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Free vibration analysis of non-prismatic beams under variable axial forces

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Abstract. Despite popularity of FEM in analysis of static and dynamic structural problems and the routine applicability of FE softwares, analytical methods based on simple mathematical relations is still largely sought by many researchers and practicing engineers around the world. Development of such analytical methods for analysis of free vibration of non-prismatic beams is also of primary concern. In this paper a new and simple method is proposed for determination of vibration frequencies of non-prismatic beams under variable axial forces. The governing differential equation is first obtained and, according to a harmonic vibration, is converted into a single variable equation in terms of location. Through repetitive integrations, integral equation for the weak form of governing equation is derived. The integration constants are determined using the boundary conditions applied to the problem. The mode shape functions are approximated by a power series. Substitution of the power series into the integral equation transforms it into a system of linear algebraic equations. Natural frequencies are determined using a non-trivial solution for system of equations. Presented method is formulated for beams having various end conditions and is extended for determination of the buckling load of non-prismatic beams. The efficiency and convergence rate of the current approach are investigated through comparison of the numerical results obtained to those obtained using available finite element software.

Keywords: vibration frequency; variable axial force; non-prismatic beam; buckling load; weak form integral equation

1. Introduction

Behavior of many natural phenomena can be mathematically modeled using variety of ordinary differential or partial differential equations. For example, the non-prismatic elastic beams vibration is formulated by partial differential equations associated with variable coefficients. According to the history of structural dynamics, Timoshenko and Bernoulli beams theory has been proposed for characterization of elastic beams vibration. If beam section dimensions are small with respect to its length, behavior is best predicted by the assumption that is governed by forces envisaged in the Bernoulli beam theory. In this theory, the effects of shear deformation and rotational inertia in the governing differential equation are not considered. However, if the beam's dimensions are

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considerably large, Timoshenko beam theory yields the best predictive results. In the Timoshenko theory, the effects of shear deformation and rotational inertia are taken into account (shooshtari *et al*. 2010, Yavari *et al.* 2001, Antes 2003, Li 2008).

The subject of free vibration of non-prismatic beams has been paid attention by many researchers. However, few of them have considered effects of axial forces on beam vibration frequencies. Most such studies include simplifications in calculation process such that the governing differential equation becomes manageable to solve. For example, in references (Li *et al.* 2000, Li 2001), the exponential or power functions are used for distributions of bending stiffness, mass per unit length and axial forces along the beam length.

Li *et al.* (2000) investigated free vibration of cantilevered tall structures under various axial loads. They estimated behavior of such structures as a cantilever beam and reduced the governing differential equation to Bessel's equations. Elfelsoufi *et al.* (2005) undertaken an investigation into buckling, flutter and vibration analysis of the non-prismatic Bernoulli beam on elastic foundation subjected to lateral excitation. Caruntu (2009) investigated dynamic modal characteristics of transverse vibrations of cantilevers of parabolic thickness. Fourth order governing differential equations of transverse vibrations was factored to reach a pair of second order differential equations which led to general solutions in terms of hyper geometric functions. Rahai *et al.* (2008) formulated a procedure for the buckling analysis of tapered column members. They calculated the buckling loads using modified vibrational mode shape and energy method. Pan *et al.* (2011) introduced a new perturbation method for determination of natural frequencies and vibration modes of a non-prismatic Timoshenko beam. They used the natural modes of vibration of its corresponding prismatic Euler-Bernoulli beam with the same length and boundary conditions as Ritz base functions with necessary modifications to account for shear strain in the Timoshenko beam. Bahadir Yuksel (2012) investigated the behavior of non-prismatic beams with symmetrical parabolic haunches (NBSPH) having the constant haunch length ratio of 0.5 using finite element analyses (FEA). Kaviani *et al.* (2008) developed an approximate method for determination of the natural periods of multistory buildings. They reduced the governing differential equations to Bessel's equations. The resulting frequencies were finally combined using an approximate method. Huang *et al.* (2010) presented an approach to solve natural frequencies of freely vibrating beams having variable flexural rigidity and mass density. They transformed the governing equation of varying coefficients to Fredholm integral equation.

The effects of axial force on the vibration frequencies and mode shapes of non-prismatic beams have been paid less attention by investigators (Arboleda-Monsalve *et al.* 2007, Elfelsoufi *et al.* 2006). Since the compressive axial force reduces the beam stiffness, it appears necessary to develop new and simple approaches for analysis and calculation of frequencies as well as mode shapes of beams under axial forces. Application of the FEM in such analysis may appear a common solution since the use of FE softwares in analysis of various structural problems has been increased in recent decades by practicing engineers or even researchers all around the world. However, many other still prefer use of simple mathematical relations, if any exists, for solution of such problems and numerous interested academics are still in the search for developments of faster, simpler, more accurate, and reliable analytical solution procedures.

In this paper, a new and simplified method is presented in which the effect of axial force on the vibration frequencies of non-prismatic beams is effectively taken into account through simple mathematical relationships. The governing differential equation for free vibration of a non-prismatic Bernoulli beam under variable axial forces is derived first. According to harmonic vibration principles, the derived equation is converted into a single variable equation in terms of location.

Through repetitive integrations, the integral equation of the weak form for the governing equation is obtained. Boundary conditions are applied next, and the integration constants are determined. Mode shape functions of the vibration are approximated using a power series. Substitution of the power series into the integral equation results in a system of linear algebraic equations. Natural frequencies are determined through calculation of a non-trivial solution for system of equations. Formulations are further provided for beams having various end supports. The proposed method is also extended for determination of buckling load of non-prismatic beams.

2. Non-prismatic beam vibration analysis under variable axial force

2.1 Conversion of the governing differential equation to its weak form

Neglecting damping terms, the governing differential equation for vibration of a non-prismatic Bernoulli beam under variable axial forces and transverse forces (Fig. 1) is given by (Clough and Penzien 1975)

$$\frac{\partial^2}{\partial x^2}\left[D(x)\frac{\partial^2}{\partial x^2}\vartheta(x,t)\right]+m(x)\frac{\partial^2}{\partial t^2}\vartheta(x,t)+\frac{\partial}{\partial x}\left[N(x)\frac{\partial}{\partial x}\vartheta(x,t)\right]=P(x,t) \tag{1}$$

In which $D(x) = EI(x)$ is bending stiffness which depends on both young's modulus E and the inertial moment of cross-sectional area $I(x)$. $N(x), m(x), \vartheta(x,t)$ and $P(x,t)$ are the axial force, the mass per unit length, transverse displacement and transverse force respectively. Axial force includes a concentrated axial force at free end of the beam and variably distributed axial force. Setting

