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Free vibration analysis of combined system with variable cross section in tall buildings

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Abstract. This paper deals with determining the fundamental frequency of tall buildings that consist of framed tube, shear core, belt truss and outrigger systems in which the framed tube and shear core vary in size along the height of the structure. The effect of belt truss and outrigger system is modeled as a concentrated rotational linear spring at the belt truss and outrigger system location. Many cantilevered tall structures can be treated as cantilevered beams with variable cross-section in free vibration analysis. In this paper, the continuous approach, in which a tall building is replaced by an idealized cantilever continuum representing the structural characteristics, is employed and by using energy method and Hamilton's variational principle, the governing equation for free vibration of tall building with variable distributed mass and stiffness is obtained. The general solution of governing equation is obtained by making appropriate selection for mass and stiffness distribution functions. By applying the separation of variables method for time and space, the governing partial differential equation of motion is reduced to an ordinary differential equation with variable coefficients with the assumption that the transverse displacement is harmonic. A power-series solution representing the mode shape function of tall building is used. Applying boundary conditions yields the boundary value problem; the frequency equation is established and solved through a numerical process to determine the natural frequencies. Computer program has been developed in Matlab (R2009b, Version 7.9.0.529, Mathworks Inc., California, USA). A numerical example has been solved to demonstrate the reliability of this method. The results of the proposed mathematical model give a good understanding of the structure's dynamic characteristics; it is easy to use, yet reasonably accurate and suitable for quick evaluations during the preliminary design stages.

Keywords: free vibration; framed tube; shear core; belt truss; outrigger system; variable cross section

1. Introduction

Free vibration analysis plays an important role in structural design of tall buildings, especially for the first mode because of its dominance in describing the response to wind and earthquake-induced vibrations of tall buildings. Over the past decades, great deal of attention has been paid to studying the behavior of tall buildings subjected to lateral loading (Wang 1978, Eisenberger 1991a, b, 1994, Wang 1996a, b, Swaddiwudhipong *et al.* 2002, Kuang and Ng 2004, 2009, Bozdogan 2006, 2009,

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Kaviani *et al.* 2008, Lee *et al.* 2008).

Lee (2007) obtained an approximate solution procedure for free vibration analysis of tube-in-tube tall buildings. The governing partial differential equation of motion was reduced to an ordinary differential equation with general variable coefficients on the assumption that the transverse displacement is harmonic. The power-series solution was utilized for solving the ordinary differential equation. Lee *et al.* (2008) presented the governing equations of wall-frame structures with outriggers through a continuum approach and the whole structure was idealized as a shear-flexural cantilever with rotational springs. Smith and Salim (1983) presented formulae that were developed for estimating the optimum levels of outriggers to minimize the drift in outrigger braced buildings. Rutenberg and Tal (1987) presented the results of an investigation on drift reduction in uniform and non-uniform belted structures with rigid outriggers under several lateral load distributions which were likely to be encountered in practice. Design aids in the form of graphical presentations of the somewhat complex solutions were provided.

Kuang and Ng (2009) presented free vibration analysis of asymmetric-plan frame structures. They emphasized on analysis of lateral-torsional vibration of the structures, where lateral shear vibrations in two orthogonal directions were coupled with St. Venant torsional vibration. The governing equation for coupled vibration of the problem was derived, and the corresponding eigenvalue equation was obtained. A theoretical solution method was proposed for solving the eigenvalue problem and a general solution was given to determine the natural frequencies and associated mode shapes of the structure. A simplified analytical method for outrigger structure was presented earlier by Smith and Coull (1991), Taranath (1988). Hoenderkamp and Bakker (2003) presented a graphical method of analysis for the preliminary design of tall building structures comprising of braced frames and outrigger trusses subjected to horizontal loading. They also presented a simple procedure for obtaining the optimum location of the outrigger along structure's height and a rapid assessment of the impact of the outrigger on the behavior of the high-rise structure.

In this paper a simple mathematical model for calculating the first natural frequency of combined framed tube, shear core and belt truss system with variable cross section along the height of structure is presented. Framed tube system consists of closely spaced exterior columns along the periphery interconnected by deep spandrel beams at each floor. This produces a system of rigidly connected, jointed orthogonal frame panels forming a rectangular tube which acts as a cantilevered hollow box (Coull and Bose 1975, Coull and Ahmad 1978, Connor and Pouangare 1991). The effect of belt truss and outrigger system is modeled as a concentrated rotational linear spring at the location of belt truss and outrigger system (Taranath 1988, Rahgozar and Sharifi 2009, Rahgozar *et al.* 2011, Malekinejad and Rahgozar 2011). Here by adopting Hamilton's variational principle (HVP), partial differential equation for vibration of structure, boundary displacements (kinematic boundary conditions), boundary forces (natural boundary conditions) and eigenvalue solution are derived and by selecting suitable functions for distribution of mass, flexural stiffness and shear stiffness along the height of the structure and assumption of harmonic motion, the partial differential equation is reduced to ordinary differential equation with variable coefficients. By using power-series solution method and applying boundary conditions, the eigenvalue problem for finding the first natural frequency of tall building is obtained. In order to illustrate the efficiency and accuracy of the proposed method a numerical example has been carried out by the proposed method and SAP2000 (Advanced 12.0.0, Computers and Structures, Berkeley, California, USA).

