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Estimating the Region of Attraction via collocation for autonomous nonlinear systems

M. Rezaiee-Pajand* and B. Moghaddasie

Department of Civil Engineering, Ferdowsi University of Mashhad, Mashhad 91175-1111, Iran

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Abstract. This paper aims to propose a computational technique for estimating the region of attraction (RoA) for autonomous nonlinear systems. To achieve this, the collocation method is applied to approximate the Lyapunov function by satisfying the modified Zubov's partial differential equation around asymptotically stable equilibrium points. This method is formulated for *n*-scalar differential equations with two classes of basis functions. In order to show the efficiency of the suggested approach, some numerical examples are solved. Moreover, the estimated regions of attraction are compared with two similar methods. In most cases, the proposed scheme can estimate the region of attraction more efficient than the other techniques.

Keywords: autonomous systems; Lyapunov function; Region of Attraction (RoA); modified Zubov's PDE; collocation method

1. Introduction

The characteristics of equilibrium points can explain some unpredictable behaviors of nonlinear dynamical systems. Time integration techniques reveal the system behavior for a particular initial condition (Rezaiee-Pajand and Alamatian 2008, 2010). Therefore, they are not able to draw a general picture of properties for nonlinear systems including initial independent parameters. Lyapunov based a powerful stability concept in nonlinear dynamical systems (Khalil 2002). Many efforts were made by researchers in this subject theoretically and practicaly in scientific areas and engineering (Lewis 2002, 2009, Tylikowski 2005, Pavlović *et al.* 2007).

A substantial issue in stability problems is to estimate the Region of Attraction (Stability Domain) around stable equilibrium points. For this purpose, several techniques have been proposed (Genesio *et al.* 1985). Some methods try to find an optimal Lyapunov function which gives less conservative estimation of stability domains (Tan 2006, Hafstein 2005, Chesi *et al.* 2005). These approaches are known as Lyapunov methods, which are extensively discussed in the literature (see, for example, Chesi 2007, Johansen 2000, Kaslik *et al.* 2005b). One applicable choice for estimating Lyapunov functions is the implementation of sum-of-squares with polynomial terms (Tan and Packard 2008, Peet 2009).

Zubov showed that the Lyapunov function giving the entire region of attraction satisfies a certain partial differential equation (Kormanik and Li 1972, Camilli *et al.* 2008). In most cases, it is impossible to find a closed form solution for this kind of PDE. But, it can be approximated by

*Corresponding author, Professor, E-mail: mrpajand@yahoo.com

power series (Margolis and Vogt 1963, Dobljević and Kazantzis 2002, Fermín Guerrero-Sánchez *et al.* 2009), rational solution (Vannelli and Vidyasagar 1985, Hachicho 2007, Giesl 2007) and other numerical techniques (see, for instance, Kaslik *et al.* 2005a, O'Shea 1964). Most of the numerical methods are applicable to autonomous systems (Sophianopoulos 1996, 2000). For nonautonomous systems, the averaging technique can be employed to transform these systems into autonomous ones with an acceptable accuracy (see, for example, Gilsinn 1975, Yang *et al*. 2010, Hetzler *et al*. 2007).

The focus of this paper is on the approximate solution of Zubov's partial differential equation by the collocation approach. For this purpose, some basis functions are introduced and formulated for *n*-dimensional problems. By using these basis functions, one can obtain a *smooth* Lyapunov function in the vicinity of the stable equilibrium point. Afterwards, the residual error in Zubov's partial differential equation is vanished in a number of discrete points through the collocation method. Finally, the unknown coefficients in the approximate Lyapunov function are calculated, and the conservative estimation of the stability domain is computed by solving an optimization problem. *The proposed scheme is applicable to both polynomial and non-polynomial systems*.

Section 2 illustrates Lyapunov theorem with some basic definitions. Furthermore, the procedure of estimating the stability domain around a stable equilibrium point is described. This procedure needs to be followed by finding a convenient Lyapunov function and using a proper global optimization method. Then, the theory of moments is presented in Section 3. It is noteworthy that this theory is useful for polynomial nonlinear systems with a polynomial Lyapunov function. Section 4 demonstrates Zubov's partial differential equation. In this section, the construction procedure is investigated for polynomial (Margolis and Vogt 1963) and rational Lyapunov functions (Vannelli and Vidyasagar 1985). Afterwards, the procedure of the collocation technique is explained in Section 5. Furthermore, two types of basis functions for *n*-dimensional problems are introduced in this section. The computational steps of the suggested technique are presented in Section 6. Then, some numerical examples are solved by the proposed method in Section 7. Concluding remarks are given in Section 8.

2. Region of attraction

In this section, the region of attraction (RoA) which is relative to a Lyapunov function is investigated. In this way, it is preferred to transform the governing equations of a dynamical system into a finite number of coupled first-order ordinary differential equations (Khalil 2002, Wiggins 2003)

$$\dot{x} = f(x, t), \quad x \in \mathrm{R}^{\mathrm{n}}, \quad t \in \mathrm{R} \tag{1}$$

where, $\dot{x}$ represents the time derivative of x. The vector differential Eq. (1) is called nonautonomous (or time dependent) system. This study is restricted to a subclass of (1) which is known as autonomous (or not time dependent) systems

$$\dot{x} = f(x), \quad x \in \mathrm{R}^{\mathrm{n}}, \quad t \in \mathrm{R} \tag{2}$$

Another noteworthy concept in stability theories is the equilibrium point. If $x = x_e$ stays at its initial position $x(t_0)$ (or x_0) for all future time, x_e is an equilibrium point of that dynamical system.

Based on this definition, the real roots of the following equations are the equilibrium points

$$f(x_e) = 0 \tag{3}$$

For more simplification, it is assumed that the origin is an equilibrium point ($f(0) = 0$). In the following, two necessary definitions are given.

Definition 2.1 *Stable, unstable and asymptotically stable* (Khalil 2002): The equilibrium point $x = 0$ of (2) is

- stable, if $\forall \varepsilon > 0$, there is a $\delta > 0$ such that

$$\|x_0\| < \delta \Rightarrow \|x(t, x_0)\| < \varepsilon, \ \forall t \geq t_0 \tag{4}$$

- unstable, if it is not stable.
- asymptotically stable, if it is *stable* and there is a $\delta > 0$ such that

$$\|x_0\| < \delta \Rightarrow \operatorname*{Lim}_{t \to \infty} x(t, x_0) = 0 \tag{5}$$

Here, $x(t,x_0)$ represents the solution of (2) starting at x_0.

Definition 2.2 *Region of attraction (Stability domain)* (Khalil 2002): The domain S is the RoA of the asymptotically stable equilibrium point $x = 0$ for autonomous systems (2), if S contains all x_0 such that

$$S = \left\{ x_0 \in \mathrm{R}^{\mathrm{n}} \middle| \operatorname*{Lim}_{t \to \infty} x(t, x_0) = 0 \right\} \tag{6}$$

Now, Theorems 2.3 and 2.4 illustrate the Lyapunov stability theorem and the guaranteed estimation of RoA, respectively.

