

Dynamic response analysis of generally damped linear system with repeated eigenvalues

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Abstract. For generally damped linear systems with repeated eigenvalues and defective eigenvectors, this study provides a decomposition method based on residue matrix, which is suitable for engineering applications. Based on this method, a hybrid approach is presented, incorporating the merits of the modal superposition method and the residue matrix decomposition method, which does not need to consider the defective characteristics of the eigenvectors corresponding to repeated eigenvalues. The method derived in this study has clear physical concepts and is easily to be understood and mastered by engineering designers. Furthermore, this study analyzes the applicability of step-by-step methods, including the Newmark beta and Runge-Kutta methods for dynamic response calculation of defective systems. Finally, the implementation procedure of the proposed hybrid approach is illustrated by analyzing numerical examples, and the correctness and the effectiveness of the formula are judged by comparing the results obtained from the different methods.

Keywords: damped system; repeated eigenvalues; transfer function; residue matrix; modal superposition method; defective system

1. Introduction

The modal superposition method is widely used in the dynamic response analysis of linear systems. This method can simplify the dynamic response analyses through decoupling the vibration equation based on the orthogonality of eigenvectors. The resulting complex multi-degree-of-freedom (MDOF) system can be turned into the linear superposition of the independent dynamic responses of a series of single degree of freedom (SDOF) systems subjected to identical ground motion. Also the modal analysis of structure can be used to carry out other research (Baek *et al.* 2011, Tan *et al.* 2011). In current seismic designs, based on proportional or classical damping assumptions, the square root of the sum of the squares (SRSS) and the complete quadratic combination (CQC) methods are proposed to calculate the dynamic response (Caughey 1960). However, there are many structures whose damping is non-uniform, for instance, soil-structure interacting systems and structures equipped with supplemental linear viscous dampers, such as oil dampers. Takewaki (2004) demonstrated that the structural energy transfer function and displacement transfer function

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will be underestimated if the over-damped modes are neglected. Over the years, a number of researchers have conducted extensive studies on developing complex modal superposition methods for systems that do not satisfy classical damping condition. Villaverde and Newmark (1980) developed a deterministic formulation for non-classically damped system by using complex frequencies and mode shapes. Igusa *et al.* (1984) studied the stationary response of multi-degrees-of-freedom (MDOF) non-classically damped linear systems subjected to stationary input excitations. Veletsos and Ventura (1986) presented a critical review of the modal superposition method of evaluating the dynamic response of non-classically damped structure. Singh *et al.* (1986) studied the modal time history analysis approach for the non-classically damped structures subjected to seismic forces. Yang *et al.* (1988) used a real-valued canonical transformation approach to decouple a non-classically damped system from a set of second order differential equations to a set of first order ones, and then performed a time history analysis as well as a response spectrum analysis. Zhou *et al.* (2004), Song *et al.* (2007a, b) derived the complex complete quadratic combination (CCQC or GCQC) method completely in real form for generally non-classically damped linear systems, and the complex square root of the sum of the squares (CSRSS) method for systems with well separated natural frequencies, in which the responses among different modes can be regarded as uncorrelated. However, for either classically damped systems or non-classically damped systems, the problems of repeated eigenvalues have not yet attracted sufficient attention from the earthquake engineering community. In the current seismic design code for structures, it is usual to design structures without equal or very close natural frequencies, mainly in terms of the following two considerations. First, if the amplitudes of the two responses are equal, the structural responses corresponding to the two equal natural frequencies will be the sum of the individual response, and the dynamic response will then be doubled. On the other hand, if the natural frequencies corresponding to the two responses are unequal, the dynamic response will only increase by 1.4 times. When a structure has repeated natural frequencies, the traditional method is to change the stiffness distribution of the structure slightly and make the natural frequencies differ from each other (Friswell *et al.* 2005). Second, when the structure has two or more sets of repeated eigenvalues and the corresponding eigenvectors are defective, there still have no appropriate response spectrum modal superposition method to calculate the earthquake response. Due to the first consideration, engineers in the earthquake engineering field usually do not need to deal with the problem of the dynamic response of the repeated eigenvalues. However, these issues change gradually along with the growth in structural size and configuration, and the variety and complexity of the systems. Very close and repeated frequencies are no longer unusual and sometimes even become inevitable. Moreover, it is worth considering whether the earthquake responses corresponding to the two equal frequencies can be offset each other or not. In fact, it is based on this consideration the tuned mass damper (TMD) was developed (Fujino and Abe 1993, Tsai 1993). In addition, problems of repeated eigenvalues have become common issues of many electrical and control systems. Inevitably, the methods of dealing with the dynamic responses of the system with repeated eigenvalues evolved, and a generalized modal analysis method has been proposed (Zhen 2002, Katsuhiko 2006, Chen 2007), which becomes the basis and reference for our study in this paper.

There are two purposes for doing this research. Firstly, we attempt to extend the method of the modal superposition method, which is widely used in earthquake engineering and structural dynamics, in systems with repeated eigenvalues. Secondly, we attempt to check if the widely used analysis methods for structural dynamic response, such as the Newmark beta and Runge-Kutta methods, can be applied directly to the cases when there are repeated eigenvalues, especially when

there are defective eigenvectors in the structural system. Hence, this paper provides a hybrid method based on orthogonality between different modes and on residue matrix calculations. This method fully uses the merits of the modal superposition method and the residue matrix decomposition method (Zhen 2002, Katsuhiko 2006), and does not consider the algebraic and geometric multiplicity of corresponding repeated eigenvalues, which makes it be easy for engineers to understand and master. It should be pointed out that the hybrid method is the basis for further research on modal response superposition methods based on the design response spectrum, that is, the CCQC and CQC methods. In fact, the residue matrix decomposition method can be treated as complementary to the modal superposition method, which gives the same result as the generalized modal superposition method. For a generally damped linear MDOF system, a formula for the dynamic response in time fields is obtained in this study, and the applicability of the step-by-step methods, including the Newmark beta and Runge-Kutta methods, for the dynamic response calculation of defective systems is discussed and verified. Finally, the steps to calculate structural response adopting the hybrid method derived in this study are given by analyzing numerical examples, and the validity and effectiveness of the formula are judged by comparing the results obtained from different methods of calculation. It is worth noting that the method presented in this paper is also based on general modal analysis, therefore, it is suitable for linear structures, electrical systems, proportional damping systems and non-proportional damping systems and can be regarded as an alternative generalized complex mode analysis method (Li 1985).

2. Decomposition technology of generally damped linear systems

It is well known that for a discrete system, with N degrees of freedom, the equations of motion in terms of nodal displacements are expressed as

$$\mathbf{M}\ddot{\mathbf{y}} + \mathbf{C}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = -\mathbf{M}\mathbf{e}\ddot{y}_g(t) \quad (1)$$

Here \mathbf{M} , \mathbf{C} and \mathbf{K} are the $N \times N$ mass, damping and stiffness matrices, which are symmetric matrices; \mathbf{y} is a $N \times 1$ nodal displacement vector which describes the dynamic response of the structure, and N is an arbitrarily large integer; \mathbf{e} is unit vector with dimension $N \times 1$, and $\ddot{y}_g(t)$ is the arbitrary time history of ground acceleration. If we introduce the equation $-\mathbf{K}\dot{\mathbf{y}} + \mathbf{K}\mathbf{y} = \mathbf{0}$, Eq. (1) can be rewritten as a group of first-order linear differential equations, that is

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\ddot{y}_g(t) \quad (2)$$

in which

$$\mathbf{A} = \begin{bmatrix} -\mathbf{M}^{-1}\mathbf{C} & -\mathbf{M}^{-1}\mathbf{K} \\ \mathbf{I} & \mathbf{0} \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} -\mathbf{e} \\ 0 \end{bmatrix}, \quad \mathbf{x} = \begin{bmatrix} \dot{\mathbf{y}} \\ \mathbf{y} \end{bmatrix} \quad (3)$$

and \mathbf{I} is identity matrix with dimension $N \times N$.

2.1 Repeated eigenvalues and generalized eigenvectors

The eigenvalues corresponding to the system expressed by Eq. (2) can be divided into two types,

i.e., distinct eigenvalues and repeated eigenvalues. According to linear system theory (Zhen 2002, Katsuhiko 2006), the multiplicity of an eigenvalue λ_i is called as algebraic multiplicity k_i , and the number of independent eigenvectors corresponding to the eigenvalue λ_i is termed as the geometric multiplicity g_i . Then $k_i = g_i = 1$ for the distinct eigenvalue λ_i , and $k_i \geq g_i \geq 1$ for the repeated eigenvalue. In fact, if the geometric multiplicity g_i is equal to algebraic multiplicity k_i , all the eigenvectors corresponding to the repeated eigenvalue are linearly independent and mutually orthogonal, and in this case, although the repeated eigenvalue exists, the modal superposition method based on the orthogonality of eigenvectors is still effective. For example, for a translation-torsion system with repeated eigenvalues, the eigenvector corresponding to the translation will be independent of the vector related to the torsion.