2. Formulation and solution

In this section by considering the following assumptions and using Hamilton's variational principle (HVP), the partial differential equation for vibration of the combined system of framed tube, shear core, belt truss and outrigger system is derived. Assumptions are as follows: a) floor slabs are rigid, b) the effect of belt truss and outrigger system is considered as a rotational linear spring with constant rotational stiffness which acts on the position of belt truss and outrigger system, c) spacing of columns and beams are constant throughout building's height and at each storey dimensions all of beams and columns are the same, d) shear core and columns are fully fixed at the base, e) connections between the members of outrigger system are assumed to be rigid and connections for members of belt truss are assumed to be pinned, f) material of the structure is linearly elastic, homogeneous and obeys Hook's law, g) the structure is assumed symmetric in plan and height and therefore cannot twist, h) dimension of shear core members, columns and beams vary smoothly along the height of structure, i) the dimension of members of belt truss and outrigger system are constant and do not vary along the structure's height, j) mass of the outrigger and belt truss system is considered to be distributed uniformly along building's height.

With above assumptions, the structure can be modeled as a beam with a variable box cross section along the height (see Fig. 1) and by using Hamilton's principle, the differential equation for vibration of combined system can be obtained. Vibration of structures can be conveniently formulated in terms of Hamilton's variational principle (HVP). HVP is an energy based functional, the diverse areas of structural dynamics, numerical solutions of partial differential equation, finite element methods and functional analysis are all be linked in a single development. Structure's energy is expressed in terms of the Lagrangian energy functional, and HVP requires that in an equilibrium configuration, this energy functional to assume a stationary value. The term functional is used to denote a general expression for a continuous function of the domain V of the structure in

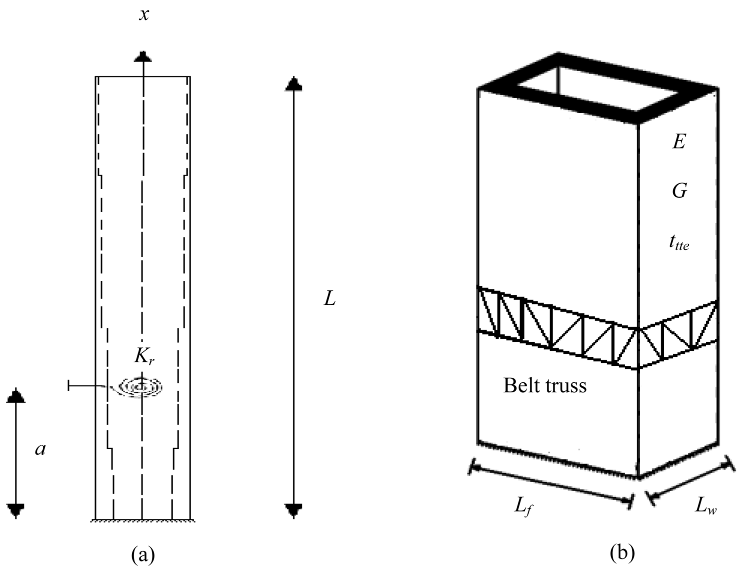

Fig. 1 Approximate models of tall buildings: (a) elevation of combined system, (b) 3D model of combined system

space and time (Piersol and Paez 2010).

Consider a combined system of framed tube, shear core, belt truss and outrigger system as a continuous beam with flexural stiffness $EI(x)$, shear stiffness $AG(x)$, mass per unit height $m(x)$, dynamic displacement $w(x, t)$ and total height L (see Fig. 1). The effect of belt truss and outrigger system is considered as a rotational linear spring at the location of belt truss and outrigger system.

The structure is defined over the closed domain $0 \le x \le L$, where x is the spatial position of any material point of the system at time t as shown in Fig. 1. In Fig. 1, dashed line show the variation of thickness along structure's height.

In (Fig. 1), the parameters E, G and t_{tte} are modulus of elasticity, shear modulus and total equal thickness of the shear core and framed tube in flange or web panel.