Theorem 2.3 *Lyapunov stability theorem* (Khalil 2002): If the origin is an equilibrium point of (2), and $V(x): D \to \mathrm{R}$ is a continuously differentiable function (where, $D \subset \mathrm{R}^{\mathrm{n}}$ is a domain containing $x = 0$) such that

$$\begin{cases} V(x) = 0, \ x = 0 \\ V(x) > 0, \ x \in D - \{0\} \end{cases} \tag{7}$$

The equilibrium point $x = 0$ is

- stable, if $\dot{V}(x) \leq 0$ in D.
- asymptotically stable, if $\dot{V}(x) < 0$ in $D - \{0\}$.

Here, $\dot{V}(x)$ shows the time derivative of $V(x)$ and could be obtained by Eq. (8)

$$\dot{V}(x) = \frac{\partial V(x)}{\partial x} \cdot \dot{x} = \nabla V(x) \cdot f(x) \tag{8}$$

Theorem 2.4 *Guaranteed estimation of RoA* (Khalil 2002, Hachicho 2007): If $V(x)$ is a Lyapunov function for system (2), and Ω is a domain such that:

$$\Omega = \{x \in D | \dot{V}(x) \leq 0\} \tag{9}$$

with the condition that there is no solution of (2) that can stay identically in the set of points $\{x \in D - \{0\} | \dot{V}(x) = 0\}$. Then, the guaranteed estimation of the stability domain is as follows

$$S = \{x \in D | V(x) < c^*\} \tag{10}$$

where, c^* is the largest positive value that keeps S in Ω. In addition, $\{x \in D | V(x) = c^*\}$ displays the boundary of the estimated stability domain.

Finding c^* is an optimization problem. Consequently, one can rewrite Eqs. (9) and (10) as a global constrained optimization problem (Hachicho 2007)

$$\begin{cases} c^* = \min V(x) \\ \dot{V}(x) = 0 \\ x \neq 0 \end{cases} \tag{11}$$

3. Global optimization of polynomials

In most cases, the exact calculation of the stability domain of an asymptotically stable equilibrium point in nonlinear systems (2) is quite difficult or impossible. But, as it is mentioned in the previous section, it is possible to obtain a subset of RoA by finding a convenient Lyapunov function and solving the optimization problem (11). In the case of polynomial systems with a polynomial Lyapunov function, using the theory of moments could be helpful to transform this global optimization into a sequence of convex linear matrix inequality (LMI) problem (Lasserre 2001, Hachicho 2007). For smooth non-polynomial systems, one can approximate $f(x)$ in Eq. (2) by using Taylor expansion in the vicinity of the asymptotically stable equilibrium point.

The general scheme of the optimization problem is as follows

$$\begin{cases} \min p(x) \\ g_i(x) \geq 0 \ , \ i = 1, \ldots, r \end{cases} \tag{12}$$

Here, $p(x) : \mathrm{R}^{\mathrm{n}} \to \mathrm{R}$ and $g_i(x) : \mathrm{R}^{\mathrm{n}} \to \mathrm{R}$ are real-valued polynomials of degrees at most m and w_i, respectively. By comparing Eqs. (11) and (12), one can equate $p(x)$ with $V(x)$ and replace the constrains $g_i(x) \geq 0$ with $\dot{V}(x) = 0$ and $x \neq 0$. For more simplification, the following notation is applied for polynomials

$$p(x) = \sum_{\alpha} p_\alpha x^\alpha \ , \ x^\alpha = \prod_{j=1}^{n} x_j^{\alpha_j} \ , \ \sum_{j=1}^{n} \alpha_j \leq m \tag{13}$$

In a similar way, $g_i(x)$ can be written as follows

$$g_i(x) = \sum_{\beta} g_\beta x^\beta \ , \ x^\beta = \prod_{j=1}^{n} x_j^{\beta_j} \ , \ \sum_{j=1}^{n} \beta_j \leq w_i \ , i = 1, \ldots, r \tag{14}$$

Now, the vector $y = \{y_\alpha\}$, where y_α is the α-order moment for some probability measure μ, is defined. In addition, its first element ($y_{0,\ldots,0}$) is equal to 1. For example, Eq. (15) illustrates y_α for two-dimensional ($n = 2$) problem

$$y_{i,j} = \int x_1^i x_2^j \mu(d(x_1, x_2)) \tag{15}$$

After computing all elements of vector y, one can be able to establish the corresponding moment matrix $M_m(y)$. In case of two-dimensional problems, $M_m(y)$ is a block matrix

$$M_m(y) = \{M_{i,j}(y)\}_{0 \le i,j \le 2m} \tag{16}$$

where, each block is a $(i + 1) \times (j + 1)$ matrix

$$M_{i,j}(y) = \begin{bmatrix} y_{i+j,0} & y_{i+j-1,1} & \cdots & y_{i,j} \\ y_{i+j-1,1} & y_{i+j-2,2} & \cdots & y_{i-1,j+1} \\ \vdots & \vdots & \ddots & \vdots \\ y_{j,i} & y_{j-1,i+1} & \cdots & y_{0,i+j} \end{bmatrix} \tag{17}$$

On the other hand, if the entry (i, j) of the matrix $M_m(y)$ ($M_{i,j}(y)$) is defined as y_s (where, the subscript S is a function of i and j) and $q(x)$ is a polynomial function ($q(x) = \sum_\alpha q_\alpha x^\alpha$), then the elements of moment matrix $M_m(qy)$ are defined as follows

$$M_{i,j}(qy) = \sum_\alpha q_\alpha y_{S+\alpha} \tag{18}$$

Afterwards, the LMI optimization problem (19) is considered

$$\begin{cases} \inf_y \sum_\alpha p_\alpha y_\alpha \\ M_N(y) \ge 0 \\ M_{N-\overline{w}_i}(g_i\, y) \ge 0\ ,\ i = 1, \ldots, r \end{cases} \tag{19}$$

where, $\overline{w}_i = \lceil w_i/2 \rceil$ is the smallest integer larger than $w_i/2$, and N should satisfy the following conditions

$$\begin{cases} N \ge \left\lceil \dfrac{m}{2} \right\rceil \\ N \ge \overline{w}_i\ ,\ i = 1, \ldots, r \end{cases} \tag{20}$$

Theorem 3.1 *Convergence of LMI optimization problem (19)* (Lasserre 2001, Hachicho 2007): If $N \to \infty$ in LMI problem (19), then the infimum value of $\sum_\alpha p_\alpha y_\alpha$ converges from below to the minimum value of $p(x)$ in the global constrained optimization problem (12).

As a result, by using the theory of moments, the optimization problem (11) can be transformed into a simple LMI optimization problem for polynomial systems with polynomial Lyapunov function. For this purpose, one can equate $p(x)$ with $V(x)$ and replace the constrains $g_i(x) \ge 0$ with $\dot{V}(x) = 0$ and $x \ne 0$.