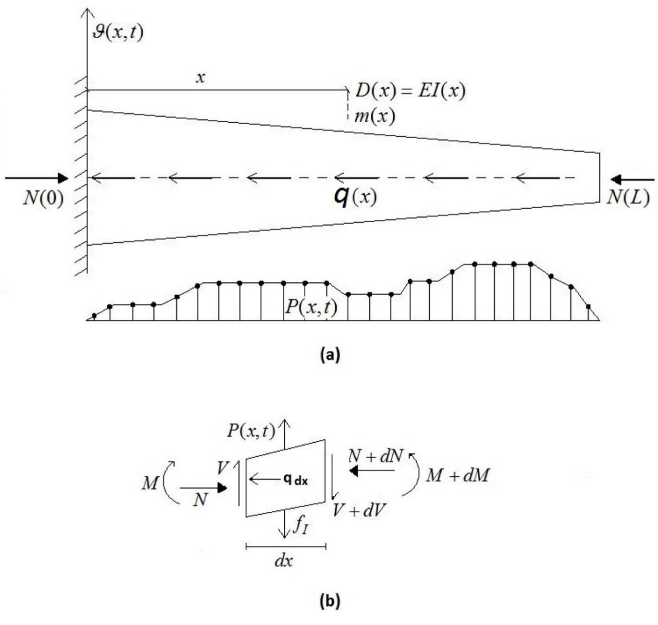

Fig. 1 (a) Bernoulli beam vibration under variable axial forces and transverse forces, (b) resultant forces acting on a differential element

$p(x,t) = 0$, the free vibration equation is obtained. If motion is represented by a harmonic vibration, the transverse displacement is obtained using the following relation

$$\vartheta(x,t) = \phi(x)e^{i\omega t} \tag{2}$$

Where $\phi(x)$ and ω are mode shape function and circular natural frequency of the beam, respectively. Substitution of relationship (2) into Eq. (1) leads to a single-variable equation in terms of location, as follows

$$\frac{d^2}{dx^2}\left[D(x)\frac{d^2\phi}{dx^2}\right] - \omega^2 m(x)\phi(x) + \frac{d}{dx}\left[N(x)\frac{d\phi}{dx}\right] = 0 \qquad 0 \le x \le L \tag{3}$$

In which L is the beam length. For further convenience, the following variables are introduced

$$\xi = \frac{x}{L}, \qquad k = \omega^2 L^4 \tag{4}$$

Substituting variables (4) into Eq. (3) leads to

$$\frac{d^2}{d\xi^2}\left[D(\xi)\frac{d^2\phi}{d\xi^2}\right] - km(\xi)\phi(\xi) + L^2\frac{d}{d\xi}\left[N(\xi)\frac{d\phi}{d\xi}\right] = 0, \qquad 0 \le \xi \le 1 \tag{5}$$

Eq. (5) is, in fact, the free vibration equation of a non-prismatic Bernoulli beam under variable axial forces based on the non-dimensional variable ξ. In order to trnsform Eq. (5) to its weak form, both sides of Eq. (5) are integrated twice with respect to ξ within the range 0 to ξ. The resulting integral equations are as follows

$$\frac{d}{d\xi}\left[D(\xi)\frac{d^2\phi}{d\xi^2}\right] - k\int_0^{\xi} m(s)\phi(s)ds + L^2 N(\xi)\frac{d\phi}{d\xi} = C_1 \tag{6}$$

$$D(\xi)\frac{d^2\phi}{d\xi^2} + L^2 N(\xi)\phi(\xi) - \int_0^{\xi}\left[k(\xi - s)m(s) + L^2 N'(s)\right]\phi(s)ds = C_1\xi + C_2 \tag{7}$$

Further, integration from both sides of Eq. (7) twice with respect to ξ from 0 to ξ yields

$$D(\xi)\frac{d\phi}{d\xi} - D'(\xi)\phi(\xi) + \int_0^{\xi}\left[D''(s) - \frac{k}{2}(\xi - s)^2 m(s) + L^2 N(s) - L^2(\xi - s)N'(s)\right]\phi(s)ds$$
$$= \frac{C_1}{2}\xi^2 + C_2\xi + C_3 \tag{8}$$

$$D(\xi)\phi(\xi) + \int_0^{\xi}\left[(\xi - s)D''(s) - \frac{k}{6}(\xi - s)^3 m(s) - 2D'(s) + L^2(\xi - s)N(s) - \frac{L^2}{2}(\xi - s)^2 N'(s)\right]\phi(s)ds$$
$$= \frac{C_1}{6}\xi^3 + \frac{C_2}{2}\xi^2 + C_3\xi + C_4 \tag{9}$$

In Eq. (9) C_1, C_2, C_3 and C_4 are the integration constants which are determined through boundary conditions of both ends of the beam. Eq. (9) is the integral equation of the weak form for free vibration of a non-prismatic Bernoulli beam under variable axial force. As can be seen, Eq. (5) includes a fourth order derivative of the mode shape function, (after four successive integrations) and only the mode shape function itself. Eqs. (6)-(9) are applicable for various end supports to determine the integration constants. Further substitution of the resulting integration constants into Eq. (9) yields an integral equation in $\phi(\xi)$.

2.2 Boundary conditions

For a non-prismatic beam under variable axial force, including a concentrated axial force at the end of the beam and a variably distributed axial force, the beam rotation (θ), the bending moment (M) and the shear force (V) can be stated as by the following relations

$$\begin{cases} \theta(x,t) = \dfrac{\partial}{\partial x}\vartheta(x,t) \\ M(x,t) = D(x)\dfrac{\partial^2}{\partial x^2}\vartheta(x,t) + N(L)\left[\vartheta(L,t) - \vartheta(x,t)\right] + \displaystyle\int_x^L q(\eta)\left[\vartheta(\eta,t) - \vartheta(x,t)\right]d\eta \\ V(x,t) = \dfrac{\partial}{\partial x}\left[D(x)\dfrac{\partial^2}{\partial x^2}\vartheta(x,t)\right] + N(x)\dfrac{\partial}{\partial x}\vartheta(x,t) \end{cases} \tag{10}$$

In which $q(x)$ is the axial force per unit length. Regarding the relationship (2) and variables introduced in Eq. (4), the relations (10) can be expressed as follows

$$\begin{cases} \theta(\xi,t) = \dfrac{d}{Ld\xi}\phi(\xi)e^{i\omega t} \\ M(\xi,t) = \left[\dfrac{D(\xi)}{L^2}\dfrac{d^2}{d\xi^2}\phi(\xi) + N(1)\left[\phi(1) - \phi(\xi)\right] + L\displaystyle\int_\xi^1 q(\eta)\left[\phi(\eta) - \phi(\xi)\right]d\eta\right]e^{i\omega t} \\ V(\xi,t) = \left[\dfrac{d}{L^3 d\xi}\left(D(\xi)\dfrac{d^2}{d\xi^2}\phi(\xi)\right) + \dfrac{N(\xi)}{L}\dfrac{d}{d\xi}\phi(\xi)\right]e^{i\omega t} \end{cases} \tag{11}$$

2.2.1 Cantilever beam (C-F)

For a cantilever beam (clamped-free) the boundary conditions are established as follows

$$\begin{cases} \xi = 0, \quad \vartheta = 0, \quad or \quad \phi = 0 \\ \xi = 0, \quad \theta = 0 \quad or \quad \left[\dfrac{d}{Ld\xi}\phi(\xi)\right]_{\xi=0} = 0 \\ \xi = 1, \quad V = 0 \quad or \quad \left[\dfrac{d}{L^3 d\xi}\left(D(\xi)\dfrac{d^2}{d\xi^2}\phi(\xi)\right) + \dfrac{N(\xi)}{L}\dfrac{d}{d\xi}\phi(\xi)\right]_{\xi=1} = 0 \\ \xi = 1, \quad M = 0 \quad or \quad \left[\dfrac{D(\xi)}{L^2}\dfrac{d^2}{d\xi^2}\phi(\xi)\right]_{\xi=1} = 0 \end{cases} \tag{12}$$