For the dynamic system considered, the kinetic energy is as follows (Piersol and Paez 2010)

$$T(x,t) = \frac{1}{2}\int_0^L m(x)[\dot{w}(x,t)]^2 dx \tag{1}$$

and the potential energy is

$$V(x,t) = \frac{1}{2}\int_0^L (EI(x)[w''(x,t)]^2 + AG(x)[w'(x,t)]^2 + K_r[w'(a,t)]^2)dx \tag{2}$$

Here K_r is the stiffness for rotational linear spring that models the effect of belt truss and outrigger system on framed tube acting at $x = a$; and primes and dots for w denote partial derivatives with respect to x and t, respectively.

Total energy of the system is defined (Piersol and Paez 2010)

$$B(x,t) = T(x,t) - V(x,t) \tag{3}$$

The action or principle function of dynamics, A, can be expressed as the time integral of $B(x, t)$ between times t_1 and t_2 (Piersol and Paez 2010).

$$A = \int_{t_1}^{t_2} B(x,t)dt = \int_{t_1}^{t_2} [T(x,t) - V(x,t)] \tag{4}$$

HVP states that A has a stationary value expressed as $\delta A = 0$ where δ is the variation operator. The δ operator over a structure domain V implies that complete description of structure's deformed shape is a linear combination of an infinite number of degree of freedom-one-degree of freedom for each natural shape. HVP requires that the action defined in Eq. (4) have a minimum or stationary value over all possible shape variations, that is (Piersol and Paez 2010)

$$\delta A = \delta\int_{t_1}^{t_2} B(x,t)dt = \int_{t_1}^{t_2} \delta B(x,t)dt = \int_{t_1}^{t_2} \delta[T(x,t) - V(x,t)]dt = 0 \tag{5}$$

This expression of HVP in terms of δA utilizing the δ operator properties and integration by parts provides the (1) differential equation(s) of motion termed the Lagrange's equation(s), (2) boundary displacements (kinematic boundary conditions), (3) boundary forces (natural boundary conditions), and (4) eigenvalue solution form (Piersol and Paez 2010).

By substituting Eqs. (1) and (2) into Eq. (5) one can obtain

$$\delta A = \int_{t_1}^{t_2}\int_0^L [m(x)\dot{w}(x,t)\delta\dot{w} - EI(x)w''(x,t)\delta w'' - AG(x)w'(x,t)\delta w']dxdt - \int_{t_1}^{t_2} K_r w'(a,t)\delta w' dt \tag{6}$$

Applying integration by parts to Eq. (6) with respect to time and space, yields

$$\delta A = -\int_{t_1}^{t_2}\int_0^L \left[m(x)\dot{w}(x,t)\delta\dot{w} + \frac{\partial}{\partial x^2}(EI(x)w''(x,t)) - \frac{\partial}{\partial x}(AG(x)w'(x,t)) \right]\delta w dxdt - \int_{t_1}^{t_2} K_r w'(a,t) + EI(x)w''(x,t)]\delta w'|_0^L dt + \int_{t_1}^{t_2}\left[\left(\frac{\partial}{\partial x}EI(x)w''(x,t)\right] - AG(x)w'(x,t)\right)\delta w|_0^L dt \tag{7}$$

The equation of vibration and boundary conditions from Eq. (7) can be obtained as follows

$$m(x)\dot{w}(x,t) + \frac{\partial^2}{\partial x^2}[EI(x)w''(x,t)] - \frac{\partial}{\partial x}[AG(x)w'(x,t)] = 0 \quad \text{for} \quad 0 \le x \le L, t \ge 0 \tag{8}$$

and the boundary conditions are

$$\begin{aligned} \frac{\partial}{\partial x}[EI(x)w''(x,t)] - AG(x)w'(x,t) &= 0 \quad \text{at} \quad x = L \\ K_r w'(a,t) + EI(x)w''(x,t) &= 0 \quad \text{at} \quad x = L \\ w(x,t) &= 0 \quad \text{at} \quad x = 0 \\ w'(x,t) &= 0 \quad \text{at} \quad x = 0 \end{aligned} \tag{9}$$

where the first two equations in Eq. (9) are boundary forces and the last are boundary displacements.

Using the method of separation of variables

$$w(x,t) = W(x)\sin(\omega t) \tag{10}$$

where ω is the circular frequency and $W(x)$ is the mode shape function.