4. Zubov's PDE

In this section, the Zubov's partial differential equation (PDE) associated with nonlinear autonomous system (2) is presented. Subsequently, the power series' solution (Margolis and Vogt 1963, Kormanik and Li 1972) and rational solution (Vannelli and Vidyasagar 1985) are investigated. For this purpose, $f(x)$ is assumed to be an *infinite differentiable* vector function and the origin is an asymptotically stable equilibrium point. By these assumptions, the Zubov's PDE for an autonomous system (2) is as follows

$$\nabla W(x) \cdot f(x) = -\varphi(x)(1 - W(x)) \tag{21}$$

where, $W(x)$ represents a Lyapunov function, which is equal to zero at the origin, and $\varphi(x)$ is a positive definite function. The following equation shows the region of attraction (Margolis and Vogt 1963)

$$S = \{x \in \mathrm{R}^{\mathrm{n}} | 0 \le W(x) < 1\} \tag{22}$$

In addition, the boundary of the stability domain is obtained by Eq. (23)

$$\{x \in \mathrm{R}^{\mathrm{n}} | W(x) = 1\} \tag{23}$$

Another useful partial differential equation can be derived by defining a new Lyapunov function as follows

$$V(x) = -\ln(1 - W(x)) \tag{24}$$

Substitution of this equation into (21) makes the Zubov's PDE simpler

$$\nabla V(x) \cdot f(x) = -\varphi(x) \tag{25}$$

This equation is called the *modified* Zubov's PDE. By considering the relationship between $V(x)$ and $W(x)$, RoA from Eq. (26) can be obtained

$$S = \{x \in \mathrm{R}^{\mathrm{n}} | 0 \le V(x) < +\infty\} \tag{26}$$

Consequently, the boundary of the stability domain can be reached as follows

$$\{x \in \mathrm{R}^{\mathrm{n}} | V(x) = +\infty\} \tag{27}$$

In many cases, it is impossible to solve the modified Zubov's PDE (or Zubov's PDE) for nonlinear dynamical systems. But, the approximate solution could estimate a subset of the domain of attraction. In the case of polynomial dynamical systems ($f(x) = \sum_i f_i(x)$), $V(x)$ can be presumed as a series form (Margolis and Vogt 1963)

$$V(x) = \sum_{i=1}^{\infty} V_i(x) \tag{28}$$

where, V_i and f_i represent a homogenous polynomials relative to x of the nth power. Fig. 1 shows the required terms for $V(x_1, x_2)$ up to the 3rd power.

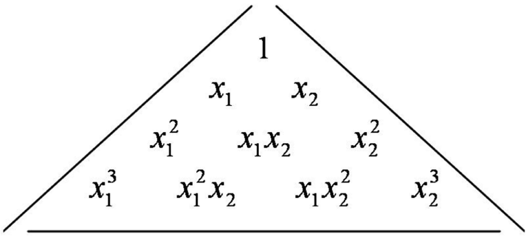


Fig. 1 The required terms for approximate V with two independent variables x_1 and x_2

After substitution of Eq. (28) into (25) and successively equating the similar terms, the unknown coefficients in V_i will be determined by solving linear equations. Therefore, the supposed Lyapunov function $V(x)$ will be in hand by Eq. (28) up to the nth power. In fact, the unknown coefficients in V_i, contain n-order partial derivatives of $V(x)$ with respect to x at the origin. *This means that the approximation error increases when x is far from the origin.*

In a similar way, $V(x)$ can be assumed as a rational function (Vannelli and Vidyasagar 1985)

$$V(x)=\frac{\sum_{i=2}^{\infty}R_i(x)}{1+\sum_{i=1}^{\infty}Q_i(x)} \tag{29}$$

where, $R_i(x)$ and $Q_i(x)$ are homogenous polynomials relative to x of the ith power. By considering Eq. (29), the modified Zubov's PDE (25) is rewritten as follows

$$\left(\left(1+\sum_{i=1}^{\infty}Q_i(x)\right)\sum_{i=2}^{\infty}\nabla R_i(x)-\left(\sum_{i=1}^{\infty}\nabla Q_i(x)\right)\sum_{i=2}^{\infty}R_i(x)\right)\cdot\left(\sum_{i=1}^{\infty}f_i(x)\right)=-\varphi\left(1+\sum_{i=1}^{\infty}Q_i(x)\right)^2 \tag{30}$$

By equating the monomials of the same degree, the following recursive relations will be achieved

$$\nabla R_2.f_1=-\varphi\,,\ k=2 \tag{31}$$

$$\nabla R_2.f_{k-1}+\sum_{j=3}^{k}\left(\nabla R_j+\sum_{i=1}^{j-2}(Q_i\nabla R_{j-i}-R_{j-i}\nabla Q_i)\right).f_{k-j+1}=-\varphi\left(2Q_{k-2}+\sum_{i=1}^{k-3}Q_iQ_{k-2-i}\right),\ k\geq 3 \tag{32}$$

Theorem 4.1 (Vannelli and Vidyasagar 1985): If $x=0$ is an asymptotically stable point of the polynomial nonlinear system (2), and the homogenous polynomials $R_i(x)$ and $Q_i(x)$ are convincing Eqs. (31) and (32), then $V_n(x)$ is a Lyapunov function for all $n\geq 2$

$$V_n(x)=\frac{\sum_{i=2}^{n}R_i(x)}{1+\sum_{i=1}^{n-2}Q_i(x)} \tag{33}$$

Since there are infinite homogenous polynomials $R_i(x)$ and $Q_i(x)$ satisfying Eqs. (31) and (32), an infinite number of $V_n(x)$ will be obtained. Vannelli and Vidyasagar (1985) demonstrated that $\dot{V}_n(x)$ can be shown in the following form

$$\dot{V}_n(x) = -\varphi + \frac{e(x)}{\left(1 + \sum_{i=1}^{n-2} Q_i(x)\right)^2} \tag{34}$$

where, $e(x)$ contains all monomials of degree greater than n. The comparison between Eqs. (8), (25) and (34) concludes that among all functions $V_n(x)$ satisfying (31) and (32), the one which minimizes $e(x)$ takes priority over all others.

5. Collocation

In this paper, the collocation method, which is a subset of weighted residual techniques, is employed to solve the modified Zubov's PDE. An advantage of this approach is that the collocation method is applicable to both *polynomial* and *non-polynomial* dynamical systems, while the numerical methods explained in the previous section can be used for polynomial systems. This approach approximate $V(x)$ by a linear combination of basis functions $N_i(x)$ which are linearly independent

$$V(x) = \sum_{i=1}^{m} N_i(x) V_i \tag{35}$$

Here, V_i represents the value of the Lyapunov function at some particular points x_i. These basis functions are satisfying the following conditions

$$N_i(x_j) = 1 \ , \ x_i = x_j \tag{36}$$

$$N_i(x_j) = 0 \ , \ x_i \neq x_j \tag{37}$$

$$\sum_{i=1}^{m} N_i(x) = 1 \tag{38}$$

It is important to remark that satisfying the condition (38) is not mandatory for all collocation problems. In the following subsections, the authors introduce two classes of basis functions which are satisfying (38) for the *whole* space of x with an acceptable accuracy.