Substituting $\phi = 0$ into (9) as well as $\phi = \theta = 0$ into (8) and setting $\xi = 0$ leads to

$$C_3 = C_4 = 0 \tag{13}$$

Similarly, Substitution of $V = 0$ into (6) and $M = 0$ into (7) as well as setting $\xi = 1$, yields

$$-k\int_0^1 m(s)\phi(s)\,ds = C_1 \tag{14}$$

$$L^2N(1)\phi(1) - \int_0^1 \left[k(1-s)m(s) + L^2N'(s)\right]\phi(s)ds = C_1 + C_2 \tag{15}$$

As can be seen in Eq. (15), $\phi(1)$ is initially unknown. Therefore, it necessitates extra equation for uniquely determination of C_1 and C_2. Setting $\xi = 1$ in Eq. (9) yields

$$6D(1)\phi(1) + \int_0^1 \left[6(1-s)D''(s) - k(1-s)^3m(s) - 12D'(s) + 6L^2(1-s)N(s) - 3L^2(1-s)^2N'(s)\right]\phi(s)ds$$
$$= C_1 + 3C_2 \tag{16}$$

Elimination of $\phi(1)$ from Eqs. (15) and (16) as well as using Eq. (14), results in the coefficient C_2 to be determined by the following relation

$$C_2 = \left(\frac{1}{6D(1) - 3L^2N(1)}\right)\int_0^1 \left[6kD(1)sm(s) - 6L^2D(1)N'(s) - kL^2N(1)m(s) - 6L^2N(1)g(s)\right]\phi(s)ds \tag{17}$$

In which

$$g(s) = (1-s)D''(s) - \frac{k}{6}(1-s)^3m(s) - 2D'(s) + L^2(1-s)N(s) - \frac{L^2}{2}(1-s)^2N'(s) \tag{18}$$

Substitution of the integration constants into (9) yields an integral equation as follows

$$D(\xi)\phi(\xi) + \int_0^\xi f_1(\xi,s)\phi(s)ds + \int_0^1 f_2(\xi,s)\phi(s)ds = 0 \tag{19}$$

In Eq. (19), f_1 & f_2 are non-dimensional functions which are calculated for the beam depending on its end condition. Functions $f_1(\xi,s)$ and $f_2(\xi,s)$ are expressed by the following relations

$$\begin{cases} f_1(\xi,s) = (\xi-s)D''(s) - \dfrac{k}{6}(\xi-s)^3m(s) - 2D'(s) + L^2(\xi-s)N(s) - \dfrac{L^2}{2}(\xi-s)^2N'(s) \\ f_2(\xi,s) = \dfrac{k}{6}\xi^3m(s) + \dfrac{\left[6kD(1)s\xi^2 - kL^2N(1)\xi^2\right]m(s) - 6L^2D(1)\xi^2N'(s) - 6L^2N(1)\xi^2g(s)}{6L^2N(1) - 12D(1)} \end{cases} \tag{20}$$

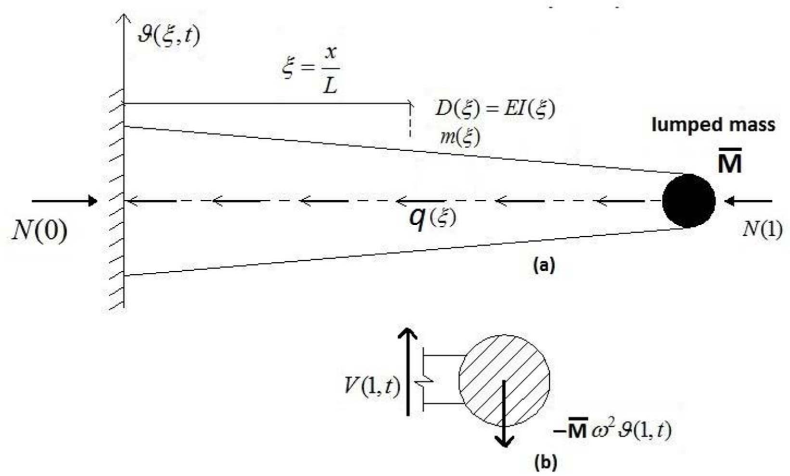

Fig. 2 (a) cantilever beam vibration with a lumped mass at free end, (b) shear force acting on the end mass

2.2.2 Cantilever beam with lumped mass at free end

In this section, it is assumed that there is a lumped mass at the free end of the cantilever beam (Fig. 2(a)). The rotational inertia effects and weight of the lumped mass have been neglected. The analysis method is exactly the same as what was stated in section 2.2.1. The difference, however, is that, the shear force is of non-zero value at the free end of the beam (Fig. 2(b)). Other boundary conditions are assumed unchanged. In this case, the boundary conditions are stated as follows (Clough and Penzien 1975)

$$\begin{cases} \xi=0, \quad \phi=0, \quad \theta=0 \\ \xi=1 \quad M=0, \quad V=-\overline{M}\omega^2\phi(1) \end{cases} \tag{21}$$

In which $\overline{M}$ is the lumped mass at free end of the beam. If the boundary conditions introduced in (21) are applied to Eqs. (6)-(9), the integration constants are determined. Introducing the integration constants into Eq. (9) yields the integral equation as follows

$$D(\xi)\phi(\xi)+\int_0^\xi f_1(\xi,s)\phi(s)ds+\int_0^1 f_2(\xi,s)\phi(s)ds=0 \tag{22}$$

In which the functions $f_1(\xi,s)$ and $f_2(\xi,s)$ can be stated as follows

$$\begin{cases} f_1(\xi,s)=(\xi-s)D''(s)-\frac{k}{6}(\xi-s)^3m(s)-2D'(s)+L^2(\xi-s)N(s)-\frac{L^2}{2}(\xi-s)^2N'(s) \\ f_2(\xi,s)=\left[-\frac{\lambda_1}{6}\xi^3-\frac{\lambda_3}{2}\xi^2\right]g_2(s)-\xi^2g_1(s)-\frac{\lambda_2}{2}\xi^2m(s) \end{cases} \tag{23}$$

Where

$$\begin{cases} g_1(s) = (1-s)D''(s) - \dfrac{k}{6}(1-s)^3 m(s) - 2D'(s) + L^2(1-s)N(s) - \dfrac{L^2}{2}(1-s)^2 N'(s) \\ g_2(s) = \left(\dfrac{3L^3 N(1) - 6LD(1)}{\overline{M}} + 3k(1-s) \right) m(s) + 6g_1(s) + 3L^2 N'(s) \end{cases} \tag{24}$$

And

$$\begin{cases} \lambda_1 = \dfrac{\overline{M}\omega^2 L^3}{6D(1) - 2\overline{M}\omega^2 L^3 - 3L^2 N(1)} \\ \lambda_2 = -\dfrac{2D(1)L}{\overline{M}} \\ \lambda_3 = \dfrac{6D(1) + \overline{M}\omega^2 L^3}{6\overline{M}\omega^2 L^3 + 9L^2 N(1) - 18D(1)} \end{cases} \tag{25}$$

It is to be mentioned that if the rotational inertia effects of the lumped mass is included, the solution procedure presented in this paper is still valid and applicable provided that the boundary conditions at the free end of the beam are defined appropriately. In such situation, the moment at the free end of the beam is no longer zero and the internal force component can be found in Clough and Penzien 1975. The weight of the lumped mass can also be considered in the analysis. In this case, the weight of lumped mass will be added to the concentrated axial force at the end of the beam.