Substituting Eq. (10) into Eq. (8) and dividing by $\sin(\omega t)$ yields

$$-m(x)\omega^2 W(x) + EI''(x)W''(x) + 2EI'(x)W'''(x) + EI(x)W''''(x) - AG'(x)W'(x) - AG(x)W''(x) = 0 \tag{11}$$

To simplify Eqs. (9) and (11), non-dimensional parameters are introduced as follows

$$\xi = \frac{x}{L} \Rightarrow \quad \text{for} \quad 0 \le x \le L \quad \text{then} \quad 0 \le \xi \le 1$$

$$\frac{d^{(n)}}{dx^{(n)}} = \frac{1}{L^n}\frac{d^{(n)}}{d\xi^{(n)}} \tag{12}$$

It is difficult to find the general solution of Eq. (11), because the structural parameters in the equation vary with the co-ordinate x. However, the general solution of Eq. (11) can be obtained by making appropriate selections for mass, flexural and shear stiffness distribution functions. As suggested by Wang (1978), Tuma and Cheng (1983), Li *et al.* (2000), the functions can be used to approximate the variation of mass and stiffness can be algebraic polynomials, exponential functions, trigonometric series, or other combinations. In this paper, the power functions considered for distribution of mass, flexural and shear stiffness are as follows

$$\begin{aligned} EI(x) &= EI_0(1+\beta x)^{m+2} \\ AG(x) &= AG_0(1+\beta x)^{m+1} \\ m(x) &= m_0(1+\beta x)^{m} \end{aligned} \tag{13}$$

In which EI_0, AG_0 and m_0 are the flexural stiffness, shear stiffness and mass per unit height at $x = 0$, respectively; β and m are parameters which can be determined by values of $EI(x)$, $AG(x)$ and $m(x)$ at $x = L/2$ and L or at other control points.

Substituting Eqs. (12) and (13) into Eq. (11), the following equation is obtained

$$\begin{aligned} &-m_0\,(1+\beta L\xi)^m L^4\omega^2\, W(\xi) + \beta^2 L^2 EI_0\,(m+2)(m+1)(1+\beta L\xi)^m\, W''(\xi) \\ &+2\,EI_0\beta\, L(m+2)\,(1+\beta L\xi)^{m+1} W'''(\xi) + EI_0(1+\beta L\xi)^{m+2}\, W''''(\xi) \\ &-\beta L^3 AG_0(m+1)(1+\beta L\xi)^m\ \ W'(\xi) - AG_0 L^2(1+\beta L\xi)^{m+1}\ \ W''(\xi) = 0 \end{aligned} \tag{14}$$

To simplify Eq. (14), new parameters are introduced as follows

$$\begin{aligned} \eta &= 1+\beta L\xi \\ \frac{d^{(n)}}{d\xi^{(n)}} &= (\beta L)^n \frac{d^{(n)}}{d\eta^{(n)}} \\ \omega^2 &= \lambda \end{aligned} \tag{15}$$

Substituting Eq. (15) into Eq. (14), the following equation is obtained

$$\begin{aligned} &-m_0\lambda\eta^m\, W(\eta) + \beta^4 EI_0\,(m+2)(m+1)\eta^m\, W''(\xi) \\ &+2\,EI_0\beta^4(m+2)\,\eta^{m+1} W'''(\xi) + \beta^4 EI_0\eta^{m+2}\, W''''(\xi) \\ &-\beta^2 AG_0(m+1)\eta^m\, W'(\xi) - \beta^2 AG_0\eta^{m+1}\ \ W''(\xi) = 0 \end{aligned} \tag{16}$$

and by substituting Eqs. (10), (12), (13) and (15) into Eq. (9), one can obtain

$$\begin{aligned} &W\,(\eta=1) = 0 \\ &W'\,(\eta=1) = 0 \\ &K_r\beta\, W'\,(\eta = 1+\beta a) + \beta^2 EI_0(\eta = 1+\beta L)^{m+2}\, W''(\eta = 1+\beta L) = 0 \\ &\beta^3(m+2)\, EI_0(\eta = 1+\beta L)^{m+1} W''\,(\eta = 1+\beta L) + \beta^3 EI_0(\eta=1+\beta L)^{m+2} W'''\,(\eta = 1+\beta L) \\ &-\beta AG_0(\eta = 1+\beta L)^{m+1} W'\,(\eta = 1+\beta L) = 0 \end{aligned} \tag{17}$$

It is evident that Eq. (16) is a homogeneous ordinary differential equation with variable coefficients. The problem can be solved exactly by the power-series solution method as follows