However, when geometric multiplicity g_i is smaller than algebraic multiplicity k_i , the number of linearly independent eigenvectors corresponding to a repeated eigenvalue λ_i is less than the multiplicity k_i of the eigenvalue, and the system becomes defective. In this case, the number of linearly independent eigenvectors, that is, the geometric multiplicity g_i , can be calculated by the rank of the characteristic matrix, i.e.

$$g_i = 2N - \text{rank}(\lambda_i \mathbf{I} - \mathbf{A}) \quad (4)$$

in which, $2N$ is the dimension of the matrix \mathbf{A} , \mathbf{I} is the identity matrix whose dimension is the same as that of the matrix \mathbf{A} , $\text{rank}(\bullet)$ represents the rank of the matrix. In practice, it is not easy to determine the geometric multiplicity by calculating the rank of the characteristic matrix for large systems (Zhang *et al.* 2006, Chi *et al.* 2004). Because a defective system seldom occurs in structural engineering, researchers in the field of structural dynamics and earthquake engineering pay less attention to the problems related to defective systems. However, the importance of studying the structural dynamic problems of defective systems has been recognized because of larger and more complex structures being constructed, and because of the wide usage of structural vibration control technology.

2.2 Partial fraction expansion method based on the residue matrix

To avoid calculating the rank of the characteristic matrix and geometric multiplicity of the repeated eigenvalues, for a system with repeated eigenvalues, the dynamic responses relevant to the repeated eigenvalues can be solved by decoupling the transfer function based on calculating the residue matrices. Using the Laplace transformation, Eq. (1) can be turned into an algebraic equation in the complex field (the shortened ‘ λ ’ field) about the parameter $\lambda = -\alpha + i\beta$, i.e.

$$(\mathbf{M}\lambda^2 + \mathbf{C}\lambda + \mathbf{K})\mathbf{Y}(\lambda) = \mathbf{F}(\lambda) \quad (5)$$

That is

$$\mathbf{Z}(\lambda)\mathbf{Y}(\lambda) = \mathbf{F}(\lambda) \quad (6)$$

in which, $\mathbf{Z}(\lambda)$ is the impedance matrix in the λ -field of the system, which is a nonsingular and symmetric matrix for the restraint system and has an inverse matrix, hence we can get

$$\mathbf{Y}(\lambda) = \mathbf{Z}(\lambda)^{-1}\mathbf{F}(\lambda) = \mathbf{G}(\lambda)\mathbf{F}(\lambda) \quad (7)$$

in which

$$\mathbf{G}(\lambda) = \mathbf{Z}(\lambda)^{-1} = \frac{\text{adj}[\mathbf{Z}(\lambda)]}{\det[\mathbf{Z}(\lambda)]} = \frac{\mathbf{J}^*(\lambda)}{D(\lambda)} \quad (8)$$

namely the transfer functions of systems with multiple inputs and outputs. When external excitation and initial condition of a system are definitive, the dynamic responses of the system at arbitrary coordinates will depend on the transfer function matrix $\mathbf{G}(\lambda)$. In Eq. (8), $D(\lambda) = \det[\mathbf{Z}(\lambda)]$ is the determinant of the matrix $\mathbf{Z}(\lambda)$, which can be denoted as a $2N$ -order real coefficient polynomial of parameter λ , that is

$$D(\lambda) = \sum_{m=0}^{2N} b_m \lambda^m = b_0 + b_1 \lambda + b_2 \lambda^2 + \dots + b_{2N} \lambda^{2N} \quad (9)$$

Since $\mathbf{J}^*(\lambda) = \text{adj}[\mathbf{Z}(\lambda)]$ is the companion matrix of matrix $\mathbf{Z}(\lambda)$, which is a $N \times N$ symmetric matrix and the element $J_{pq}^*(\lambda)$ in the p -th row and q -th column of matrix $\mathbf{J}^*(\lambda)$ can be written as a $2(N-1)$ -order polynomial of parameter λ , that is

$$\mathbf{J}_{pq}^*(\lambda) = \sum_{m=0}^{2N-2} a_{pq}^m \lambda^m = a_{pq}^0 + a_{pq}^1 \lambda + a_{pq}^2 \lambda^2 + \dots + a_{pq}^{2N-1} \lambda^{2N-2} \quad (10)$$

Meanwhile the element $G_{pq}(\lambda)$ of the transfer function matrix $\mathbf{G}(\lambda)$ can be denoted as

$$G_{pq}(\lambda) = \frac{J_{pq}^*(\lambda)}{D(\lambda)} = \frac{a_{pq}^0 + a_{pq}^1 \lambda + a_{pq}^2 \lambda^2 + \dots + a_{pq}^{2N-1} \lambda^{2N-2}}{b_0 + b_1 \lambda + b_2 \lambda^2 + \dots + b_{2N} \lambda^{2N}} \quad (11)$$

namely all the elements $G_{pq}(\lambda)$ of the transfer function $\mathbf{G}(\lambda)$ of the system are rational functions in the λ -field. Physically $G_{pq}(\lambda)$ represents the response transfer coefficient of the system at p caused by a unit input $e^{\lambda t}$ at q . The equality $D(\lambda) = 0$ is called the characteristics equation of the system combined from the matrices \mathbf{M} , \mathbf{C} and \mathbf{K} , and we can get $2N$ roots, that is, eigenvalues or poles, in the complex field. Because the matrices \mathbf{M} , \mathbf{C} and \mathbf{K} of Eq. (1) are all real symmetric matrices, the eigenvalues obtained by calculating the characteristic equation of the corresponding Eq. (1) normally occur in complex conjugate pairs, and the distinct root can be regarded as a special case of repeated root of order 1. Moreover, the real root can also be regarded as a complex conjugate pair for which the imaginary part is zero.

Because $G_{pq}(\lambda)$ is a rational function of the parameter λ , Eq. (11) can be expanded as a partial fraction according to the roots of equality $D(\lambda) = 0$. For a better expression in the latter part of this section, for a system with the repeated eigenvalues, it is assumed that there are z -pairs ($z \leq N$) of complex-conjugate eigenvalues, $(\lambda_1, \bar{\lambda}_1), (\lambda_2, \bar{\lambda}_2), \dots, (\lambda_z, \bar{\lambda}_z)$, whose multiplicity are k_1, k_2, \dots, k_z , respectively, in which $k_1 + k_2 + \dots + k_z = N$, and then Eq. (11) can be expressed as

$$G_{pq}(\lambda) = \frac{J_{pq}^*(\lambda)}{D_M \prod_{m=1}^z (\lambda - \lambda_m)^{k_m}} = \sum_{m=1}^z \sum_{i=1}^{k_m} \left[\frac{R_{m,i}^{pq,l}}{(\lambda - \lambda_m)^i} + \frac{\bar{R}_{m,i}^{pq,l}}{(\lambda - \bar{\lambda}_m)^i} \right], \quad (12)$$

in which, $D_M = \det[\mathbf{M}]$ is determinant of the matrix \mathbf{M} ; k_m represents the multiplicity of the pole

λ_m ; $R_{m,i}^{pq,l}$ and $\bar{R}_{m,i}^{pq,l}$ are a pair of the desired conjugate constants and can be determined by the following formula

$$R_{m,i}^{pq,l} = \lim_{\lambda \rightarrow \lambda_m} \frac{1}{D_M(l-1)!} \frac{d^{l-1}}{d\lambda^{l-1}} \left[\frac{J_{pq}^*(\lambda)(\lambda - \lambda_m)^{k_m}}{\prod_{m=1}^z (\lambda - \lambda_m)^{k_m} (\lambda - \bar{\lambda}_m)^{k_m}} \right] \quad (13)$$