2.2.3 Simply supported beam (S-S)

For a simply supported beam, the boundary conditions are established as follows

$$\xi = 0, 1 \quad \phi = 0, \quad M = 0 \tag{26}$$

Substitutio of $\phi = 0$ into Eq. (9) as well as $M = \phi = 0$ into Eq. (7) and setting $\xi = 0$ yields

$$C_4 = C_2 = 0 \tag{27}$$

Also, substitution of $\phi = M = 0$ into Eq. (7), and setting $\xi = 1$ results in

$$-\int_0^1 \left[k(1-s)m(s) + L^2 N'(s) \right] \phi(s) d(s) = C_1 \tag{28}$$

Finally, with regard to Eqs. (27) and (28), substitution of $\phi = 0$ into Eq. (9) and setting $\xi = 1$, leads to a relation for constant C_3 as follows

$$\int_0^1 \left[(1-s)D''(s) - \frac{k}{6}(1-s)^3 m(s) - 2D'(s) + L^2(1-s)N(s) - \frac{L^2}{2}(1-s)^2 N'(s) \right. \\ \left. + \frac{k}{6}(1-s)m(s) + \frac{L^2}{6} N'(s) \right] \phi(s) ds = C_3 \tag{29}$$

Introduction of the integration constants into Eq. (9), further develops the integral equation as follows

$$D(\xi)\phi(\xi)+\int_0^{\xi} f_1(\xi,s)\phi(s)ds+\int_0^{1} f_2(\xi,s)\phi(s)ds=0 \tag{30}$$

In which functions $f_1(\xi,s)$ and $f_2(\xi,s)$ are expressed by the following relations

$$\begin{cases} f_1(\xi,s)=(\xi-s)D''(s)-\dfrac{k}{6}(\xi-s)^3 m(s)-2D'(s)+L^2(\xi-s)N(s)-\dfrac{L^2}{2}(\xi-s)^2 N'(s) \\ f_2(\xi,s)=\left[\dfrac{k}{6}(1-s)\xi^3+\dfrac{k}{6}(1-s)^3\xi-\dfrac{k}{6}(1-s)\xi\right]m(s)+\left[\dfrac{L^2}{6}\xi^3+\dfrac{L^2}{2}(1-s)^2\xi-\dfrac{L^2}{6}\xi\right]N'(s) \\ \qquad\qquad +2\xi D'(s)-(1-s)\xi D''(s)-L^2(1-s)\xi N(s) \end{cases} \tag{31}$$

2.2.4 Clamped-Pinned beam (C-P)

In this section, a beam with a clamped and a pinned end is considered. It is assumed that the beam is clamped at $\xi=0$ and is pinned at $\xi=1$. The boundary conditions are introduced as follows

$$\begin{cases} \xi=0, & \phi=0, & \theta=0 \\ \xi=1, & \phi=0, & M=0 \end{cases} \tag{32}$$

Substituting $\phi=0$ as well as $\phi=\theta=0$ into Eq. (9) and Eq. (8) respectively associated with $\xi=0$ leads to

$$C_3=C_4=0 \tag{33}$$

Also, substitution of $\phi=M=0$ into Eq. (7) and setting $\xi=1$ yields

$$-\int_0^1\left[k(1-s)m(s)+L^2N'(s)\right]\phi(s)ds=C_1+C_2 \tag{34}$$

Finally, introducing $\phi=0$ into Eq. (9) and setting $\xi=1$ also results in

$$\int_0^1\left[6(1-s)D''(s)-k(1-s)^3m(s)-12D'(s)+6L^2(1-s)N(s)-3L^2(1-s)^2N'(s)\right]\phi(s)ds=C_1+3C_2 \tag{35}$$

If the integration constants from Eqs. (33-35) are determined and are introduced into Eq. (9), the integral equation is obtained as follows

$$D(\xi)\phi(\xi)+\int_0^{\xi} f_1(\xi,s)\phi(s)ds+\int_0^{1} f_2(\xi,s)\phi(s)ds=0 \tag{36}$$

In which functions $f_1(\xi,s)$ and $f_2(\xi,s)$ are expressed as follows

$$\begin{cases} f_1(\xi,s)=(\xi-s)D''(s)-\dfrac{k}{6}(\xi-s)^3 m(s)-2D'(s)+L^2(\xi-s)N(s)-\dfrac{L^2}{2}(\xi-s)^2 N'(s) \\ f_2(\xi,s)=\left[\dfrac{k}{12}(1-s)\xi^3-\dfrac{k}{12}(1-s)^3\xi^3+\dfrac{k}{6}(1-s)\xi^3-\dfrac{k}{4}(1-s)\xi^2+\dfrac{k}{4}(1-s)^3\xi^2\right]m(s) \\ +\left[3\xi^2-\xi^3\right]D'(s)+\left[\dfrac{L^2}{12}\xi^3-\dfrac{1}{4}L^2(1-s)^2\xi^3+\dfrac{L^2}{6}\xi^3-\dfrac{L^2}{4}\xi^2+\dfrac{3}{4}L^2(1-s)^2\xi^2\right]N'(s) \\ +\left[\dfrac{L^2}{2}(1-s)\xi^3-\dfrac{3}{2}L^2(1-s)\xi^2\right]N(s)+\left[\dfrac{1}{2}(1-s)\xi^3-\dfrac{3}{2}(1-s)\xi^2\right]D''(s) \end{cases} \tag{37}$$

2.2.5 Clamped-Clamped beam (C-C)

For a beam with both clamped end, the boundary conditions are established as follows

$$\xi = 0, 1, \quad \phi = \theta = 0 \tag{38}$$

If $\phi = 0$ and $\phi = \theta = 0$ are introduced into Eq. (9) as well as Eq. (8) respectively and ξ is set $\xi = 0$ one obtains

$$C_3 = C_4 = 0 \tag{39}$$

Also, substitution of $\phi = \theta = 0$ into Eq. (8) and setting $\xi = 1$ leads to

$$\int_0^1\left[D''(s)-\frac{k}{2}(1-s)^2\, m(s)+L^2 N(s)-L^2(1-s)N'(s)\right]\phi(s)ds=\frac{C_1}{2}+C_2 \tag{40}$$

Finally, introduction of $\phi = 0$ into Eq. (9) and setting $\xi = 1$ leads to

$$\int_0^1\left[(1-s)D''(s)-\frac{k}{6}(1-s)^3 m(s)-2D'(s)+L^2(1-s)N(s)-\frac{L^2}{2}(1-s)^2 N'(s)\right]\phi(s)ds$$
$$=\frac{C_1}{6}+\frac{C_2}{2} \tag{41}$$

Integration constants from Eqs. (39-41) are determined and then introduced in Eq. (9). This results in the integral equation to be obtained as follows

$$D(\xi)\phi(\xi)+\int_0^{\xi} f_1(\xi,s)\phi(s)ds+\int_0^1 f_2(\xi,s)\phi(s)ds=0 \tag{42}$$