$$W(\eta)=\sum_{n=0}^{\infty} w_n \eta^n$$

$$W'(\eta)=\sum_{n=1}^{\infty} n\, w_n \eta^{n-1}$$

$$W''(\eta)=\sum_{n=2}^{\infty} n(n-1)\, w_n \eta^{n-2}$$

$$W'''(\eta)=\sum_{n=3}^{\infty} n(n-1)(n-2)\, w_n \eta^{n-3}$$

$$W''''(\eta)=\sum_{n=4}^{\infty} n(n-1)(n-2)(n-3)\, w_n \eta^{n-4} \tag{18}$$

Eq. (16) can be expressed in power series form as follows

$$\beta^4 EI_0(m+1)(m+2)\sum_{n=2}^{\infty} n\,(n-1)\, w_n\, \eta^{n+m-2} + 2\ EI_0\,\beta^4 (m+2)$$
$$\sum_{n=3}^{\infty} n\,(n-1)\,(n-2)\, w_n\, \eta^{n+m-2} + \beta^4 EI_0 \sum_{n=4}^{\infty} n\,(n-1)\,(n-2)\,(n-3)\, w_n\, \eta^{n+m-2}$$
$$-\beta^2 AG_0\,(m+1)\sum_{n=1}^{\infty} n\, w_n\, \eta^{n+m-1} - \beta^2 AG_0 \sum_{n=2}^{\infty} n\,(n-1)\, w_n \eta^{n+m-1} - m_0\,\lambda \sum_{n=2}^{\infty} w_{n-2}\, \eta^{n+m-2} = 0 \tag{19a}$$

By shifting the indices, one can obtain a series expression for the ordinary differential with variable coefficients as

$$\sum_{n=4}^{\infty} [\beta^4 EI_0(m+1)(m+2)n(n-1)w_n + 2EI_0\beta^4 (m+2)n(n-1)(n-2)w_n$$
$$+\beta^4 EI_0 n(n-1)(n-2)(m-3)w_n - \beta^2 AG_0\,(m+1)(n-1)w_{n-1} - \beta^2 AG_0(n-1)(n-2)w_{n-1}$$
$$-m_0\,\lambda w_{n-2}]\,\eta^{n+m-2} = 0 \tag{19b}$$

Using the identity property, for n = 0, 1, 2 and 3, the coefficients can be obtained as

$$w_0 = \frac{2\beta^4 EI_0(m+1)(m+2)w_2 - \beta^2 AG_0(m+1)w_1}{m_0\lambda}$$

$$w_3 = \frac{2\beta^2 AG_0(m+2)w_2 + m_0\lambda w_1}{6\beta^4 EI_0(m+2)(m+3)}$$

$$w_1$$

$$w_2 \tag{19c}$$

and the recursion relationships for any $n \geq 4$ can be determined as

$$w_n = \frac{\beta^2 AG_0\,(n-1)\,(m+n-1)\, w_{(n-1)} + m_0\,\lambda\, w_{(n-2)}}{\beta^4 EI_0[\,(m+1)\,(m+2)\,n\,(n-1) + 2\,(m+2)\,n\,(n-1)\,(n-2) + n\,(n-1)\,(n-2)\,(n-3)]} \tag{19d}$$

The power-series solution of mode shape functions can be obtained by solving Eqs. (19c) and (19d) as follows

$$W(\eta) = C_1 W_1(\eta,\lambda) + C_2 W_2(\eta,\lambda) + C_3 W_3(\eta,\lambda) + C_4 W_4(\eta,\lambda)$$

$$W_1(\eta,\lambda) = 1$$

$$W_2(\eta,\lambda) = x$$

$$W_3(\eta,\lambda) = x^2 + \frac{m_0\, \lambda\, x^4}{\beta^4 EI_0\,[\,12\,(m+1)\,(m+2) + 48\,(m+2) + 24\,]}$$

$$[\,1 + \frac{4\,\beta^2 AG_0\,(m+4)\,x}{\beta^4\ EI_0\,[\,20\,(m+1)\,(m+2) + 120\,(m+2) + 120\,]}] + \ldots$$

$$W_4(\eta,\lambda) = x^3 + \frac{3\beta^2 AG_0(m+3)x^4}{\beta^4 EI_0\,[\,12(m+1)(m+2) + 48(m+2) + 24\,]}$$

$$[\,1 + \frac{4\beta^2 AG_0(m+4)x}{\beta^4\ EI_0\,[\,20\,(m+1)\,(m+2) + 120\,(m+2) + 120\,]}]$$

$$+ \frac{m_0\, \lambda\, x^5}{\beta^4\ EI_0\,[20\,(m+1)\,(m+2) + 120\,(m+2) + 120\,]} + \ldots \qquad (20)$$

In Eq. (20), coefficients C_1, C_2, C_3 and C_4 are determined by applying the boundary conditions. From Eq. (17), the characteristic equation of the boundary value problem can be derived to obtain non-trivial solutions of the system; to this end, the determinant $D4 \times 4$ of the matrix of coefficients is set to zero

$$D_{4\times4}(\lambda) = \begin{vmatrix} d_{11} & d_{12} & d_{13} & d_{14} \\ d_{21} & d_{22} & d_{23} & d_{24} \\ d_{31} & d_{32} & d_{33} & d_{34} \\ d_{41} & d_{42} & d_{43} & d_{44} \end{vmatrix} = 0 \qquad (21a)$$