Here the superscript p and q of $R_{m,i}^{pq,l}$ represents the p -th row and q -th column of the residue matrix, respectively; the superscript l represents the order of the derivative in determining the residue corresponding to the term $(\lambda - \lambda_m)^i$, and the values of l are in decreasing order i.e., $l = k_m, k_m - 1, \dots, 1$; here we use the subscript $m = 1, 2, \dots, z$ to number the different eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_z$; i is the power index of the corresponding term $(\lambda - \lambda_m)^i$, the values can be $i = 1, 2, \dots, k_m$, i.e., in increasing order which is the opposite to l . Specifically, Eq. (12) can be expressed as

$$G_{pq}(\lambda) = \sum_{m=1}^z \left(\frac{R_{m,1}^{pq,k_m}}{(\lambda - \lambda_m)} + \frac{\bar{R}_{m,1}^{pq,k_m}}{(\lambda - \bar{\lambda}_m)} + \dots + \frac{R_{m,k_m-1}^{pq,2}}{(\lambda - \lambda_m)^{k_m-1}} + \frac{\bar{R}_{m,k_m-1}^{pq,2}}{(\lambda - \bar{\lambda}_m)^{k_m-1}} + \frac{R_{m,k_m}^{pq,1}}{(\lambda - \lambda_m)^{k_m}} + \frac{\bar{R}_{m,k_m}^{pq,1}}{(\lambda - \bar{\lambda}_m)^{k_m}} \right) \quad (14)$$

From above discussion, we can see that if there is only one independent eigenvector corresponding to every eigenvalue, i.e., $k_m = 1$, then obviously the eigenvalue is distinct and the number of the eigenvalue is equal to that of the eigenvector, and the system is non-defective. Whereas, if the multiplicity of the eigenvalue λ_m is more than one ($k_m > 1$), but only one independent eigenvector corresponds to this eigenvalue λ_m , then there exist defective vectors corresponding to the eigenvalue, and the number of defective vectors will be $k_m - 1$. In this case, the maximum value of the power index i of the corresponding term $(\lambda - \lambda_m)^i$ in Eq. (12) can reach k_m , and all the defective vectors must be derived from the independent vector, which are related to the independent vector and can be called derived-vectors. However, sometimes the repeated eigenvalue may have several independent vectors, which is different from the discussion above. For this case, the repeated eigenvalue will correspond to smaller groups, and the maximum value of the power index i corresponding to term $(\lambda - \lambda_m)^i$ in Eq. (12) will be less than the multiplicity k_m , that is $i < k_m$. In general, the system is not defective if the number of the independent eigenvectors is the same as the multiplicity of the repeated eigenvalue, and there is no need to calculate the derived-vectors, but if the defective eigenvector exists we need to determine the derived-vectors if the number of independent eigenvectors is less than the multiplicity. It can be concluded that the non-defective or independent eigenvectors corresponding to a repeated eigenvalue λ_m are orthogonal, and these eigenvectors are orthogonal to the eigenvectors of the other eigenvalues. That is, the orthogonal condition can be satisfied among the independent eigenvectors and there is no need to consider the multiplicity of eigenvalues.

The above analysis is considered from the aspect of generalized modal decomposition. In fact, because the transfer function of every group can be expressed by the Eq. (12) or Eq. (14), the contribution of every group in Eq. (12) can be combined according to the order of term $(\lambda - \lambda_m)^i$. The combined results will automatically include the contributions of the independent vectors and derived-vectors, therefore it is not necessary to distinguish the algebraic and geometric multiplicity and to calculate the eigenvectors and consider the defective characteristic of eigenvectors, the

advantages of which are obtained by the transfer function method based on the residue matrix. It can be concluded that all elements $G_{pq}(\lambda)$ in the transfer function matrix $\mathbf{G}(\lambda)$ can be obtained through calculating the residues, therefore, the transfer function matrix $\mathbf{G}(\lambda)$ can definitely be determined, and the corresponding calculation method can be called the residue matrix decomposition method.

Suppose the multiplicity of the total complex-conjugate pair of eigenvalues is $k_m = 1$, Eq. (12) can then be rewritten as

$$G_{pq}(\lambda) = \sum_{m=1}^N \left(\frac{R_{m,1}^{pq}}{\lambda - \lambda_m} + \frac{\bar{R}_{m,1}^{pq}}{\lambda - \bar{\lambda}_m} \right) \quad (15)$$

Because $k_m = 1$, the superscript l in a pair of conjugate residue matrices $R_{m,i}^{pq,l}$ and $\bar{R}_{m,i}^{pq,l}$ in Eq. (12) can be omitted, and the power index i is equal to 1. The corresponding equation to Eq. (13) can be expressed as

$$R_{m,1}^{pq} = \lim_{\lambda \rightarrow \lambda_m} \frac{1}{D_M} \left[\frac{J_{pq}^*(\lambda)(\lambda - \lambda_m)}{\prod_{m=1}^N (\lambda - \lambda_m)(\lambda - \bar{\lambda}_m)} \right]. \quad (16)$$

It can be deduced from Eq. (12) that the system can be divided into several small systems according to the different eigenvalues by using the residue matrix decomposition method based on partial-fraction expansion, and the system responses in the λ -domain can be expressed as a linear combination of contributions from all the different eigenvalues in terms of Eq. (12). The residue according to each eigenvalue or pole can be calculated by Eq. (13), or can be calculated through commercial software, such as MATLAB, MATHEMATICA, etc. The method is especially suitable for analyzing small systems. However, for large systems with thousands of degrees of freedom, the computation of poles and residues can be a time-consuming task, and impractical even when using commercial software. For the convenience of numerical computation, we propose a hybrid approach, which is described in the following subsection.

2.3 The hybrid method combining partial fraction expansion and modal analysis

The proportion of repeated roots in the total eigenvalues and the multiplicity of each repeated eigenvalue are usually limited, even in large systems with thousands of degrees of freedom. Hence, the total structural responses can be divided into two groups according to distinct and repeated eigenvalues, and the response contributions from each group can be calculated through different methods. Suppose the distinct conjugate eigenvalues of the system are $(\lambda_m, \bar{\lambda}_m)$ ($m = 1, 2, \dots, z$) with multiplicity k_m ($k_m \geq 1$). Now separate the z_1 -pairs of distinct eigenvalues λ_m with $k_m = 1$ from all the eigenvalues, and define the multiplicity k_m of the eigenvalues ($m = 1, \dots, z_1, z_1 + 1, z$) as follows through renumbering

$$\begin{cases} k_m = 1, & m \leq z_1 \\ k_m \geq 2, & z_1 + 1 < m \leq z \end{cases}$$

The displacement responses of the structure in the time domain can then be written as

$$y(t) = y_D(t) + y_M(t) = \sum_{m=1}^{z_1} \{y_D(t)\}_m + \sum_{m=z_1+1}^z \{y_M(t)\}_m \quad (17)$$

In Eq. (17), the first part $y_D(t)$ represents the linear combination of the displacement responses of the z_1 SDOF oscillators. We recommend the employment of the modal superposition method mentioned by Zhou *et al.* (2004) to calculate the first part in Eq. (17) in order to reduce the computation amounts. It can be proven that for the dynamic response corresponding to the case with the distinct eigenvalues, the structural responses calculated by using the modal superposition method are the same as those obtained from the residue matrix decomposition method. The second part $y_M(t)$ represents the linear combination of the displacement responses of the $z - z_1$ coupled systems, in which $\{y_M(t)\}_m$ is the response of the m -th coupled system. As mentioned above, if the system is decoupled based on the generalized modal decomposition method, it is necessary to calculate the geometric multiplicity of the eigenvalue λ_m and to determine the corresponding independent vectors and derived-vectors of each repeated eigenvalue, which is a time consuming task for a large system. Therefore, in this study, the residue matrix decomposition method mentioned in the preceding discussion is selected to calculate the response $y_M(t)$, in which the coupled system corresponding to a repeated eigenvalue will be handled as a coupled-system, and no longer decomposed into smaller systems. Then, the second part $y_M(t)$ in Eq. (17) can be expressed as

$$y_M(t) = \sum_{m=z_1+1}^z \sum_{i=1}^{k_m} \{y_M(t)\}_{mi} \quad (18)$$

in which, $\{y_M(t)\}_{mi}$ is the i -th order response corresponding to repeated eigenvalue λ_m with multiplicity k_m , which corresponds to the term $(\lambda - \lambda_m)^i$ in the transfer function expressed by Eq. (12). The computation procedure for the response $y_M(t)$ will be discussed in Section 3.