In which functions $f_1(\xi,s)$ and $f_2(\xi,s)$ can be stated as follows

$$\begin{cases} f_1(\xi,s)=(\xi-s)D''(s)-\dfrac{k}{6}(\xi-s)^3 m(s)-2D'(s)+L^2(\xi-s)N(s)-\dfrac{L^2}{2}(\xi-s)^2 N'(s) \\ f_2(\xi,s)=\left[\dfrac{k}{2}(1-s)^2\xi^3-\dfrac{k}{3}(1-s)^3\xi^3+\dfrac{k}{2}(1-s)^3\xi^2-\dfrac{k}{2}(1-s)^2\xi^2\right]m(s) \\ +\left[6\xi^2-4\xi^3\right]D'(s)+\left[2L^2(1-s)\xi^3-L^2\xi^3-3L^2(1-s)\xi^2+L^2\xi^2\right]N(s) \\ +\left[L^2\xi^3(1-s)-L^2(1-s)^2\xi^3+\dfrac{3}{2}L^2(1-s)^2\xi^2-L^2(1-s)\xi^2\right]N'(s) \\ +\left[-\xi^3+2(1-s)\xi^3-3(1-s)\xi^2+\xi^2\right]D''(s) \end{cases} \tag{43}$$

It is concluded , from the discussion given in preceding sections (Sec. 2.2.1 to Sec. 2.2.4) that, f_1 & f_2 in Eq. (19) are non-dimensional functions which are calculated for the beam depending on its end condition. For example, in a cantilever beam with lumped mass at free end, these non-dimensional functions are defined by Eq. (23) whereas for simply supported beams, it is required that Eq. (31) be employed for calculation of these functions. As another example, in clamped-pinned beams (C-P), non-dimensional functions f_1 & f_2 are written based on Eq. (37). In all these equations, end conditions generally dominates the shape of non-dimensional functions f_1 & f_2. These functions, when calculated based on the desired end condition, are inserted into integral equations obtained which finally yields the linear algebraic equation system. Natural frequencies of the system are obtained through calculation of non-trivial solution of the resulting system of equations. To achieve this, the determinant of the coefficients matrix of the system has to be vanished. Accordingly, a frequency equation is introduced. The roots of frequency equation are vibrations frequencies of beam. In the following, these issues are described in details.

2.3 Establish the system of linear algebraic equations

The mode shape function $\phi(s)$ is the unknown parameter in the integral equations obtained. In order to solve the integral equations and to determine the vibration frequencies, the mode shape function is approximated by the following power series

$$\phi(\xi)=\sum_{r=0}^{R}c_r\xi^r \tag{44}$$

Where C_r are unknown coefficients and R is a given positive integer, which is adopted such that the accuracy of the results are sustained. Introducing Eq. (44) into integral equations obtained before leads to

$$\sum_{r=0}^{R}\left[D(\xi)\xi^r+\int_0^{\xi} f_1(\xi,s)s^r ds+\int_0^{1} f_2(\xi,s)s^r ds\right]C_r=0 \tag{45}$$

Both sides of (45) are multiplied by ξ^m and integrated subsequently with respect to ξ between 0 and 1. This results in a system of linear algebraic equations in C_r

$$\sum_{r=0}^{R}\left[G(m,r)+F_1(m,r)+F_2(m,r)\right]c_r=0 \qquad m=0,1,2,\dots,R \tag{46}$$

In which functions $G(m,r), F_1(m,r)$ and $F_2(m,r)$ are expressed as follows

$$\begin{cases} G(m,r)=\int_0^1 \xi^{r+m}D(\xi)d\xi \\ F_1(m,r)=\int_0^1\int_0^{\xi} f_1(\xi,s)s^r\xi^m ds d\xi \\ F_2(m,r)=\int_0^1\int_0^1 f_2(\xi,s)s^r\xi^m ds d\xi \end{cases} \tag{47}$$

The system of linear algebraic Eq. (46) may be expressed in matrix notations as follows

$$\begin{cases} [A]_{(R+1)\times(R+1)}[C]_{(R+1)\times 1}=[0]_{(R+1)\times 1} \\ [A]=\begin{bmatrix} [G(0,0)+F_1(0,0)+F_2(0,0)] & [G(0,1)+F_1(0,1)+F_2(0,1)] & \dots\dots\dots & [G(0,R)+F_1(0,R)+F_2(0,R)] \\ [G(1,0)+F_1(1,0)+F_2(1,0)] & [G(1,1)+F_1(1,1)+F_2(1,1)] & \dots\dots\dots & [G(1,R)+F_1(1,R)+F_2(1,R)] \\ . & . & . & . \\ . & . & . & . \\ [G(R,0)+F_1(R,0)+F_2(R,0)] & [G(R,1)+F_1(R,1)+F_2(R,1)] & \dots\dots\dots & [G(R,R)+F_1(R,R)+F_2(R,R)] \end{bmatrix} \\ [C]^T=[C_0 \quad C_1 \quad \dots\dots \quad C_R] \end{cases} \tag{48}$$

In which $[A]$ and $[C]^T$ are matrix coefficients and unknowns vector transpose respectivly. In order to obtain the circular natural frequencies of the beam, functions $f_1(\xi,s)$ and $f_2(\xi,s)$ are first obtained. Introducing these functions into (47), the functions $G(m,r), F_1(m,r)$, $F_2(m,r)$ associated with the coefficients of matrix [A] are obtained next. The unknown parameter in the coefficients matrix [A] is therefore the circular natural frequency of the beam. $[C]=0$ is a trivial solution for the resulting system of equations introduced in (46). The natural frequencies are determined through calculation of a non-trivial solution for resulting system of equations. To achieve this, the determinant of the coefficients matrix of the system has to be vanished. Accordingly, a frequency equation in ω (which is a polynomial function of the order $2(R+1)$) is introduced. Given the fact that the mode shape function is approximated by the power series of (44), the results accuracy is improved if more number of the series sentences are taken into account. In this case, the order of polynomial is also increased accordingly. Hence, adoption of larger R yields more accurate results. In the examples to be provided and discussed in Sec.3, it will be shown that if the number of the series sentences is increased to 8, results converge to those given by SAP2000 employing 100 or more elements. While such a number of elements employed, irrespective of much time and core space required, may appear a common practice in FE analysis, the interested readers may note that the method developed in this study deals essentially with one unique element. Thus the advantage of the method developed here is further manifested when one reach comparable and accurate results by simple mathematical relations developed here. It is also of note that adoption of more number of series sentences is far from a complex or unmanageable situation. It may only results in a slightly larger coefficient matrix [A] in the analysis.