where the coefficients in Eq. (21a) are as follows

$$a_{11} = W_1(1,\lambda) \quad a_{12} = W_2(1,\lambda) \quad a_{13} = W_3(1,\lambda) \quad a_{14} = W_4(1,\lambda)$$

$$a_{21} = W_1'(1,\lambda) \quad a_{22} = W_2'(1,\lambda) \quad a_{23} = W_3'(1,\lambda) \quad a_{24} = W_4'(1,\lambda)$$

$$a_{31} = K_r\ \beta\ W_1'(1+\beta a,\lambda) + \beta^2 EI_0(1+\beta L,\lambda)^{m+2}\ W_1''(1+\beta L,\lambda)$$

$$a_{32} = K_r\ \beta\ W_2'(1+\beta a,\lambda) + \beta^2 EI_0(1+\beta L,\lambda)^{m+2}\ W_2''(1+\beta L,\lambda)$$

$$a_{33} = K_r\ \beta\ W_3'(1+\beta a,\lambda) + \beta^2 EI_0(1+\beta L,\lambda)^{m+2}\ W_3''(1+\beta L,\lambda)$$

$$a_{34} = K_r\ \beta\ W_4'(1+\beta a,\lambda) + \beta^2 EI_0(1+\beta L,\lambda)^{m+2}\ W_4''(1+\beta L,\lambda)$$

$$a_{41} = \beta^3 (m+2)\ EI_0(1+\beta L,\lambda)^{m+1} W_1''(1+\beta L,\lambda) + \beta^3 EI_0(1+\beta L,\lambda)^{m+2} W_1'''(1+\beta L,\lambda)$$

$$-\beta AG_0(1+\beta L,\lambda)^{m+1} W_1'\ (1+\beta L,\lambda)$$

$$a_{42} = \beta^3(m+2)\, EI_0(1+\beta L,\lambda)^{m+1} W_2''(1+\beta L,\lambda) + \beta^3 EI_0(1+\beta L,\lambda)^{m+2} W_2'''(1+\beta L,\lambda)$$
$$-\beta AG_0(1+\beta L,\lambda)^{m+1} W_2'(1+\beta L,\lambda)$$
$$a_{43} = \beta^3(m+2)\, EI_0(1+\beta L,\lambda)^{m+1} W_3''(1+\beta L,\lambda) + \beta^3 EI_0(1+\beta L,\lambda)^{m+2} W_3'''(1+\beta L,\lambda)$$
$$-\beta AG_0(1+\beta L,\lambda)^{m+1} W_3'(1+\beta L,\lambda)$$
$$a_{44} = \beta^3(m+2)\, EI_0(1+\beta L,\lambda)^{m+1} W_4''(1+\beta L,\lambda) + \beta^3 EI_0(1+\beta L,\lambda)^{m+2} W_4'''(1+\beta L,\lambda)$$
$$-\beta AG_0(1+\beta L,\lambda)^{m+1} W_4'(1+\beta L,\lambda) \tag{21b}$$

Eq. (21) is the frequency equation for free vibration of a combined system consisting of framed tube, shear core, belt truss and outrigger system. The natural frequency can be determined by solving Eq. (21).

Equivalent stiffness of rotational linear spring K_r can be given by Lee *et al.* (2008)

$$K_r = 1/\theta \tag{22}$$

where θ denotes the total rotation of the outrigger and belt truss system due to the restraining moment, and can be obtained by splitting up the action (Lee *et al.* 2008)

$$\theta = \theta_a + \theta_b + \theta_s \tag{23}$$

First, the restraining forces in the exterior columns will cause rotation of the outrigger resulting from the axial lengthening and shortening of the columns. The outrigger rotation θ_a due to the resulting restraining moment can then be defined as the column's change in length divided by the length of the outrigger (d) (Lee *et al.* 2008)

$$\theta_a = (2a)/(d^2 AE) \tag{24}$$

where d and AE are the distance between center to center of exterior columns and the axial stiffness of the exterior columns, respectively (Lee *et al.* 2008).

Flexural deformation of outrigger due to the action of the column force will cause additional drifts between adjacent floors. The resulting rotation θ_b is given by (Lee *et al.* 2008)

$$\theta_b = (d)/(12EI_{oe}) \tag{25}$$

where EI_{oe} is the effective flexural stiffness of the outrigger, as though its length extended from the column to the centroid of the core. EI_{oe} can be obtained from the outrigger's actual flexural rigidity EI_{ou} by converting the flexural rigidity of a wide-column beam to that of the equivalent full-span beam as follows (Smith and Salim 1983)

$$EI_{oe} = EI_{ou}(1+((b_c/2)/((d-b_c)/2)))^3 \tag{26}$$

where b_c is length of the shear core and EI_{ou} can be calculated by using theory of parallel axes.