After substitution of this approximation into (25), the residual function $R(x)$ will be formed

$$\left(\sum_{i=1}^{m} \nabla N_i(x) V_i\right) \cdot f(x) + \varphi(x) = R(x) \tag{39}$$

The collocation method makes $R(x)$ zero at x_i

$$R(x_i) = 0 \ , \ i = 1, \ldots, m \tag{40}$$

The collocation method, as a weighted residuals technique, works appropriately when $R(x)$ is *continuous* and *smooth* in the vicinity of the origin (Giesl 2007). As a result, the dynamical systems

(2) with non-smooth behavior are *not* in the scope of this paper. Another important point in this approach is to ensure that the particular points x_i is contained in the area $\Omega = \{x \in D | \dot{V}(x) \leq 0\}$ (Giesl 2008). This point will be explained in Section 6. Since the modified Zubov's PDE is linear, the Eq. (40) obtains m linear equations and m variables V_i, which can be solved easily. It should be reminded that this set of equations is dependent when the equilibrium point is assumed to be one of the collocation points (Giesl 2008). In this case, the following boundary condition should be added to Eq. (40)

$$V(0) = \sum_{i=1}^{m} N_i(0) V_i = 0 \tag{41}$$

By doing this and considering Eqs. (36), (37) and (41), one can conclude that the value of V_o is equal to zero.

The crucial part of the analysis is to find a set of convenient and compatible basis functions to have an optimum Lyapunov function. In the following subsections, two types of basis functions are proposed in the polar and Cartesian coordinates.

5.1 Basis functions in polar coordinates

Every point in polar coordinates can be written in terms of two parameters $\{r \in \mathrm{R} | 0 \leq r < +\infty\}$ and $\{\theta \in \mathrm{R} | -\pi \leq \theta < +\pi\}$. In these coordinates, the easiest area for placing the collocation nodes in a regular way, is a circle of radius r_0

$$\Gamma = \{r \in \mathrm{R} | 0 \leq r < r_0\} \tag{42}$$

In this paper, Γ is defined as a closed area that all collocation nodes are regularly placed in. If m_1 circles of radius $r_{j_1} \leq r_0$ ($j_1 = 1,\ldots,m_1$) with a unique center at the origin are assumed, one can put $2 \times m_2$ collocation nodes in the perimeter of each. The perimetric nodes are placed on the vertices of the regular polygon circumscribed by the supposed circle. Fig. 2 shows an example of node arrangement for $m_1 = 2$ and $m_2 = 3$.

Now, the basis function suggested for the node located at (r_{j_1}, θ_{j_2}) is as follows

$$N^{j_1,j_2}(r,\theta) = N^{j_1}(r) \times N^{j_2}(\theta) \tag{43}$$

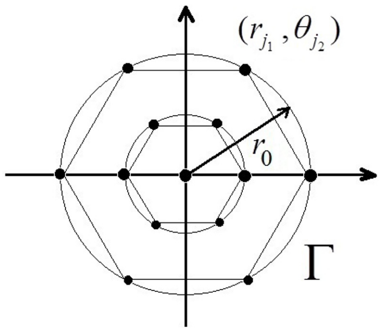


Fig. 2 An example of node arrangement for polar coordinates

Here, the superscripts denote the location of the supposed point. There are some characteristics that $N^{j_1}(r)$ and $N^{j_2}(\theta)$ (the radial and tangential parts of basis functions, respectively) should include:

1 - The values of $N^{j_1}(r)$ and $\partial N^{j_1}(r)/\partial r$ equal zero at the origin.
2 - Because of symmetry, a unique form is expected for all $N^{j_2}(\theta)$.
3 - Since $\theta_i - \pi$ and $\theta_i + \pi$ indicate the same angle (where the opposite node is located) and the tangential part is a continuous and differentiable function, $N^{j_2}(\theta)$ and its first derivative at these angles should be equal to zero.
4 - The summation of all $N^{j_1,j_2}(r, \theta)$ are equal to one for $\{r \in \mathrm{R} | 0 \le r < r_0\}$ and $\{\theta \in \mathrm{R} | -\pi \le \theta < +\pi\}$.

By considering the aforementioned characteristics, $N^{j_1}(r)$ can be assumed in the polynomial form

$$N^{j_1}(r) = \sum_{k=1}^{m_1} c_k r^{m_1+1} \tag{44}$$

In this equation, c_k represents the kth element of vector $\{C\}_{m_1 \times 1}$ obtained from Eq. (45)

$$[A]_{m_1 \times m_1} \{C\}_{m_1 \times 1} = \{B\}_{m_1 \times 1} \tag{45}$$

where

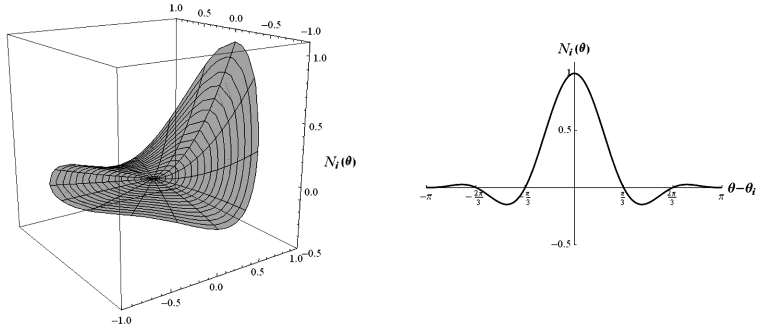
$$a_{ij} = r_i^{j+1}, \ i, j = 1, \ldots, m_1 \tag{46}$$

$$b_i = \delta_{i,j_1}, \ i = 1, \ldots, m_1 \tag{47}$$

Here, δ_{i,j_1} is the Kronecker delta. Furthermore, $N^{j_2}(\theta)$ could be written as follows

$$N^{j_2}(\theta) = \left[\prod_{k=1}^{m_2-1} \cos\left(\frac{m}{2k}(\theta - \theta_{j_2})\right) \right] \cos^2\left(\frac{1}{2}(\theta - \theta_{j_2})\right) \tag{48}$$

It can be easily shown that the suggested $N^{j_1,j_2}(r, \theta)$ satisfies the conditions (36) and (37) *precisely*, and approximately satisfies the condition (38). For $m_2 \ge 2$, the maximum error is less than one percent. Fig. 3 shows a basis function with its tangential part for the problem with six perimetric



(a) (b)

Fig. 3 (a) A basis function (b) tangential part of basis function nodes ($m_1 = 1$, $m_2 = 3$, $r_0 = 1$).

As it is mentioned, the collocation method takes the value of Lyapunov function V_i at the particular points x_i. Afterwards, Eq. (35) obtains the supposed Lyapunov function $V(x)$. This function is not polynomial. As a result, the optimization procedure mentioned in Section 3 cannot be applicable. An option for computing a polynomial Lyapunov function from the calculated V_i, is to use a regression function which is composed of a sum of homogenous polynomials with unknown coefficients (see Eq. (28)). The unknown coefficients are obtained from the least-squares approximation. Needless to say the number of coefficients should not exceed the number of V_i. Additionally, the selected terms in the regression function should satisfy the conditions described in (7).