The preceding analysis reveals that the second part response $y_M(t)$ is composed of $z - z_1$ equations of k_m -order. Because $k_m \ll N$ in most cases, the solution of these k_m -order equations will be much more convenient than the direct solution of Eq. (1). Through the above analysis we know that, for each repeated eigenvalue, if the residue matrix decomposition method is adopted to calculate the structural dynamic responses, the system corresponding to a repeated eigenvalue can be treated as a subsystem when carrying out the solution of the structural response. In this case, it is not necessary to consider the geometric multiplicity of the eigenvalue and calculate the independent eigenvalues and derived-vectors, which is undoubtedly convenient in practical applications. In this study, the method of partial-fraction expansion of the transfer function based on calculating the residue matrix is called as the residue matrix decomposition method, and the method expressed by Eq. (17) is called as the hybrid method, which makes full use of the merits of the modal superposition method and residue matrix decomposition method, and can be treated as an alternative expression form of the generalized complex mode analysis method.

3. Dynamic responses in the time domain for generally damped linear system

3.1 Structural response contributions from distinct eigenvalues

Firstly, we discuss the calculation method for the displacement response $y_D(t)$ corresponding to z_1

SDOF systems in Eq. (17). In this study, the modal superposition method reported by Zhou *et al.* (2004) is proposed for calculating the displacement response contributions $y_D(t)$ instead of the partial fraction expansion method, i.e.

$$y_D(t) = \sum_{m=1}^{z_1} \hat{\mathbf{A}}_{m,1} q_{m,1}(t) + \hat{\mathbf{B}}_{m,1} \dot{q}_{m,1}(t) \quad (19)$$

in which: $\hat{\mathbf{A}}_{m,1}$ and $\hat{\mathbf{B}}_{m,1}$ are the generalized participation factors corresponding to the m -th distinct eigenvalue λ_m . The coefficients $\hat{\mathbf{A}}_{m,1}$ and $\hat{\mathbf{B}}_{m,1}$ can, in fact, be calculated based on the orthogonality of the eigenvectors provided by the characteristic equation. $q_{m,1}(t)$ and $\dot{q}_{m,1}(t)$ can be calculated by solving the following equation

$$\ddot{q}_{m,1}(t) + 2\zeta_m \omega_m \dot{q}_{m,1}(t) + \omega_m^2 q_{m,1}(t) = -\ddot{y}_g(t) \quad (20)$$

i.e., they are the displacement response and velocity response of the m -th SDOF oscillator, which represent the contribution of the m -th mode at an arbitrary time t , in which ω_m and ζ_m are the free vibration frequency and the corresponding critical damping ratio. It is noted that the subscripts $(m, 1)$ in the generalized participation factors represent the order number of the eigenvalue and the multiplicity, respectively. In the case of distinct eigenvalues, the multiplicity is equal to 1 as previously mentioned.

It can be proved that the results obtained using the residue matrix decomposition method are exactly identical to those from the modal superposition method, which is why we can use Eq. (19) to solve $y_D(t)$ in Eq. (17). As mentioned above, for the z_1 -pair of distinct conjugate complex eigenvalues, the element of transfer function described by Eq. (12) can be rewritten as

$$G_{pq}(\lambda) = \sum_{m=1}^{z_1} \left(\frac{R_{m,1}^{pq}}{\lambda - \lambda_m} + \frac{\bar{R}_{m,1}^{pq}}{\lambda - \bar{\lambda}_m} \right) \quad (21)$$

in which λ_m and $\bar{\lambda}_m$ are a pair of complex-conjugate eigenvalues, $R_{m,1}^{pq}$ and $\bar{R}_{m,1}^{pq}$ are the residues corresponding to the poles $(\lambda_m, \bar{\lambda}_m)$, and Eq. (13) can be simplified as

$$R_{m,1}^{pq} = \lim_{\lambda \rightarrow \lambda_m} \left[\frac{J_{pq}^*(\lambda)(\lambda - \lambda_m)}{D_M \left(\prod_{m=1}^{z_1} (\lambda - \lambda_m)(\lambda - \bar{\lambda}_m) \prod_{n=z_1+1}^z (\lambda - \lambda_n)^{k_n} (\lambda - \bar{\lambda}_n)^{k_n} \right)} \right] \quad (22)$$

If all the elements for distinct roots described by Eq. (21) are determined, the transfer function matrix can be written as

$$\mathbf{G}(\lambda) = \sum_{m=1}^{z_1} \left(\frac{\mathbf{R}_{m,1}}{\lambda - \lambda_m} + \frac{\bar{\mathbf{R}}_{m,1}}{\lambda - \bar{\lambda}_m} \right) \quad (23)$$

Substituting the transfer function matrix expressed by Eq. (23) into the Eq. (7), we get

$$Y_D(\lambda) = \sum_{m=1}^{z_1} \left(\frac{\mathbf{R}_{m,1}}{\lambda - \lambda_m} + \frac{\bar{\mathbf{R}}_{m,1}}{\lambda - \bar{\lambda}_m} \right) \mathbf{F}(\lambda) \quad (24)$$

Carrying out the inverse Laplace transformation on both sides of Eq. (24), gives the corresponding time domain responses as

$$y_D(t) = \sum_{m=1}^{z_1} \left(\mathbf{R}_{m,1} \int_0^t e^{\lambda_m(t-\tau)} f(\tau) d\tau + \bar{\mathbf{R}}_{m,1} \int_0^t e^{\bar{\lambda}_m(t-\tau)} f(\tau) d\tau \right) \quad (25)$$

Suppose the poles of the system are pairs of complex-conjugate values and have negative real parts, we then have

$$\lambda_m = -\alpha_m + i\beta_m \quad \text{and} \quad \bar{\lambda}_m = -\alpha_m - i\beta_m \quad (26)$$

in which $\alpha_m = \zeta_m \omega_m$ and $\beta_m = \omega_m \sqrt{1 - \zeta_m^2}$ are the damping coefficient and the damped frequency of the m -th mode, and ω_m and ζ_m are the free vibration frequency and the corresponding critical damping ratio.

Separating the real and imaginary parts of the poles and residue matrices in Eq. (25), and combining the contributions of the pairs of conjugate values, the structure response expressed by Eq. (25) can then be written as

$$y_D(t) = 2 \sum_{m=1}^{z_1} \left(\{u_m\} \int_0^t e^{-\alpha_m(t-\tau)} \cos \beta_m(t-\tau) \ddot{y}_g(\tau) d\tau - \{v_m\} \int_0^t e^{-\alpha_m(t-\tau)} \sin \beta_m(t-\tau) \ddot{y}_g(\tau) d\tau \right) \quad (27)$$

in which

$$\{u_m\} = \text{Re}(\mathbf{R}_{m,1}) \mathbf{M} \mathbf{e} \quad (27a)$$

$$\{v_m\} = \text{Im}(\mathbf{R}_{m,1}) \mathbf{M} \mathbf{e} \quad (27b)$$

Eq. (27) consists of two Duhamel integrations with respect to the sine and cosine terms and the Duhamel integration in terms of the sine is obviously the solution of Eq. (20), i.e.

$$q_{m,1}(t) = \frac{-1}{\beta_m} \int_0^t e^{-\zeta_m \omega_m(t-\tau)} \sin(\sqrt{1 - \zeta_m^2} \omega_m(t-\tau)) \ddot{y}_g(\tau) d\tau = - \int_0^t h_m(t-\tau) \ddot{y}_g(\tau) d\tau \quad (28)$$

Here $h_m(t)$ is the corresponding impulse response function, that is

$$h_m(t) = \frac{1}{\beta_m} e^{-\zeta_m \omega_m t} \sin \beta_m t \quad (28a)$$

By taking the derivative of the preceding Eq. (28), the Duhamel integration for $\cos \beta_m(t)$ in Eq. (27) can be expressed as

$$\int_0^t e^{-\zeta_m \omega_m(t-\tau)} \cos(\sqrt{1 - \zeta_m^2} \omega_m(t-\tau)) \ddot{y}_g(\tau) d\tau = -\zeta_m \omega_m q_{m,1}(t) - \dot{q}_{m,1}(t) \quad (29)$$

and thus Eq. (27) can be written as

$$y_D(t) = \sum_{m=1}^{z_1} \hat{\mathbf{A}}_{m,1}^* q_{m,1}(t) + \hat{\mathbf{B}}_{m,1}^* \dot{q}_{m,1}(t) \quad (30)$$

in which

$$\widehat{\mathbf{A}}_{m,1}^* = -2(\zeta_m \omega_m \{u_m\} - \beta_m \{v_m\}) \quad (30a)$$

$$\widehat{\mathbf{B}}_{m,1}^* = -2\{u_m\} \quad (30b)$$

The generalized participation factors given by Eqs. (30a) and (30b) are, in fact, consistent with the results obtained from Eq. (19), that is $\widehat{\mathbf{A}}_{m,1}^* = \mathbf{A}_{m,1}$ and $\widehat{\mathbf{B}}_{m,1}^* = \mathbf{B}_{m,1}$, which can be briefly explained as follows.