2.4 Determination of the buckling load of the non-prismatic beams

When the axial force acting on a beam approaches the buckling load, the beam vibration frequency tends to zero. The proposed formulation in this paper can, therefore, be used for determination of the buckling load of non-prismatic beams. For this purpose, a constant axial force is assumed to act on the beam (i.e., $N(\xi) \equiv constant \equiv N$) and ω is set zero (i.e., ω = 0). These conditions are subsequently applied to functions $f_1(\xi,s)$ and $f_2(\xi,s)$. These functions, for various end support conditions, are simplified as follows

$$\begin{cases} \textit{for all cases} & f_1(\xi,s) = (\xi-s)D''(s) - 2D'(s) + NL^2(\xi-s) \\ \textit{for cantilever beam}: & f_2(\xi,s) = \left(\dfrac{NL^2\xi^2}{2D(1)-NL^2}\right)\left[(1-s)D''(s) - 2D'(s) + NL^2(1-s)\right] \\ \textit{for simply supported beam}: & f_2(\xi,s) = 2\xi D'(s) - (1-s)\xi D''(s) - NL^2(1-s)\xi \\ \textit{for clamped - pinned beam}: & f_2(\xi,s) = \left[\dfrac{1}{2}(1-s)\xi^3 - \dfrac{3}{2}(1-s)\xi^2\right]D''(s) + \left[3\xi^2 - \xi^3\right]D'(s) \\ & \quad + \dfrac{1}{2}NL^2(1-s)\xi^3 - \dfrac{3}{2}NL^2(1-s)\xi^2 \\ \textit{for clamped - clamped beam}: & f_2(\xi,s) = \left[-4\xi^3 + 6\xi^2\right]D'(s) + \left[-\xi^3 + 2(1-s)\xi^3 - 3(1-s)\xi^2 + \xi^2\right]D''(s) \\ & \quad - NL^2\xi^3 + 2NL^2(1-s)\xi^3 - 3NL^2(1-s)\xi^2 + NL^2\xi^2 \end{cases} \tag{49}$$

Functions $G(m,r), F_1(m,r)$, and $F_2(m,r)$ as well as coefficients matrix [A] are obtained if values for functions $f_1(\xi,s)$ and $f_2(\xi,s)$ obtained from Eq. (49) are introduced into Eq. (47). It is of note that unknown parameter in this matrix is, therefore, the constant axial force N. Further, the beam buckling load is determined through calculation of a non-trivial solution developed for resulting system of equations.

3. Numerical results

Five numerical examples are presented in this section in which vibration frequencies of the non-prismatic beams under variable axial forces are calculated based on the present approach. The results of the analysis are then compared to those obtained using *SAP-2000* software. The "beam element" is adopted from elements library to model the problem. In all examples, the elastic modulus (E), the weight per unit volume (γ) and the mass per unit volume (ρ) are adopted as: 210×10^6 KN/m^2, 200 KN/m^3 and, 20.3943 ton/m^3 respectively.

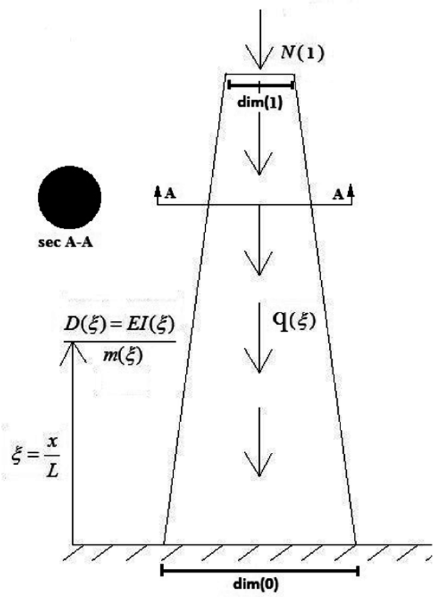

Fig. 3 The non-prismatic cantilever beam with circular cross section under variable axial forces

3.1 Example 1: Cantilever beam with circular cross section

In this example, the vibration frequencies of a non-prismatic cantilever beam with circular cross section under variable axial forces are investigated (Fig. 3). The beam is assumed to have a linearly varying diameter as follows

$$\dim(x) = -\frac{8}{L}x + 10 \quad or \quad \dim(\xi) = -8\xi + 10 \qquad 0 \le \xi \le 1 \tag{50}$$

In which $\dim(\xi)$ and L are the cross sectional diameter and the beam length respectively. The latter is adopted as 50 m in this example. The concentrated axial force at the free end of the beam $N(1)$ is also taken equal to 5×10^6 KN. The function defining the variable axial force, including a concentrated axial force at the free end of the beam, as well as variably distributed axial force due to beam weight is obtained as follows

$$N(\xi) = N(1) + \gamma L \int_{\xi}^{1} A(\xi) d(\xi) = 5 \times 10^6 + \frac{200 \times 50 \times \pi}{4} \int_{\xi}^{1} [\dim(\xi)]^2 \, d(\xi) \tag{51}$$

In which $A(\xi)$ is cross sectional area. The first five frequencies of the beam are calculated. The results are summarized in Table 1. The beam's buckling load is obtained equal to 10.91×10^6 KN for $R = 8$ while it is evaluated by SAP-2000 as 10.89×10^6 KN using 100 elements. With increase of the concentrated axial force at free end of the beam to 7×10^6 KN, the vibration frequencies of the beam have been calculated and shown in Table 2.

Table 1 The beam vibration frequencies of the example 1

			Vibration frequencies (rad/sec)				
			ω_1	ω_2	ω_3	ω_4	ω_5
With effects of axial forces	Analysis results	$R = 2$	21.133	130.24	-	-	-
		$R = 4$	16.836	52.91	125.9	373	-
		$R = 5$	16.975	52.978	119.805	242.522	671.484
		$R = 8$	16.974	52.869	121.05	221.21	347.7
	SAP 2000 results	10 *elements*	16.58	50.47	112.98	202.6	319.6
		25 *elements*	16.91	52.47	119.69	218.5	348.7
		50 *elements*	16.953	52.77	120.69	220.97	353.72
		100 *elements*	16.965	52.84	120.94	221.59	354.98
Without effects of axial forces	Analysis results	$R = 2$	20.1	82.77	617.2	-	-
		$R = 5$	19.88	58.7	124.9	248.8	682.6
		$R = 8$	19.884	58.9974	127.82	228.	354.2
	SAP 2000 results	10 *elements*	19.53	56.7	119.7	208.8	324.5
		25 *elements*	19.826	58.63	126.53	225.35	355.3
		50 *elements*	19.869	58.91	127.5	227.79	360.36
		100 *elements*	19.88	58.974	127.74	228.4	361.6

Table 2 The beam vibration frequencies of the example 1 under increased concentrated axial force

			Vibration frequencies (rad/sec)				
			ω_1	ω_2	ω_3	ω_4	ω_5
With effects of axial forces	Analysis results	$R = 2$	20.381	113.3	-	-	-
		$R = 4$	13.706	48.793	119.186	359.8	-
		$R = 5$	14.899	49.929	117.35	239.2	
		$R = 8$	14.958	49.708	117.99	218.308	345
	SAP 2000 results	10 *elements*	14.556	47.365	110.1	200.12	317.62
		25 *elements*	14.885	49.314	116.65	215.62	345.95
		50 *elements*	14.933	49.604	117.64	218.08	350.98
		100 *elements*	14.945	49.677	117.89	218.7	352.24

3.2 Example 2: clamped-pinned beam with hollow circular cross section

In this example, the vibration frequencies of a non-prismatic beam with a clamped-pinned ends having hollow circular cross section subjected to variable axial forces are evaluated (Fig. 4). The beam section is assumed to have a linearly varying external diameter and a constant thickness defined by the following relations