5.2 Basis functions in Cartesian coordinates

Another technique for calculation of basis functions is presented in this section. In the n-dimensional Cartesian coordinates, the easiest shape for the domain Γ, which includes the collocation nodes in a regular way, is defined as an n-dimensional cuboid

$$\Gamma = \{x \in \mathrm{R}^{\mathrm{n}} \mid |x_i - x_{i0}| \le L_i,\ i = 1,......,n\} \tag{49}$$

where, x_{i0} is the ith component of the cuboid's center, and L_i represents the half length of the side dealing with the direction x_i. In order to simplify the process, the following transformation is suggested

$$\xi_i = \frac{x_i - x_{i0}}{L_i},\quad i = 1,..\,,n \tag{50}$$

Now, the area of Γ is redefined as follows

$$\Gamma = \{\xi \in \mathrm{R}^{\mathrm{n}} \mid -1 \le \xi_i \le +1,\quad i = 1,......,n\} \tag{51}$$

As a result, Γ represents an n-dimensional cube in space of ξ. This cube can be divided into n-dimensional sub-spaces by a regular grid. Then, a series of internal and external nodes can be placed on the vertices of these subspaces. In order to have a node at the origin, the number of nodes in each principal direction should be an odd number. In addition, the total number of nodes (m) is

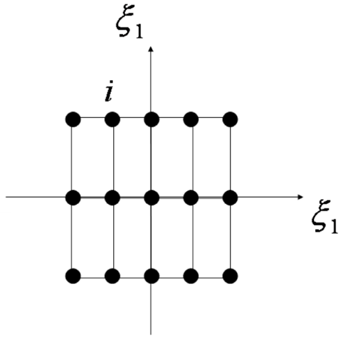


Fig. 4 An example of node arrangement for 2D problems

equal to the product of the numbers of nodes lied on each axis (m_j). Fig. 4 shows an example of node arrangement for 2D problems ($m_1 = 5$, $m_2 = 3$).

A systematic method for generation of basis functions can be achieved by the product of a number of independent polynomials in n-coordinates (Zienkiewicz and Taylor 2000)

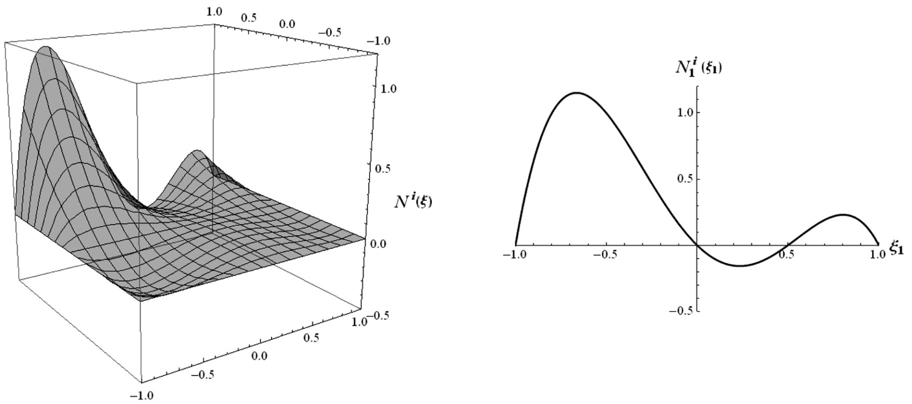
$$N^i(\xi) = \prod_{j=1}^{n} N_j^i(\xi_j), \quad i = 1\ldots, m \tag{52}$$

Here, the superscripts denote the location of the supposed point. The basis function $N^i(\xi)$ convinces the properties (36)-(38) *precisely*, as each $N_j^i(\xi_j)$ does in its own axis. In this way, the Lagrange polynomials are presented (Ralston and Rabinowitz 1978)

$$N_j^i(\xi_j) = \prod_k \frac{\xi_j - \xi_j^k}{\xi_j^i - \xi_j^k}, \quad \begin{cases} i = 1, \ldots, m \\ i = 1, \ldots, n \end{cases} \tag{53}$$

where, ξ_j^i illustrates the value of ξ_j at the particular point i and a set of points k represents all nodes at the direction ξ_j, except the point i. Fig. 5 displays the basis function of point i and its component $N_1^i(\xi_1)$ shown in Fig. 4.

The terms used in Lagrange polynomials are important. Although the maximum power of x_i in basis functions has a direct relationship to the number of nodes on its axis minus one, the product



(a) (b)

Fig. 5 (a) A basis function (b) a component of basis function

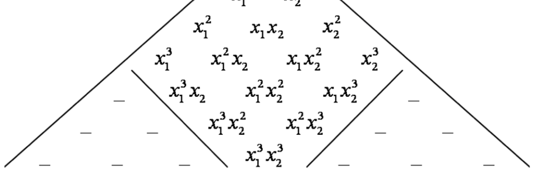


Fig. 6 The required terms for a 2D problem with 4×4 nodes

of Lagrange polynomials in different direction generates some higher order terms. Fig. 6 shows the required terms for a 2D problem with 4×4 nodes.

The comparison between Fig. 1 and Fig. 6 shows that the additional terms could enhance the estimation of the Lyapunov function. It is noteworthy that the linear change of variables, which can cause shift, rotation and scaling in Γ, can impress the final result of the suggested method. Therefore, the union of the estimated stability domains is the largest conservative region of attraction.

6. Computational steps

In this section, the steps of the suggested method are given:

Step 1: Estimation of the domain Γ is the first step of the proposed method. Γ should be contained in $\Omega = \{x \in D\dot{V}(x) \leq 0\}$ (Giesl 2008). In this way, one can compute an initial estimation of the Lyapunov function $V(x)$ by using other simple methods explained in Section 4. Then, an initial estimation of Ω is calculated. In order to explain why Γ needs to be contained in Ω, it should be noticed that the collocation method tries to find a Lyapunov function $V(x)$, of which the time derivative $\dot{V}$ of V along solution trajectories is approximately equal to the negative definite function $-\varphi(x)$ over $\Gamma - \{0\}$. On the other hand, according to Eqs. (9) and (10), there exists at least one point in the boundary of the computed stability domain which $\dot{V}$ in this point is equal to zero. Consequently, there should be a huge jump in the time derivative of Lyapunov function around this point. This can cause an inappropriate impression on the collocation method.

Step 2: As it is mentioned, the Lyapunov function $V(x)$ is replaced by a linear combination of basis functions $N_i(x)$. Depending on the nature of the dynamical system, a set of convenient basis functions is chosen. These functions are convincing the conditions (36)-(38). In addition, according to the modified Zubov's partial differential equation, $N_i(x)$ should be *continuously differentiable* all over the domain Γ.