Suppose the residue matrix $\mathbf{R}_{m,1}$ corresponding to the distinct eigenvalue λ_m can be expressed as

$$\mathbf{R}_{m,1} = \frac{\mathbf{J}^*(\lambda_m)}{D'(\lambda_m)} \quad (31)$$

Multiplying both sides of the Eq. (8) by $\mathbf{D}(\lambda)\mathbf{Z}(\lambda)$, gives

$$\mathbf{D}(\lambda)\mathbf{I} = \mathbf{Z}(\lambda)\mathbf{J}^*(\lambda) \quad (32a)$$

$$\mathbf{D}(\lambda)\mathbf{I} = \mathbf{J}^*(\lambda)\mathbf{Z}(\lambda) \quad (32b)$$

If we let $\lambda = \lambda_m$, we obtain

$$\mathbf{Z}(\lambda_m)\mathbf{J}^*(\lambda_m) = 0 \quad (33a)$$

$$\mathbf{J}^*(\lambda_m)\mathbf{Z}(\lambda_m) = 0 \quad (33b)$$

Because λ_m is the zero value point, substituting $\lambda = \lambda_m$ into $(\lambda^2\mathbf{M} + \lambda\mathbf{C} + \mathbf{K})\Phi = 0$, we have

$$(\mathbf{M}\lambda_m^2 + \mathbf{C}\lambda_m + \mathbf{K})\Phi_m = \mathbf{Z}(\lambda_m)\Phi_m = 0 \quad (34)$$

Comparing Eq. (34) and Eq. (33a), we see that any column of the adjoining matrix $\mathbf{J}^*(\lambda_m)$ is linearly dependent on eigenvector Φ_m , that is, only a constant factor exists between any column of $\mathbf{J}^*(\lambda_m)$ and Φ_m . In addition, it can be seen from Eq. (31) that there is a constant factor between every column of the residue matrix $\mathbf{R}_{m,1}$ and eigenvector Φ_m . Similarly, we can prove that every row of the adjoining matrix is linearly dependent on the left eigenvector Δ_m^T . Due to the symmetrical characteristic of the associated mass, damping and stiffness matrices, we have $\Delta_m = \Phi_m^T$, that is, only a constant factor exists between the row of residue matrix $\mathbf{R}_{m,1}$ and the transpose eigenvector Φ_m^T of vector Φ_m . Based on the preceding discussion, the relationship between the adjoining matrix $\mathbf{J}^*(\lambda_m)$ and vector Φ_m can be expressed as

$$\mathbf{J}^*(\lambda_m) = \eta_m \Phi_m \Phi_m^T \quad (35)$$

in which η_m is a constant that needs to be determined and the following relationship between the residues matrix $\mathbf{R}_{m,1}$ and η_m can be obtained via simple derivation.

Taking the first-order derivative of λ on both sides of Eq. (32b), and letting $\lambda \rightarrow \lambda_m$, gives

$$\mathbf{D}'(\lambda_m)\mathbf{I} = [\mathbf{J}^*(\lambda_m)]'\mathbf{Z}(\lambda_m) + \mathbf{J}^*(\lambda_m)(2\lambda_m\mathbf{M} + \mathbf{C}) \quad (36)$$

Multiplying both sides of Eq. (36) by the eigenvector Φ_m , we obtain

$$\mathbf{D}'(\lambda_m)\Phi_m = [\mathbf{J}^*(\lambda_m)]'\mathbf{Z}(\lambda_m)\Phi_m + \mathbf{J}^*(\lambda_m)(2\lambda_m\mathbf{M} + \mathbf{C})\Phi_m \quad (37)$$

Substituting Eq. (35) into Eq. (37), and noting that $\mathbf{Z}(\lambda_m)\Phi_m = 0$, we get

$$\mathbf{D}'(\lambda_m)\Phi_m = \eta_m \bar{\eta}_m \Phi_m \quad (38)$$

in which

$$\bar{\eta}_m = \Phi_m^T(2\lambda_m\mathbf{M} + \mathbf{C})\Phi_m \quad (38a)$$

because the eigenvalue λ_m is a distinct eigenvalue, $\mathbf{D}'(\lambda_m) \neq 0$. Moreover, because the eigenvector $\Phi_m \neq 0$, we obtain the following equation from the Eq. (38) as

$$\eta_m = \frac{D'(\lambda_m)}{\bar{\eta}_m} \quad (39)$$

Substituting Eq. (39) into Eq. (35), we obtain

$$\mathbf{J}^*(\lambda_m) = \frac{D'(\lambda_m)}{\bar{\eta}_m} \Phi_m \Phi_m^T \quad (40)$$

and substituting Eq. (40) into Eq. (31), the relationship between the residue matrix $\mathbf{R}_{m,1}$ and vector Φ_m can be expressed as

$$\mathbf{R}_{m,1} = \frac{\Phi_m \Phi_m^T}{\bar{\eta}_m} \quad (41)$$

where

$$\bar{\eta}_m = \Phi_m^T(2\lambda_m\mathbf{M} + \mathbf{C})\Phi_m = a_m + ib_m \quad (42)$$

If we substitute Eq. (41) into $\{u_m\} = \text{Re}(\mathbf{R}_{m,1})\mathbf{M}\mathbf{e}$ and $\{v_m\} = \text{Im}(\mathbf{R}_{m,1})\mathbf{M}\mathbf{e}$ of Eq. (27), via necessary simplifications, it is easy to show that Eq. (27) is exactly identical to that of the modal superposition method proposed by Zhou *et al.* (2004).

3.2 Structural response corresponding to repeated eigenvalues

Next, we discuss the calculation of the displacement response contribution $y_M(t)$ corresponding to $z - z_1$ groups of repeated eigenvalues or poles, which can be obtained using the residue matrix decomposition method based on the partial-fraction expansion of the transfer function. If the multiplicity of the m -th pole of the structure is k_m , then for this coupled subsystem, the dynamic response of the structure can be written as

$$\{Y_M(\lambda)\}_m = \sum_{i=1}^{k_m} \left(\frac{\mathbf{R}_{m,i}}{(\lambda - \lambda_m)^i} + \frac{\bar{\mathbf{R}}_{m,i}}{(\lambda - \bar{\lambda}_m)^i} \right) \mathbf{F}(\lambda) = \sum_{i=1}^{k_m} \mathbf{G}_{mi}(\lambda) \mathbf{F}(\lambda) \quad (43)$$

in which λ_m and $\bar{\lambda}_m$ are pairs of conjugate poles or eigenvalues; $\mathbf{R}_{m,i}$ and $\bar{\mathbf{R}}_{m,i}$ ($i=1,2,\dots,k_m$) are the conjugate residue matrices of the corresponding terms $(\lambda-\lambda_m)^i$ and $(\lambda-\bar{\lambda}_m)^i$ respectively. For the pole with multiplicity k_m , each of the elements in the residue matrices can be determined by using Eq. (13). It is not difficult to calculate the corresponding residue through Eq. (13), provided all the poles of the system are determined; $\mathbf{G}_{mi}(\lambda)$ on the right side of Eq. (43) is the transfer function corresponding to the term $(\lambda-\lambda_m)^i$.