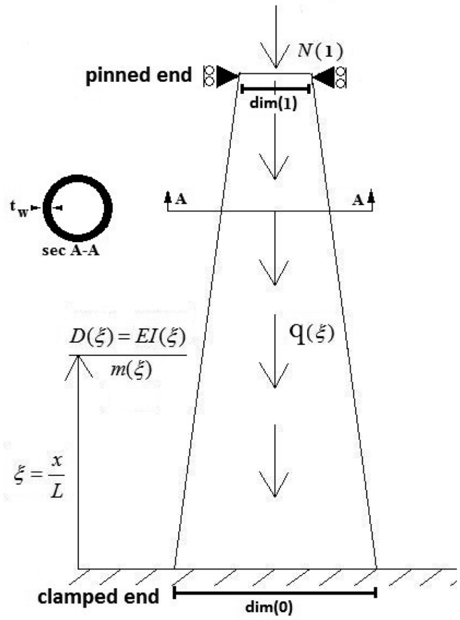

Fig. 4 The non-prismatic clamped-pinned beam with hollow circular cross section, under variable axial forces

Table 3 The beam vibration frequencies of the example 2

			Vibration frequencies (rad/sec)				
			ω_1	ω_2	ω_3	ω_4	ω_5
With effects of axial forces	Analysis results	$R = 4$	27.33	81.21	187.72	-	-
		$R = 5$	32.019	103.198	194.976	360.159	923.453
		$R = 6$	32.329	106.32	220.148	367.358	607
		$R = 8$	32.238	105.793	221.741	378.078	585.262
	SAP 2000 results	5 *elements*	32.446	107.11	233.63	454.09	8451.6
		10 *elements*	32.246	105.74	221.37	378.42	577.62
		25 *elements*	32.222	105.73	221.56	379.25	578.75
		50 *elements*	32.221	105.73	221.56	379.3	578.94
		100 *elements*	32.221	105.73	221.56	379.3	578.95
Without effects of axial forces	Analysis results	$R = 4$	40.074	101.744	200.807	544.812	-
		$R = 5$	40.739	115.745	214.401	373.324	927.467
		$R = 6$	40.685	116.593	232.487	385.545	621.124
		$R = 8$	40.628	116.024	232.793	389.853	596
	SAP 2000 results	5 *elements*	40.586	116	239.86	456.58	8451
		10 *elements*	40.619	115.9	232.08	388.9	586.83
		25 *elements*	40.621	115.97	232.56	390.64	590.36
		50 *elements*	40.62	115.98	232.57	390.7	590.57
		100 *elements*	40.62	115.97	232.57	390.7	590.58

$$\begin{cases} \dim_{ex}(x) = -\frac{8}{L}x + 10 \quad or \quad \dim_{ex}(\xi) = -8\xi + 10 \quad 0 \le \xi \le 1 \\ t_w = 0.5 \text{ m} \end{cases} \tag{52}$$

In which $\dim_{ex}(\xi)$ and t_w are the cross sectional external diameter and the thickness of section, respectively. The concentrated axial force at the free end of the beam and beam length are also adopted as 10×10^6 KN and 50 m respectively. The first five frequencies for various R's are shown in Table 3 associated with those obtained from SAP-2000. The beam's buckling load is evaluated as high as 22.829×10^6 KN for $R = 8$ while SAP-2000 reaches more or less the same result as 22.8059×10^6 KN if 100 elements are introduced to model the problem.

3.3 Example 3: Simply supported beam with square cross section

In this example, the vibration frequencies of a simply supported and non-prismatic beam having square cross section under variable axial forces are determined. The beam cross section is assumed to have a linearly varying width and height as follows (Fig. 5)

$$b(\xi) = d(\xi) = 4[1 - 0.5\xi] \quad 0 \le \xi \le 1 \tag{53}$$

In which $b(\xi)$ and $d(\xi)$ are section's width and height, respectively. The concentrated axial force at free end of the beam and beam length are taken as 6×10^6 KN and 30 m respectively. The first five frequencies of the beam calculated are shown in Table 4. The beam buckling load is obtained

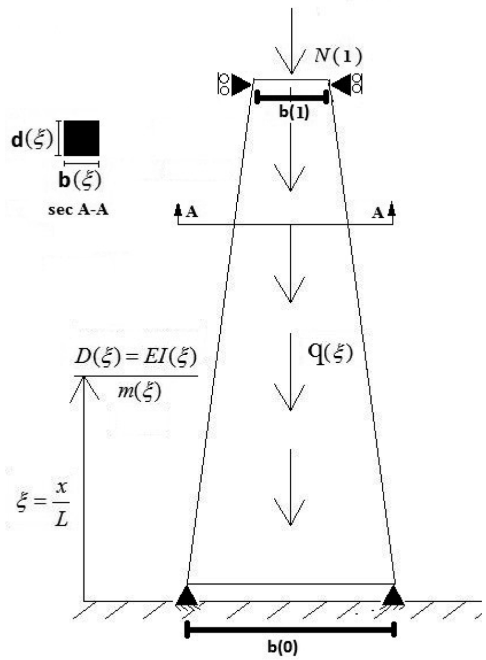

Fig. 5 The non-prismatic simply supported beam with square cross section, under variable axial forces

Table 4 The beam vibration frequencies of the example 3

			Vibration frequencies (rad/sec)				
			ω_1	ω_2	ω_3	ω_4	ω_5
With effects of axial forces	Analysis results	$R = 2$	18.767	106.28	-	-	-
		$R = 4$	20.61	111.96	280.841	601.132	1976.75
		$R = 5$	20.61	112.13	259.896	537.326	1090.81
		$R = 8$	20.61	112.189	260.772	468.422	748.524
	SAP 2000 results	15 *elements*	20.612	112.16	260.66	467.93	733.68
		30 *elements*	20.613	112.18	260.75	468.23	734.76
		60 *elements*	20.614	112.19	260.76	468.27	734.84
Without effects of axial forces	Analysis results	$R = 2$	27.82	116.07	-	-	-
		$R = 4$	28.649	119.698	287.923	610.746	1979.48
		$R = 5$	28.641	119.845	267.845	543.596	1100.69
		$R = 8$	28.64	119.848	268.539	476.216	756.667
	SAP 2000 results	15 *elements*	28.634	119.81	268.42	475.73	741.49
		30 *elements*	28.639	119.84	268.52	476.05	742.6
		60 *elements*	28.64	119.84	268.53	476.09	742.69

as 12.282×10^6 KN for $R = 5$ while this load is evaluated by SAP-2000 almost equally as 12.282301×10^6 KN when 60 elements are used to model the beam.

3.4 Example 4: clamped-clamped beam with hollow square cross section

In this example, vibration frequencies of a non-prismatic beam with clamped-clamped ends having a hollow square cross section are evaluated subjected to variable axial forces. Fig. 6 shows the geometry and loading conditions adopted in this example. The beam section has been assumed to have both linearly varying width and height with uniform thickness defined by the following relation

$$\begin{cases} d(\xi) = b(\xi) = 4[1-0.5\xi] \\ t_w = 0.5 \text{ m} \quad \text{constant thickness} \end{cases} \tag{54}$$

The concentrated axial force at free end, $N(1)$, and beam length are adopted as 20×10^6 KN and 30 m, respectively. The first five frequencies of the beam have calculated using current approach as well as those given by SAP-2000 are summarized in Table 5. As can be seen in the table, the beam buckling load for $R = 8$ is obtained as 40.672×10^6 KN using current approach . If 60 elements are used to model the same beam in SAP-2000, it converges to an almost identical buckling load of 40.671×10^6 KN. Negligible difference is seen between the results obtained by two approaches which shows the accuracy and efficiency of the proposed method.