Step 3: After constructing the approximate Lyapunov function, one can vanish the residual error $R(x)$ through the collocation method in a number of discrete points x_i. For this purpose, the linear equations (40) are solved. As it is mentioned in Section 5, the Eq. (40) are not linearly independent when a collocation point is located on the equilibrium point (Giesl 2008). In order to achieve the value of the Lyapunov function at particular points x_i, the boundary condition (41) is added to (40). As a result, the approximation of $V(x)$ will be derived either by using Eq. (35) or by employing the regression technique explained in Section 5.

Step 4: At the final step, a new Ω is obtained. In this domain, the time derivative of the Lyapunov function $\dot{V}(x)$ is negative semi-definite. Afterward, the region of attraction will be in hand by finding the maximum value of c^* which keeps S in Ω. In this way, the theory of moments transforms the global optimization problem (11) into a sequence of convex LMI problem (19) for polynomial systems with a polynomial Lyapunov function.

7. Numerical examples

The estimation of RoA and its application in engineering fields has been discussed in previous efforts (see, for example, Genesio *et al.* 1985). In this section, the proposed method is examined by five examples, which are excessively used in the literature and compared with the results of V&V

procedure (Vannelli and Vidyasagar 1985) and the Giesl method (Giesl 2007).

In the examples, the estimated RoAs, which are obtained by the suggested method with different basis functions, are shown in diagrams. The number n illustrates the order of the relative Lyapunov function. This could be a proper criterion for observing the rate of convergence. Furthermore, the domain Γ, the results of V&V and the Giesl method (dot-dashed) and the exact solution (dotted) are given. The estimated RoAs in the 3D example are shown in three 2D diagrams for three distinct values of the third variable.

In this paper, it is preferred to use regression technique to achieve Lyapunov function when polar basis functions are applied. In addition, the Jordan transform is employed before using the proposed method. As it is mentioned at the end of Section 5, the linear change of variables could reshape Γ. Therefore, the domain Γ changes from a circle or a rectangle into an oval or a parallelogram, respectively. In order to have a better comparison between various methods, a same form is assumed for the function $\varphi(x)$ in the modified Zubov's PDE

$$\varphi(x) = \sum_{i=1}^{n} x_i^2 \tag{54}$$

It is needless to say that all approaches mentioned in this paper obtain the region of attraction conservatively. Therefore, the largest estimated stability domain is preferred.

7.1 Example 1

An example of the asymptotically stable equilibrium point confined by an unstable limit circle can be found in Vander Pol oscillator (Vannelli and Vidyasagar 1985, Grosman and Lewin 2009)

$$\ddot{x} - \varepsilon(x^2 - 1)\dot{x} + x = 0 \tag{55}$$

This equation can be rewritten in the following form for $\varepsilon = 1$

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = x_1 + (x_1^2 - 1)x_2 \end{cases} \tag{56}$$

By using the Jordan decomposition ($\{x\} = [\mathrm{J}]\{z\}$), this system transforms into a new space (z). Eq. (57) shows the transformation matrix

$$[J] = \begin{bmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ 1 & 0 \end{bmatrix} \tag{57}$$

It is noteworthy that the eigenvalues of the linear part are complex. In this example, r_0 (the radius of Γ in the polar coordinates) is equal to $\sqrt{2.5}$. Moreover, L_1 and L_2 (the half of sides' length of Γ in Cartesian coordinates) are 1.2. Other properties of Lyapunov functions are given in Table 1. Fig. 7 illustrates the computed regions of attraction by the suggested Method.

As it can be seen, when the order of Lyapunov function increases, the proposed method in Cartesian

Table 1 Some properties of Lyapunov functions in 2D examples

Diagram	Polar Coordinates				Cartesian Coordinates		
	m_1	m_2	Reg. Order	No. of Terms	m_1	m_2	No. of Terms
$n=2$	1	3	2	3	3	3	9
$n=4$	3	9	4	12	5	5	25
$n=6$	5	14	6	15	7	7	49
$n=8$	7	18	8	42	9	9	81
$n=10$	10	30	10	63	11	11	121

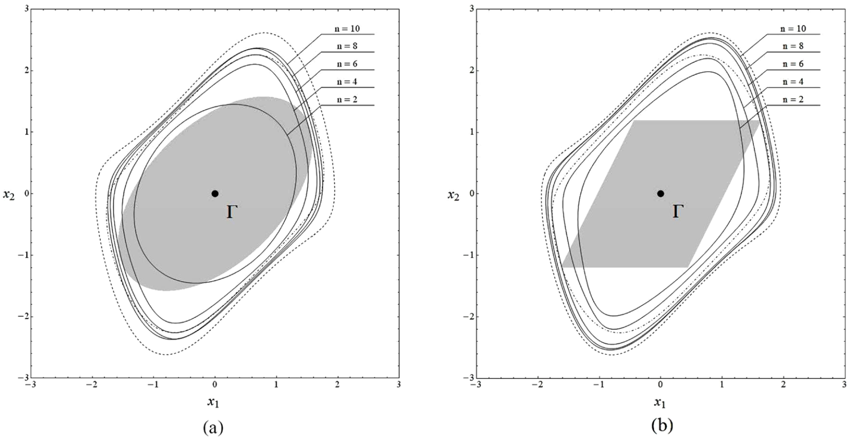


Fig. 7 Example 1 ($\varepsilon = 1$): (a) Regions of attraction computed by the suggested method in polar and (b) Cartesian coordinates (solid), Vannelli and Vidyasagar (1985) (dot-dashed) and the exact solution (dashed)

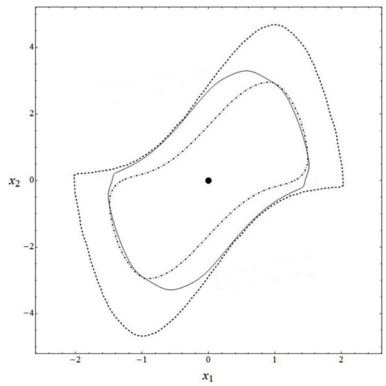


Fig. 8 Example 1 ($\varepsilon = 3$): Regions of attraction computed by the suggested method in Cartesian coordinates (solid), Giesl (2007) (dot-dashed) and the exact solution (dashed)

coordinates (Fig. 7 (b)) demonstrates a higher rate of convergence to the exact solution in comparison with the polar one (Fig. 7 (a)).

In the following, Fig. 8 compares the regions of attractions computed by the proposed scheme and the method described in Giesl (2007) for Vander Pol oscillator with $\varepsilon = 3$.

The Giesl method applies the collocation technique by using Wendland functions (Wendland 2005, Buhmann 2003) which are a subclass of radial basis functions (Giesl 2007, Giesl and Wendland 2011). Here, the number of collocation nodes used in the Giesl method is 236, while this number for the proposed method in Cartesian coordinates is 225. As Fig. 8 shows, the suggested method gives a larger estimation of RoA in comparison with the other method.