It can be seen from Eq. (43) that when there is a pole with multiplicity k_m in the structural system, the term with high order $(\lambda-\lambda_m)^i$ ($i>1$) will be included in the partial fraction of the transfer function which is different from that of distinct poles. If the inverse Laplace transformation is carried out on Eq. (43), the structural response corresponding to the m -th coupled system in the time domain can be expressed as

$$\{y_M(t)\}_m = \sum_{i=1}^{k_m} \left(\mathbf{R}_{m,i} \int_0^t (t-\tau)^{i-1} e^{\lambda_m(t-\tau)} f(\tau) d\tau + \bar{\mathbf{R}}_{m,i} \int_0^t (t-\tau)^{i-1} e^{\bar{\lambda}_m(t-\tau)} f(\tau) d\tau \right) \quad (44)$$

It is noted that the displacement vectors expressed by Eq. (44) are the result of the integration of the product of the residue matrix and load vector to time t . When $i > 1$, the equation contains the dimensional scalar function $(t-\tau)^{i-1}$, which means the residue matrix $\mathbf{R}_{m,i}$ in Eq. (44) has different dimensions at $i=1$ and $i>1$. For convenience of comparison, we assume the dimensionless factor $(t-\tau)^{i-1}$, and Eq. (44) can then be rewritten as

$$\{y_M(t)\}_m = \sum_{i=1}^{k_m} \left\{ \frac{1}{\beta_m^{i-1}} \left[\mathbf{R}_{m,i} \int_0^t [\beta_m(t-\tau)]^{i-1} e^{\lambda_m(t-\tau)} f(\tau) d\tau + \bar{\mathbf{R}}_{m,i} \int_0^t [\beta_m(t-\tau)]^{i-1} e^{\bar{\lambda}_m(t-\tau)} f(\tau) d\tau \right] \right\} \quad (45)$$

Similarly, we separate the real and imaginary parts of the poles and residue matrices in Eq. (45), and combine the contribution of pairs of conjugate values. Then the structure response given by Eq. (45) can be written as

$$\{y_M(t)\}_m = \sum_{i=1}^{k_m} \{y_M(t)\}_{mi} \quad (46)$$

in which, $\{y_M(t)\}_{mi}$ is the i -th order response of corresponding repeated eigenvalue λ_m with multiplicity k_m and can be expressed as

$$\begin{aligned} \{y_M(t)\}_{mi} &= 2 \{u_{mi}\} \int_0^t [\beta_m(t-\tau)]^{i-1} e^{-\alpha_m(t-\tau)} \cos \beta_m(t-\tau) \ddot{y}_g(\tau) d\tau \\ &\quad - 2 \{v_{mi}\} \int_0^t [\beta_m(t-\tau)]^{i-1} e^{-\alpha_m(t-\tau)} \sin \beta_m(t-\tau) \ddot{y}_g(\tau) d\tau \end{aligned} \quad (47)$$

in which

$$\{u_{mi}\} = \frac{1}{\beta_m^{i-1}} \text{Re}(\mathbf{R}_{m,i}) \mathbf{M} \mathbf{e} \quad (48a)$$

$$\{v_{mi}\} = \frac{1}{\beta_m^{i-1}} \text{Im}(\mathbf{R}_{m,i}) \mathbf{M} \mathbf{e} \quad (48b)$$

For the i -th term (the value of i is varied between 1 and k_m) of the repeated root, the

corresponding impulse response function can be written as

$$h_{mi}(t) = \frac{1}{\beta_m} (\beta_m t)^{i-1} e^{-\zeta_m \omega_m t} \sin \sqrt{1 - \zeta_m^2} \omega_m t \quad (49)$$

Note that the impulse transfer function of Eq. (49) has an additional dimensionless term $(\beta_m t)^{i-1}$ compared to the case with distinct roots.

For the convenience of calculating in the time domain, the i -order displacement response of the m -th repeated eigenvalue, which is the Duhamel integration for the sine term, can be written as

$$q_{m,i}(t) = -\frac{1}{\beta_{m0}} \int_0^t [\beta_m(t-\tau)]^{i-1} e^{-\zeta_m \omega_m(t-\tau)} \sin(\sqrt{1 - \zeta_m^2} \omega_m(t-\tau)) \ddot{y}_g(\tau) d\tau \quad (50)$$

To eliminate the Duhamel integration for $\cos \beta_m(t)$ in Eq. (47), we take the derivative of the preceding Eq. (50), and then obtain

$$\int_0^t [\beta_m(t-\tau)]^{i-1} e^{-\zeta_m \omega_m(t-\tau)} \cos \beta_m(t-\tau) \ddot{y}_g(\tau) d\tau = (i-1) \beta_m q_{m,i-1} - \zeta_m \omega_m q_{m,i} - \dot{q}_{m,i} \quad (51)$$

$$\text{Here } q_{m,i-1} = -\frac{1}{\beta_{m0}} \int_0^t [\beta_m(t-\tau)]^{i-2} e^{-\zeta_m \omega_m(t-\tau)} \sin(\sqrt{1 - \zeta_m^2} \omega_m(t-\tau)) \ddot{y}_g(\tau) d\tau$$

Obviously, Eq. (47) can be rewritten as

$$\{y_M(t)\}_{mi} = \tilde{\mathbf{E}}_{m,i} q_{m,i-1}(t) + \tilde{\mathbf{A}}_{m,i} q_{m,i}(t) + \tilde{\mathbf{B}}_{m,i} \dot{q}_{m,i}(t) \quad (52)$$

in which

$$\tilde{\mathbf{E}}_{m,i} = 2(i-1) \beta_m \{u_{mi}\} \quad (52a)$$

$$\tilde{\mathbf{A}}_{m,i} = -2 \zeta_m \omega_m \{u_{mi}\} + 2 \beta_m \{v_{mi}\} \quad (52b)$$

$$\tilde{\mathbf{B}}_{m,i} = -2 \{u_{mi}\} \quad (52c)$$

Because vectors $\tilde{\mathbf{E}}_{m,i}$, $\tilde{\mathbf{A}}_{m,i}$, $\tilde{\mathbf{B}}_{m,i}$ are the functions of the modal displacement in physical coordinates, they can also be used to express the other responses, such as the internal force and stress through linear transformation, where $q_{m,i-1}(t)$, $q_{m,i}(t)$, $\dot{q}_{m,i}(t)$ are time-varying functions. Hence, the structural response of the $z-z_1$ composite systems in the time domain can be expressed by the double summation of Eq. (52) as

$$\mathbf{y}_M(t) = \sum_{m=z_1+1}^z \sum_{i=1}^{k_m} (\tilde{\mathbf{E}}_{m,i} q_{m,i-1} + \tilde{\mathbf{A}}_{m,i} q_{m,i} + \tilde{\mathbf{B}}_{m,i} \dot{q}_{m,i}) \quad (53)$$

To sum up, the equation of the displacement response for the structure corresponding to Eq. (1) can now be written as

$$\mathbf{y}(t) = \sum_{m=1}^{z_1} \hat{\mathbf{A}}_{m,1} q_{m,1}(t) + \hat{\mathbf{B}}_{m,1} \dot{q}_{m,1}(t) + \sum_{m=z_1+1}^z \sum_{i=1}^{k_m} (\tilde{\mathbf{E}}_{m,i} q_{m,i-1} + \tilde{\mathbf{A}}_{m,i} q_{m,i} + \tilde{\mathbf{B}}_{m,i} \dot{q}_{m,i}) \quad (54)$$

in which the parameters are defined as previously mentioned. Eq. (54) shows that when $k_m = 1$, the generalized participation factor expressed by Eq. (52a) will be $\tilde{\mathbf{E}}_{m,i} = 0$. Eq. (53) can then be written as the single-summation formula as is shown in Eq. (19), and the dynamic response can be calculated by the method based on the orthogonality of the distinct eigenvalues (Zhou *et al.* 2004). Because the structural responses of the distinct eigenvalues can be treated as the case when the multiplicity of the eigenvalue is $k_m = 1$, the Eq. (54) can be generalized as the equation of the double-summation expression, i.e.

$$\mathbf{y}(t) = \sum_{m=1}^z \sum_{i=1}^{k_m} (\mathbf{E}_{m,i} q_{m,i-1} + \mathbf{A}_{m,i} q_{m,i} + \mathbf{G}_{m,i} \dot{q}_{m,i}) \tag{55}$$

in which $\mathbf{E} = [\mathbf{0}, \dots, \mathbf{0}, \tilde{\mathbf{E}}_{z_1+1,1}, \dots, \tilde{\mathbf{E}}_{z_1+1,k_{z_1+1}}, \dots, \tilde{\mathbf{E}}_{z,1}, \dots, \tilde{\mathbf{E}}_{z,k_z}]$

$$\mathbf{A} = [\hat{\mathbf{A}}_{1,1}, \dots, \hat{\mathbf{A}}_{z_1,1}, \hat{\mathbf{A}}_{z_1+1,1}, \dots, \hat{\mathbf{A}}_{z_1+1,k_{z_1+1}}, \dots, \hat{\mathbf{A}}_{z,1}, \dots, \hat{\mathbf{A}}_{z,k_z}]$$

$$\mathbf{A} = [\hat{\mathbf{B}}_{1,1}, \dots, \hat{\mathbf{B}}_{z_1,1}, \hat{\mathbf{B}}_{z_1+1,1}, \dots, \hat{\mathbf{B}}_{z_1+1,k_{z_1+1}}, \dots, \hat{\mathbf{B}}_{z,1}, \dots, \hat{\mathbf{B}}_{z,k_z}]$$

It is noted that although Eq. (55) incorporates the structural responses corresponding to the distinct and repeated eigenvalues, it does not hinder or prevent us from calculating the structural responses separately using different methods, such as the modal superposition method and residue matrix decomposition method. In fact, while using the proposed hybrid approach to conduct the dynamic analysis of large complex structures, we can first use any software to calculate all the eigenvalues and then distinguish the distinct and repeated ones. The next step is to calculate their corresponding eigenvectors and then to calculate dynamic responses by the modal superposition method i.e. Eq. (27) for all the distinct eigenvalues. The final step is to calculate the remaining dynamic responses using Eq. (53) corresponding to the repeated eigenvalues based on the residue matrix method. It is worth noting that in the case of using the residue matrix and partial fraction expansion method we do not need to distinguish the algebraic and geometric multiplicity. To calculate the independent eigenvector and derived-vectors for each repeated eigenvalue is not only time consuming, but can also be very difficult. This disadvantage can be avoided in the proposed hybrid approach.