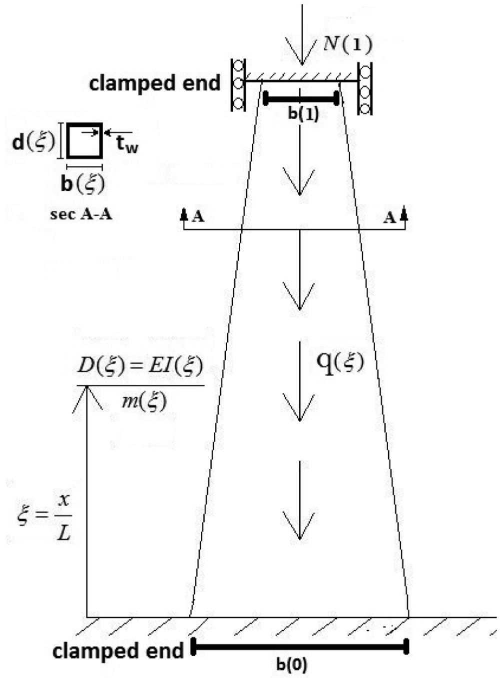

Fig. 6 The non-prismatic clamped-clamped beam with hollow square cross section, under variable axial force

Table 5 The beam vibration frequencies of the example 4

			Vibration frequencies (rad/sec)				
			ω_1	ω_2	ω_3	ω_4	ω_5
With effects of axial forces	Analysis results	R = 4	57.027	189.294	492.392	1109	-
		R = 5	58.072	192.356	402.057	909	2003
		R = 6	58.154	193.062	406.338	699.098	1510
		R = 8	58.154	193.042	402.61	684.495	1097.8
	SAP 2000 results	15 *elements*	58.15	193	402.53	682.78	1032
		30 *elements*	58.153	193.03	402.64	683.4	1035
		60 *elements*	58.151	193.03	402.64	683.42	1035
Without effects of axial forces	Analysis results	R = 4	80.89	220.87	507	1102	-
		R = 5	80.21	221.52	435.27	923.61	1993
		R = 6	80.196	220.77	437.415	733	1524
		R = 8	80.196	220.72	432.35	715.15	1129
	SAP 2000results	15 *elements*	80.192	220.69	432.22	713.73	1064
		30 *elements*	80.194	220.71	432.36	714.42	1067
		60 *elements*	80.193	220.71	432.36	714.44	1067

Table 6 The vibration frequencies of the beam of the example 5

			Vibration frequencies (rad/sec)				
			ω_5	ω_4	ω_3	ω_2	ω_1
With effects of axial forces	Analysis results	R = 3	5.56	40.14	118.14	390	-
		R = 5	5.56	40.057	113.98	234	426
		R = 6	5.56	40.057	113.96	227.55	405
		R = 8	5.56	40.057	113.95	227.064	381.32
	SAP 2000 results	10 *elements*	5.536	39.725	112.75	224.28	374.5
		30 *elements*	5.557	40.019	113.81	226.76	379.75
		60 *elements*	5.559	40.047	113.91	227	380.14
Without effects of axial forces	Analysis results	R = 3	7.434	41.87	119	392	-
		R = 5	7.434	41.778	115.47	235.87	427.15
		R = 6	7.434	41.778	115.438	228.95	406
		R = 8	7.434	41.778	115.427	228.471	382.69
	SAP 2000 results	10 *elements*	7.404	41.428	114.21	225.66	375.84
		30 *elements*	7.431	41.739	115.29	228.16	381.12
		60 *elements*	7.433	41.768	115.39	228.39	381.52

3.5 Example 5: Cantilever beam with a lumped mass at free end

In this example, the vibration frequencies of a non-prismatic cantilever beam with a lumped mass at its free end under variable axial force, shown in Fig. 2, are evaluated. It has been assumed that the beam has a rectangular cross section with a linearly varying width and a uniform height as defined by the following relations

$$\begin{cases} b(\xi) = 4(1-0.5\xi) \\ d = 2\ \text{m} \quad \text{uniform height} \end{cases} \tag{55}$$

Concentrated axial force, lumped mass at free end of the beam, and beam length are adopted in this example as: 580000 KN, 300 ton and 30 m respectively. The first five frequencies of the beam resulted from the current approach and those obtained from SAP-2000 are all presented in Table 6 it is of note that in this example, rotational inertia effects and weight of lumped mass are neglected.

4. Conclusions

Despite popularity of FEM in analysis of static and dynamic structural problems and the routine applicability of FE softwares, analytical methods based on simple mathematical relations is still largely sought by many researchers and practicing engineers around the world. Development of such analytical methods for analysis of free vibration of non-prismatic beams is also of primary concern.

The vibration frequency of non-prismatic beams under variable axial forces is analytically and numerically investigated. The proposed method is based on the conversion of the governing differential equation into its weak form integral equation. The mode shape function has been approximated by a power series, which allows the weak form integral equation to be transformed into a system of linear algebraic equations. The natural frequencies are determined by calculation of a non-trivial solution for system of equations. Proposed method has also been extended for determination of the buckling load of the non-prismatic beams. It is shown that when the mode shape function is approximated by a power series, the accuracy of results will increase depending on the increase in the number of terms applied in the series. It is, in fact, the merit of the work presented in this paper that the method essentially provides solution procedure and accurate results by adopting a unique element representing the beam which is solved by a simple mathematical relation. As can be inferred from practical examples provided in sec. 3, FEM requires perhaps 100 or more elements (hence more time and core space) to reach the same accuracy as that provided by current approach.

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Notations

$A(\xi)$	: cross sectional area of the beam.
$b(\xi)$	: beam section width.
C_i	: integration constants $(i = 1, \ldots, 4)$.
$d(\xi)$	: beam section height.
$D(\xi)$	: bending stiffness ($=EI(\xi)$).
$\dim(\xi)$	: beam section diameter.
$\dim_{ex}(\xi)$	: external diameter of the beam section.
E	: young's modulus.
$I(\xi)$	: inertial moment of cross-sectional area.
$k = \omega^2 L^4$	
L	: beam length.
$m(\xi)$	: mass per unit length ($=\rho A(\xi)$).
$M(\xi, t)$	: bending moment.
$\overline{M}$	: lumped mass at free end of the beam.
$N(\xi)$	: axial force.
$p(x, t)$	: transverse force.
$q(\xi)$	: axial force per unit length.
R	: a certain positive integer.
t	: time.
t_w	: cross sectional thickness.
$V(\xi, t)$	: shear force.
x	: coordinate along the axis of the beam.
$\vartheta(x, t)$	: beam transverse displacement.
$\phi(\xi)$	: mode shape function.
θ	: beam rotation, due to bending of the centroidal line of the beam.
ξ	: non-dimensional variable ($= x/L$).
ρ	: specific mass.
γ	: specific weight.
ω	: natural angular frequency of the beam.