7.2 Example 2

In this example, the region of attraction for the following dynamical system (a toy example) is investigated

$$\begin{cases} \dot{x}_1 = -x_1 + x_1^3 \\ \dot{x}_2 = -0.5x_2 + x_1^2 \end{cases} \tag{58}$$

This system includes on asymptotically stable point at the origin and two saddle points at (± 1,2). The stability domain for the origin is $S = \{(x_1,\ x_2) \in \mathrm{R}^2 | -1 < x_1 < +1\}$ (Giesl 2007, 2008). The eigenvectors of the linearized system are in the same directions of x_1 and x_2 coordinates. Consequently, the Jordan transform is not required. Here, the proposed method with Cartesian basis functions is used with different Γ (Fig. 9). The value of L_2 is constant and equal to 0.5, while the length of L_1 is dependent on the parameter ε ($L_1 = 1 - \varepsilon$). Fig. 9 displays the estimated RoAs by the suggested method for $\varepsilon = 0.5$, 0.05 and 0.005.

As it can be seen, by decreasing the parameter ε, the proposed method obtains a larger estimation of the stability domain. In this example, nine collocation nodes are considered, while 122 nodes are used in Giesl method.

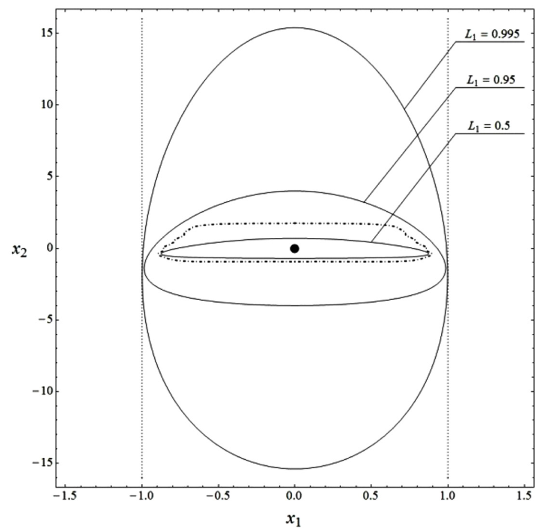


Fig. 9 Example 2: Regions of attraction computed by the suggested method in Cartesian coordinates (solid), Giesl (2007) (dot-dashed) and the exact solution (dashed)

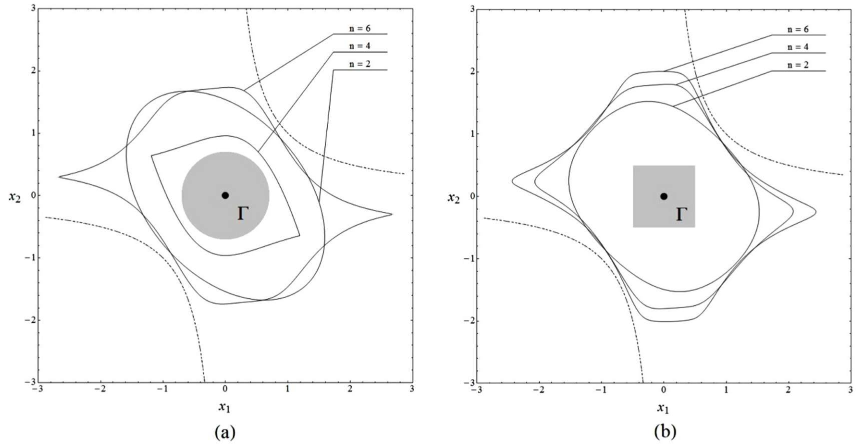


Fig. 10 Example 3: (a) Regions of attraction computed by the suggested method in polar and (b) Cartesian coordinates (solid), Vannelli and Vidyasagar (1985) and the exact solution (dot-dashed)

7.3 Example 3

The nonlinear dynamical system (59) includes an asymptotically stable equilibrium point at the origin (Margolis and Vogt 1963, Vannelli and Vidyasagar 1985)

$$\begin{cases} \dot{x}_1 = -x_1 + 2x_1^2 x_2 \\ \dot{x}_2 = -x_2 \end{cases} \tag{59}$$

Similar to the previous example, the eigenvalues of the linear part are real and its eigenvectors coincide with x_1 and x_2 coordinates. The presumed values of r_0, L_1 and L_2 are equals to $\sqrt{2}/2$, 1/2 and 1/2, respectively. More information about computed Lyapunov functions can be observed in Table 1. The estimated RoAs are drawn in Fig. 10.

Fig. 10 (b) shows a gradual improvement in estimation of stability domain, while the proposed method in the polar coordinates does not follow any regular rule. Since the exact solution of modified Zubov's PDE is rational (Margolis and Vogt 1963), V&V procedure gives the whole region of attraction.

7.4 Example 4

Another example of the asymptotically stable equilibrium point with complex eigenvalues can be seen in Eq. (60) (Vannelli and Vidyasagar 1985, Grosman and Lewin 2009)

$$\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -\frac{1}{2}x_2 - \left(\sin\left(x_1 + \frac{\pi}{3}\right) - \sin\left(\frac{\pi}{3}\right)\right) \end{cases} \tag{60}$$

Furthermore, the transformation matrix for the Jordan decomposition is as follows

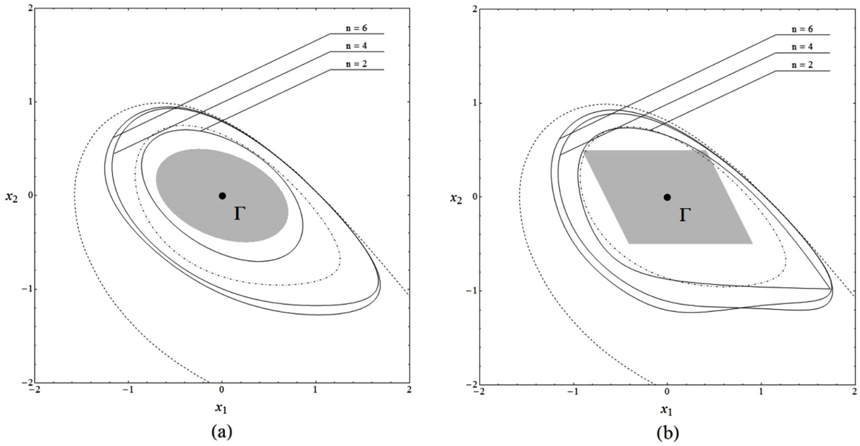


Fig. 11 Example 4: (a) Regions of attraction computed by the suggested method in polar and (b) Cartesian coordinates (solid), Vannelli and Vidyasagar (1985) (dot-dashed) and the exact solution (dashed)

$$[J] = \begin{bmatrix} \frac{1}{4} & -\frac{\sqrt{7}}{4} \\ -\frac{1}{2} & 0 \end{bmatrix} \tag{61}$$

In this example, r_0, L_1 and L_2 are equal to 1. In addition, other properties of Lyapunov functions are similar to the first example (see Table 1). The computed RoAs by the suggested method are displayed in Fig. 11.