4. Discussion on the applicability of numerical integration computation to the response of a defective system

For a linear MDOF system, the popular calculation methods can be divided into two categories, that is, the direct numerical integration computation methods (Newmark beta method, Runge-Kutta method, etc) and the modal superposition method based on generalized coordinates. For a system without defective characteristics, the direct numerical integration computation methods are suitable for the calculation of response under the earthquake input. In this section, we will give an example to demonstrate the applicability of these methods to defective systems by comparing the calculation results obtained by direct numerical integration computation methods and theoretical analysis.

The physical parameters of the system with the repeated eigenvalues are expressed as

Table 1 Maximum displacements calculated by the different methods (units: cm)

Mass	Exact value	Eq. (53)	RK method	Newmark- β method
m_1	0.035656	0.035656	0.035656	0.035660
m_2	0.143440	0.143440	0.143440	0.143450

$$\mathbf{M} = \begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 4 & 2\sqrt{2} \\ 2\sqrt{2} & 6 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 36 & \\ & 81 \end{bmatrix}$$

Transforming the motion equation into first-order differential equations as is shown in Eq. (2), the eigenvalues of a system can be obtained from the system state matrix \mathbf{A} as, $\lambda_{1,2} = -\frac{5}{2} \pm \frac{\sqrt{191}}{2}i$, which

is a pair of conjugate repeated eigenvalues. Because the system has two-degrees of freedom, it is clearly a defective system when multiple eigenvalues appear (Chen 1988).

Based on Eq. (13), the residue matrices $\mathbf{R}_{1,1}$ and $\mathbf{R}_{1,2}$ corresponding to the terms $(\lambda - \lambda_1)$ and $(\lambda - \lambda_1)^2$ can be obtained as

$$\mathbf{R}_{1,1} = \begin{bmatrix} -0.0909i & -0.0054i \\ -0.0054i & -0.0606i \end{bmatrix}, \quad \mathbf{R}_{1,2} = \begin{bmatrix} -0.1283 - 0.0362i & -0.0370 + 0.1023i \\ -0.037 + 0.1023i & 0.0812 + 0.0362i \end{bmatrix}$$

In Table 1, column 2 shows the theoretical solution of the displacements for two masses when the acceleration amplitude of a rectangular wave input is 1.0 (gal) over the time range (0- t), in which the total time length is $t = 5$ s and the time interval is $\Delta t = 0.01$ s. To verify Eq. (53) derived in this paper, we carried out a numerical analysis under the same input, and the results of the generalized participation factors are: $\tilde{\mathbf{E}}_{1,1} = 0$, $\tilde{\mathbf{E}}_{1,2} = [-0.3306, 0.0883]'$, $\tilde{\mathbf{A}}_{1,1} = [-1.3306, -0.9117]'$, $\tilde{\mathbf{A}}_{1,2} = [0.2519, 0.2451]'$, $\tilde{\mathbf{B}}_{1,1} \approx 0$ and $\tilde{\mathbf{B}}_{1,2} = [0.0478, -0.0128]'$.

For the decoupled SDOF system, the piecewise exact method (Clough and Penzien 1993) is selected to calculate the displacement and velocity responses. The results show that the displacements of the mass calculated by Eq. (53) are completely consistent with the theoretical solution.

For the defective system, we also use the common step-by-step integration methods, the Runge-kutta (RK) and Newmark beta methods, to analyze the dynamic responses directly in order to discuss their applicability to a defective system. Only the maximum values of displacements are listed in columns 4 and 5 of Table 1. The results show that the two methods are also suitable for the analysis the dynamic responses of the MDOF system with defective characteristic. It should be noted that the calculation results derived from the Runge-kutta method get closer to the theoretical solution, because of its fourth-order accuracy of calculation.

5. Numerical analysis and examination

Example 1:

To illustrate the calculation procedure of the hybrid method derived in this study, we first

introduce a three-degrees of freedom system and analyze its dynamic response. The physical parameters of the three-degrees of freedom system are

$$\mathbf{M} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{C} = \begin{bmatrix} 2 & -1 & \sqrt{3}/2 \\ -1 & 2 & -1/2\sqrt{3} \\ \sqrt{3}/2 & -1/2\sqrt{3} & 7/6 \end{bmatrix}, \quad \mathbf{K} = \begin{bmatrix} 4 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 5/6 \end{bmatrix}$$

The calculation obtained a pair of conjugate distinct eigenvalues at $\lambda_1 = -0.5833 \pm 2.1586i$ and repeated eigenvalues of order 2 at $\lambda_2 = -1 \pm 1i$ of the system. The corresponding structural circular frequencies are: $\omega_1 = 2.2361$ (rad/s) and $\omega_2 = 1.4142$ (rad/s), and the damping ratios are: $\zeta_1 = 0.2609$ and $\zeta_2 = 0.7071$. Moreover, it is known, by calculating the algebraic and geometric multiplicity of the eigenvalue λ_2 , that the system is defective (Chen 1988).

For this system, the displacement response corresponding to the distinct eigenvalue can be calculated by using Eq. (19), and the corresponding generalized participation factors are listed in column 2 of Table 2. The displacement response corresponding to the repeated eigenvalue is calculated using Eq. (53), in which the residue matrices corresponding to the terms of $(\lambda - \lambda_2)$ and $(\lambda - \lambda_2)^2$ are

$$\mathbf{R}_{2,1} = \begin{bmatrix} 0.0854 - 0.3360i & 0.0299 + 0.2160i & -0.0223 - 0.1958i \\ 0.0299 + 0.2160i & -0.0668 - 0.1069i & -0.0435 + 0.1142i \\ -0.0223 - 0.1958i & -0.0435 + 0.1142i & -0.0106 - 0.9810i \end{bmatrix}$$

$$\mathbf{R}_{2,2} = \begin{bmatrix} 0.0402 + 0.2153i & -0.0591 - 0.0781i & -0.2704 + 0.2048i \\ -0.0591 - 0.0781i & 0.0393 + 0.0194i & 0.0672 - 0.1360i \\ -0.2704 + 0.2048i & 0.0672 - 0.1360i & -0.4711 - 0.2329i \end{bmatrix}$$

The generalized participation factors derived from Eqs. (52a), (52b) and (52c) corresponding to $i = 1$ and $i = 2$ are listed in the columns 3 and 4 of Table 2.

To test the relationship of the modal superposition method and residue matrix decomposition method, the residue matrix of the distinct eigenvalue $\lambda_1 = -0.5833 + 2.1586i$ is calculated as

$$\mathbf{R}_{1,1} = \begin{bmatrix} -0.0854 - 0.0738i & -0.0299 - 0.1332i & 0.0223 - 0.0303i \\ -0.0299 - 0.1332i & 0.0668 - 0.1510i & 0.0435 - 0.0134i \\ 0.0223 - 0.0303i & 0.0435 - 0.0134i & 0.0106 + 0.0067i \end{bmatrix}$$

The generalized participation factors corresponding to λ_1 derived from Eqs. (52a), (52b) and (52c) are the same as the results obtained using Eq. (19), as are shown in column 2 of Table 2.