The rotation due to shear force in the outrigger and belt truss system θ_s results from strain in diagonals, and can be expressed, (Lee *et al.* 2008), as

$$\theta_s = [1/(hAG_{ou})] \tag{27}$$

where h is the height of the outrigger and AG_{ou} is racking shear stiffness of the outrigger and belt truss system. This racking shear stiffness can be calculated for specific outrigger truss type. The racking shear stiffness is a property whose method of determination is most particular to the type of bent. It depends on deformation of web members as structure racks under the action of shear. It should be noted that the vertical members do not have any influence on racking shear stiffness of the segment. For various types of bent, value of AG_{ou} has been given by Smith and Coull (1991).

The value of K_r which corresponds to stiffness of the spring located at $x = a$ can be derived as follows

$$K_r = [(2a)/(d^2AE) + (d)/(12EI_{oe}) + (1/hAG_{ou})]^{-1} \tag{28}$$

where A is the area of the exterior columns at the position of belt truss and outrigger system that are perpendicular to the direction of structure's dynamic displacement.

3. Accuracy of the results

A numerical example is given to demonstrate the ease of application and illustrate the accuracy of the proposed approximate method. A high-rise 40 storey reinforced concrete consists of framed tube, shear core and belt truss is analyzed. All dimensions of beam and column members that vary with structure's height have been listed in Table 1. All dimensions of the outrigger and belt truss system are the same and equal to 0.8×0.8 m. Member spacing of the outrigger and belt truss system as shown in (Fig. 2) are $s_v = s_h = 5$ m, $s_{0h} = s_{ov} = 2.5$ m, $s_b = 2.5$ m. Height of each storey is 3.0 m and the center-to-center spacing of the columns (S) is 2.5 m. The Young's and shear modulai

Table 1 Properties of the combined system of framed tube, shear core, belt truss and outrigger system

No. Storey	1	40
Height from the base of the structure (m)	3	120
Dimensions of beams (cm)	100	61
Dimensions of columns (cm)	100	61
A_{eft} (m^2)	24	8.9304
A_{sc} (m^2)	6.72	2.2968
I_{sc} (m^4)	50.72	18.3688
EI_t (kg·m^2)	1.6002×10^{13}	6.0985×10^{12}
AG_t (kg)	1.2271×10^{10}	2.7377×10^{9}
t_{sc} (cm)	60	22
G_{eft} (kg/m^2)	2.8281×10^{8}	9.6695×10^{7}
I_{eft} (m^4)	7793.2992	2971.0851

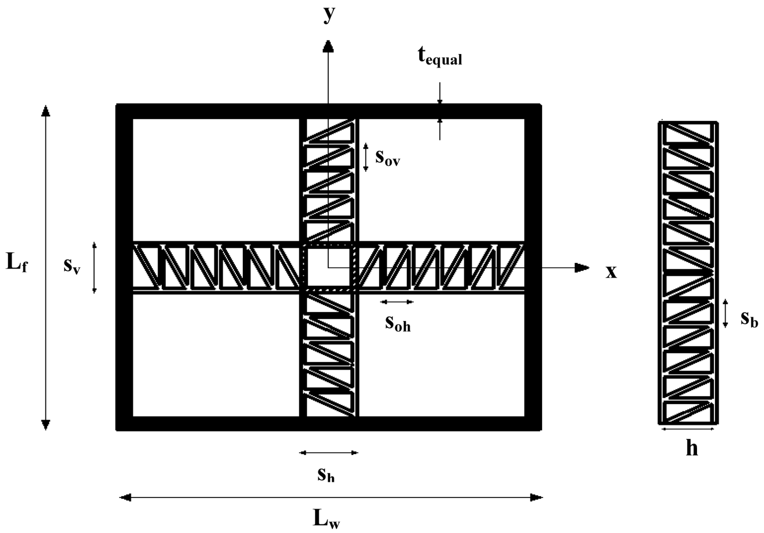

Fig. 2 Plane of outrigger and belt truss system

of flange and web panels are $E = 20.4 \times 10^8$ kg/m^2 and $G = 8.16 \times 10^8$ kg/m^2, respectively.

Other specifications that are used in the example are as follows:

$$L_f = 35 \text{ m}, L_w = 30 \text{ m}, \quad s = 2.5 \text{ m}, \quad \rho = 2400 \text{ kg/m}^3$$
$$t_{slab} = 0.3 \text{ m}, \quad L = 120 \text{ m}, \quad y = 3.0 \text{ m}$$

where y, t_{slab} and ρ are height of each storey, slab thickness for each storey and mass per unit volume of materials of tall building, respectively. Dimensions of core are 5.0 × 5.0 m. Thicknesses of shear core (t_{sc}) vary in height of structure. The Poisson ratio assumed to be 0.25.