Similar to Example 1, the suggested method in Cartesian coordinates estimates RoAs with higher rate of convergence to the exact solution compare to the polar one. Needless to say, all predicted RoAs, except $n = 2$ in Fig. 11(a), contain the stability domain estimated by V&V procedure.

7.5 Example 5

The 3D dynamical system (62) includes an asymptotically stable equilibrium point at the origin (Davison and Kurak 1979, Vannelli and Vidyasagar 1985)

$$\begin{cases} \dot{x}_1 = -x_2 \\ \dot{x}_2 = -x_3 \\ \dot{x}_3 = -0.915x_1 + (1 - 0.915x_1^2)x_2 - x_3 \end{cases} \tag{62}$$

Here, two eigenvalues of the linear part are complex and the third one is a real number. Consequently, the authors employ polar basis functions for the complex part and Lagrange polynomials for the direction relative to the real eigenvalue. In addition, the proposed method in the 3D Cartesian coordinates is applied. The following equation denotes the transformation matrix for the Jordan decomposition

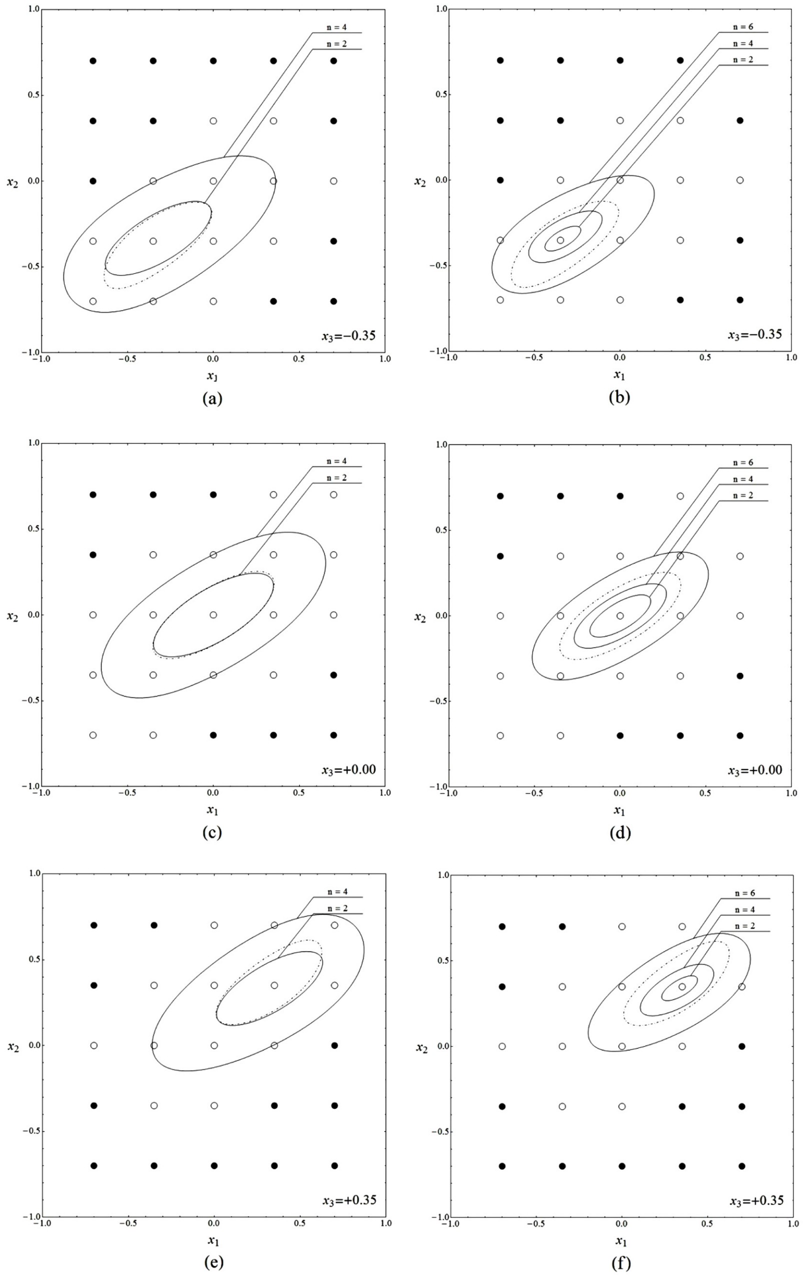


Fig. 12 Example 5: (a, c and e) Regions of attraction computed by the suggested method in polar-Cartesian and (b, d and f) Cartesian coordinates (solid), Vannelli and Vidyasagar (1985) (dot-dashed)

$$[J] = \begin{bmatrix} 1 & 0.977286 & 1.09515 \\ -0.933854 & 1 & 1.0465 \\ -0.999035 & -0.891364 & 1 \end{bmatrix} \tag{63}$$

Table 2 Some properties of Lyapunov functions in the 3D example

Diagram	Polar-Cartesian Coordinates					3D Cartesian Coordinates			
	m_1	m_2	m_3	Reg. Order	No. of Terms	m_1	m_2	m_3	No. of Terms
$n = 2$	1	3	3	2	6	3	3	3	27
$n = 4$	3	9	5	4	31	5	5	5	125
$n = 6$	-	-	-	-	-	7	7	7	343

All values of r_0 and L_3 for Γ in polar-Cartesian coordinates and L_1, L_2 and L_3 in 3D Cartesian coordinates are assumed to be 0.1. Other supplementary information about Lyapunov functions are given in Table 2.

Fig. 12 illustrates the RoAs computed by the suggested method and V&V procedure for three distinct values of x_3. In this figure, the points marked with " ○ " belong to the region of attraction, while those marked with " ● " do not (Grosman and Lewin 2009).

As it can be observed, the proposed method in polar-Cartesian coordinates estimates the RoA much larger than the one in the 3D coordinates and V&V procedure with fewer terms. Nevertheless, when the order of Lyapunov function increases, the suggested method in the 3D coordinates keeps its gradual improvement in calculation of the stability domain.

8. Conclusions

In this paper, a novel technique is proposed to solve the modified Zubov's partial differential equation approximately. In this way, the collocation method is applied in order to achieve a convenient Lyapunov function. For this purpose, two types of basis functions are proposed. The former is formulated in polar coordinate systems. These basis functions are proper for analysis of nonlinear systems which include an asymptotically stable equilibrium point at the origin. The latter type of basis functions is formulated in Cartesian coordinates. In this formulation, Lagrange polynomials are employed to obtain a simple and practical form of basis functions. An advantage of the proposed approach is that it is applicable to both *polynomial* and *non-polynomial* systems. It is important to remark that the suggested basis functions are not the best ones and can be improved depending on the characteristics of the dynamical stability problems.

The numerical examples indicate that the Lyapunov functions obtained by the proposed method in Cartesian coordinates can estimate larger stability domains with higher rate of convergence to the exact solution in comparison with the method formulated in the polar one coordinate system. In addition, the stability domains estimated by Vannelli and Vidyasagar (1985) and Giesl (2007) are contained in the stability domains computed by the suggested method in most cases.

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