Table 2 Generalized modal participation factors

E	[0 0 0]'	[0 0 0]'	[-0.5788 0.0947 -1.3487]'
A	[-0.91623 -1.3787 -024888]'	[-0.81744 0.60748 -1.9723]'	[1.2627 -0.48406 1.0205]'
B	[0.1859 -0.1609 -0.1530]'	[-0.1859 0.1609 0.1530]'	[0.5788 -0.0947 1.3487]'

Table 3 Maximum displacements calculated by the different methods (units: cm)

Time interval	Mass	Newmark- β method	RK method	Eq. (55)
$\Delta t = 0.005$ s	m_1	0.075509	0.075512	0.075511
	m_2	0.085980	0.085983	0.085984
	m_3	0.10205	0.10205	0.10205
$\Delta t = 0.010$ s	m_1	0.075482	0.075494	0.075483
	m_2	0.085943	0.085958	0.085960
	m_3	0.10202	0.10200	0.10200

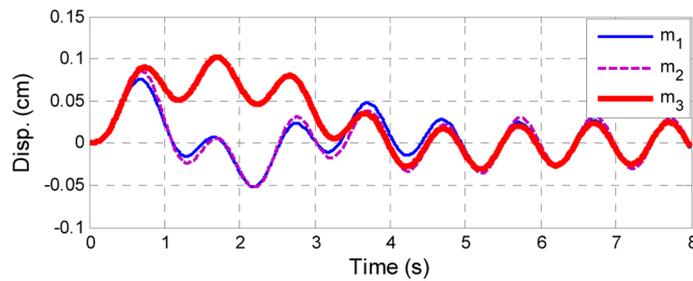


Fig. 1 Time histories of the displacements for three masses

Table 3 shows the maximum displacements of three masses calculated by the three methods when subjected to sine wave inputs, in which the input amplitude is $\sin 2\pi t$ (gal), and the time intervals are $\Delta t = 0.005$ s and $\Delta t = 0.010$ s, respectively. The piecewise exact method (Clough and Penzien 1993) is selected to calculate the dynamic response for the SDOF systems. It is clear that the maximum values of displacements are in accord with each other for all three methods, the differences between the calculation results can be identified in previous analysis. It is noted that the smaller the time interval Δt , the closer the results obtained from all the methods. Fig. 1 gives the displacement time histories of the three masses calculated by Eq. (55), in which only one of the three calculation results is shown because the three results are similar to each other, as is discussed previously.

Example 2:

A single-sphere network shell is considered in this example, whose diameter is 10 m and whose height is 5 m. The lattice layout of the network shell is of the ‘sunflower’ form. The material is seamless steel pipe with sectional dimensions given as: $\Phi 35 \times 2$ (dimensions of the bars of the inner ring of the network shell) and $\Phi 15 \times 2$ (the diagonal brace dimensions). Young’s modulus is 2.1×10^{11} N/m² for the materials and the uniformly distributed load is 200 kg/m². In addition, the bearing of the network is a fixed hinge bearing, and the plain and three-dimensional views are shown in Fig. 2 and Fig. 3. The serial numbers of the structural nodes are shown in Fig. 2. Six degrees of freedom were considered for every node. Therefore, there are 36 degrees of freedom in total. We defined the damping matrix as: $\mathbf{C} = \delta_1 \mathbf{M} + \delta_2 \mathbf{K}$, in which \mathbf{M} and \mathbf{K} are the structural mass and stiffness matrices. We let $\delta_1 = 0.1757$ (1/s) and $\delta_2 = 0.00173$ (s), and the first two damping ratios are then equal to 0.02.

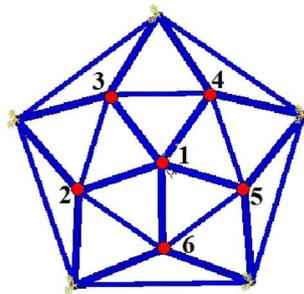


Fig. 2 Plan view and node layout

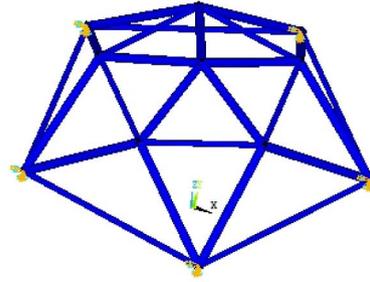


Fig. 3 Three-dimensional structural view

Table 4 Modal properties of the structure

Mode number	Frequency	Damping ratio (%)	Mode number	Frequency	Damping ratio (%)	Mode number	Frequency	Damping ratio (%)
1	25.399	2.017	13	33.361	2.096	25	552.60	39.29
2	25.399	2.017	14	37.303	2.174	26	552.60	39.29
3	26.391	2.018	15	37.303	2.174	27	824.67	33.02
4	26.391	2.018	16	58.004	2.758	28	974.11	39.03
5	26.705	2.019	17	64.642	3.003	29	974.11	39.03
6	28.305	2.030	18	64.642	3.003	30	976.34	42.82
7	28.305	2.030	19	265.46	11.28	31	976.34	42.82
8	30.012	2.047	20	311.53	22.62	32	1157.5	48.90
9	30.012	2.047	21	311.53	22.62	33	1172.6	47.54
10	31.336	2.064	22	410.65	40.20	34	1172.6	47.54
11	32.281	2.078	23	410.65	40.20	35	1651.0	66.84
12	32.281	2.078	24	450.41	39.80	36	1651.0	66.84

We then transform the network to a non-proportionally damped system by equipping it with supplemental dampers in the vertical direction of the nodes (2, 3, 4, 5, 6), which results in abrupt changes both in the stiffness and damping. The corresponding changes to the \mathbf{K} and \mathbf{C} matrices are such that: $\mathbf{K}(i, i) \leftarrow 1.05\mathbf{K}(i, i)$ and $\mathbf{C}(i, i) \leftarrow 2\mathbf{C}(i, i)$.

The NS component of the El-Centro earthquake acceleration was recorded in the May 18, 1940 earthquake in California, which contained energy over a broad range of frequencies, was used as a ground motion input. The dynamic response analysis is carried out in a Matlab platform by using the calculation method given by this study. The modal properties of the structure are given in Table 4, from which we can conclude that for this monolayer net-shell construction, there are 14 pairs of repeated frequencies, which include the first two modes. According to the hybrid method derived in this study, it is not necessary to consider defective characteristics for 14 pairs of repeated eigenvalues, and the system is divided into several small systems based on the different eigenvalues.

We calculated the dynamic responses according to the distinct eigenvalues and the repeated eigenvalues separately through the methods given in the study. Table 5 shows the maximum displacements in the X , Y and Z directions of the structure, among which columns 2 to 4 show the

Table 5 Maximum displacements in the X , Y and Z directions ($\times 10^{-3}$) (units: cm)

Node	Eq. (55)			Runge_Kutta method		
	X direction	Y direction	Z direction	X direction	Y direction	Z direction
1	4.5613	4.5613	7.0677	4.5613	4.5613	7.0677
2	4.9232	4.0443	4.8026	4.9232	4.0443	4.8026
3	1.2085	3.2500	2.9500	1.2085	3.2500	2.9500
4	5.2074	1.7091	1.4069	5.2074	1.7091	1.4069
5	3.0407	5.8071	1.9463	3.0407	5.8071	1.9463
6	1.9587	1.5278	1.9172	1.9587	1.5278	1.9172

maximum displacements of nodes that are calculated using Eq. (55). To justify the correctness of the method, the authors also calculated the dynamic response using the Runge-Kutta method, and the maximum displacements of the nodes are listed in columns 5 to 7 in Table 5. After comparing the results obtained from the two methods, it can be seen that the results from the generalized modal superposition method outlined in this paper are the same as those from the numerical integration computation method.

6. Conclusions

According to theoretical analysis and numerical investigations in this study, some important results and conclusions are obtained as follows:

- 1) For the generally damped linear MDOF system with repeated eigenvalues, a hybrid method combining the modal superposition method and the residue matrix decomposition method is proposed. The combination Eq. (55), completely in real form, is deduced, incorporating the effect and contribution of the repeated eigenvalues on the structural responses. The new algorithm decomposes the system according to the different eigenvalues, calculates the response corresponding to the distinct roots through the modal superposition method and calculates the responses related to the repeated roots through the residue matrix decomposition method based on partial-fraction expansion of the transfer function. Consequently, the process for identifying the algebraic and geometric multiplicity and calculating the independent and derived vectors of the multiple eigenvalues are omitted. This hybrid method is not only concise, but also easily to be understood by engineers.
- 2) The applicability of the step-by-step integration computation method, including the Newmark beta and Runge-Kutta methods, is analyzed for the dynamic response calculation of defective systems. The calculation steps of the hybrid method proposed in the paper are illustrated via analyzing numerical examples, and the correctness and effectiveness of the formula are formally verified by comparing the results obtained from the different calculation methods.
- 3) The hybrid method proposed in this paper is a universal method. It is suitable for linear structures, electrical systems, proportional and non-proportional damping systems. Consequently, it can be treated as an alternative expression form for the generalized complex mode analysis method.

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