By Using the method presented by Kwan (1994), and considering Fig. 1, the effective equivalent section area of plan for framed tube structure (A_{eft}) that varies in height of the structure, have been listed in Table 1 for two stories. In Fig. 1, the dimension of web and flange panel are shown and they do not vary with structure's height; but the total equivalent thickness of framed tube and shear core vary with the height of the structure and are determined as follows:

$$t_{tte} = t_{eft} + t_{sc}$$
$$t_{eft} = A_c/s$$

where t_{tte}, t_{eft} and t_{sc} are the total thickness of flange and web panels in the structure, the equivalent thickness of framed tube and the thickness of shear core, respectively and A_c is section area of column.

The moment of inertia for the plan about the Y-axis in shear core (I_{sc}) and moment of inertia of framed tube (I_{eft}) that vary with structure's height, have been listed in Table 1 for two stories.

The effective section area of shear core (A_{sc}) varies with structure's height and has been listed in Table 1 for two stories.

Equivalent elastic parameters for the analogous orthotropic membrane tube (G_{eft}), as evaluated by Kwan (1994), varies with height of the structure, and is listed in Table 1 for two stories.

Total flexural stiffness (EI_t) and the total shear stiffness (AG_t), vary with the height of the structure are listed in Table 1 for two stories.

$$EI_t = E \times (I_{SC} + I_{eft})$$
$$AG_t = (A_{Sc} \times G) + (A_{eft} \times G_{eft})$$
$$A_{Sc} = 2 \times t_{sc} \times (b_c + t_{sc})$$
$$A_{eft} = 2 \times L_w \times t_{eft}$$

By using Eq. (28), the value of K_r is calculated as follows

$$K_r = 9.4776 \times 10^9 \text{ kg m}, \quad AE = 253.3986 \times 10^8 \text{ kg}, \quad d = 30 \text{ m}$$
$$EI_{oe} = 1.7943 \times 10^{11} \text{ kg·m}^2, \quad AG_{out} = 936884843.9 \text{ kg}, \quad a = 30 \text{ m}$$

The structure has been analyzed and value of the first natural frequency in each case using the proposed approximate method are compared with the results of SAP2000 software (Advanced 12.0.0, Computers and Structures, Berkeley, California, USA). Where the properties of the structure at its base and top of the building are listed in Table 1. It should be noted that the dimensions of all sections such as columns, beams and shear core have decreased by 1 cm for each storey starting from the base of the structure. In other words, the thickness of each structural members vary at the rate of 1 cm per storey.

From Table 1, constants EI_0, AG_0, m_0, β and m are determined as

$$EI_0 = 1.6002 \times 10^{13} \text{ kg·m}^2, \quad AG_0 = 1.2271 \times 10^{10} \text{ kg}, \quad m_0 = 409931.1699 \text{ kg/m}$$
$$\beta = -0.0039055, \quad m = -0.4397047442$$

In this example the outrigger and belt truss system located at stories 9-11 or at height 27-33 m from the base of the structure.

By applying the data into Eqs. (20-28) and solving Eq. (21) through a numerical process, natural frequency ω_1 = 2.38 rad/s can be obtained. Using SAP2000 program (Advanced 12.0.0, Computers and Structures, Berkeley, California, USA) ω_1 is defined to be 2.47 rad/s. The proposed method, underestimates the natural frequency by 3.7%. The proposed method demonstrates the structural behavior rather well; it is easy to use, yet reasonably accurate and suitable for quick evaluations during the preliminary design stage. The main source of error in the proposed approximate method as compared to the finite element method are as follows: (1) modeling of the frame panels as equivalent orthotropic membranes (framed tube), so it can be analyzed as a continuous structure, (2) the equivalent elastic properties derived for the frame tube, shear core, and belt truss, (3) the equivalent stiffness of the rotational spring used to model the effect of belt truss and outrigger system on frame tube, (4) the approximation of EI and AG along the height of the structure, (5) the effect of shear lag has been neglected in approximate method.

4. Conclusions

In this paper, the simple approximate method has been developed for determining the first natural frequency of combined systems consisting of framed tube, shear core, belt truss and outrigger system. Using the Hamilton's variational principle, the governing equation for dynamic motion of the structure is obtained; and by selecting suitable functions for distribution of mass, flexural stiffness and shear stiffness, the governing equation is reduced to an ordinary differential equation with variable coefficients; which is solved using power-series functions. Accuracy of the proposed method is demonstrated by a numerical example. The numerical example shows that the approximate value of natural frequency obtained by the proposed method appears to be 3.7% smaller than the more accurate finite element method. From the point of view of a structural engineer, this error is within the acceptable range of engineering practice and therefore the proposed method may be used to estimate the natural frequency at preliminary stages of the structural's design which demands less time.

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