solution does not coincide with composite solutions from obtained graphics. This situation becomes apparent by increasing transverse rigidity. Because, outer expansion solution only contains order 1 solution, have not order e terms. Therefore composite solution has much sensitive results than outer expansion solution for high values of transverse rigidity. For all situations that studied, an outer expansion solution does not satisfy boundary conditions in comparing incline and moment variations. Composite solution is obtained for eliminating this problem and the results show that composite solution satisfies all boundary conditions. Increases of natural frequency values with increasing transverse rigidity value at fixed-fixed supported case are more than simply supported cases. And again amplitude values also higher at fixed-fixed supported cases. Other than this if we investigated displacement variation graphics, we can see maximum amplitudes increase with increasing of mean velocity.

At the end of study we obtained different frequency values for different boundary conditions. We investigated effects of variable velocity and showed that assuming velocity as constant is acceptable. Frequency values, mode shape, incline and moment values are obtained with enough sensibility for most boundary conditions.

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