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A modified modal perturbation method for vibration characteristics of non-prismatic Timoshenko beams

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Abstract. A new perturbation method is introduced to study the undamped free vibration of a non-prismatic Timoshenko beam for its natural frequencies and vibration modes. For simplicity, the natural modes of vibration of its corresponding prismatic Euler-Bernoulli beam with the same length and boundary conditions are used as Ritz base functions with necessary modifications to account for shear strain in the Timoshenko beam. The new method can transform two coupled partial differential equations governing the transverse vibration of the non-prismatic Timoshenko beam into a set of nonlinear algebraic equations. It significantly simplifies the solution process and is applicable to non-prismatic beams with various boundary conditions. Three examples indicated that the new method is more accurate than the previous perturbation methods. It successfully takes into account the effect of shear deformation of Timoshenko beams particularly at the free end of cantilever structures.

Keywords: Timoshenko beams; Euler-Bernoulli beams; perturbation; eigenvalue; natural mode; vibration

1. Introduction

Non-prismatic structural members are widely used in civil engineering due to their economic and aesthetic features in various applications. For example, chimneys, towers, tall buildings, and traffic light and sign supported cantilever arms are often built with varying sizes of cross sections. The flexural rigidity and mass per unit length of such structures change along their longitudinal direction. Therefore, it is practically meaningful to cost-effectively analyze non-prismatic beams and obtain their vibration characteristics in various engineering applications (Khaji *et al*. 2009, Lin 2009, Ruge and Birk 2007).

Beam structures can be expressed by two theories: Euler-Bernoulli beam and Timoshenko beam

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theories. The response and behavior of a classical Euler-Bernoulli beam are governed by its flexural deformation. The effects of rotational inertia and shear deformation are neglected. As such, the Euler-Bernoulli beam theory will lead to erroneous results when applied to short beams with significant shear deformation, particularly for vibration modes of high order (Weaver *et al*. 1990). On the other hand, the Timoshenko beam theory accounts for the effects of both rotational inertia and shear deformation, which is applicable to all beam structures but requires more efforts to arrive at any solution.

The characteristic equations governing the transverse vibration of a Timoshenko beam are two coupled partial differential equations. A closed-form solution of the equations can be obtained only when the beam has a constant cross section. In the past, both numerical and semi-analytic methods were utilized mainly to obtain the approximate solution of characteristic equations. Examples include the finite element method (Rossi *et al*. 1990, Cleghorn and Tabarrok 1992), dynamic stiffness method (Leung and Zhou 1995, Eisenberger 1995, Yuan *et al*. 2007), Rayleigh-Ritz method (Gutierrez *et al*. 1991, Auciello and Ercolano 2004), step-reduction method (Tong *et al*. 1995), Lagrangian method (Auciello 1993, 2000), numerical assembly method (Wu and Chen 2001, Lin and Tsai 2006, Lin 2009), and perturbation method (Lou *et al*. 2005). Among them, Lou *et al*. (2005) proposed a modal perturbation method (MPM) and expanded the response of a Timoshenko beam (referred to as the modified system in Lou *et al*. 2005) into a linear combination of special Ritz vectors associated with its corresponding Euler-Bernoulli beam (referred to as the original system in Lou *et al*. 2005). The main advantage of the MPM is that it can transform the differential characteristic equation governing the transverse vibration into a set of nonlinear algebraic equations. Numerical simulations demonstrated its excellent stability and accuracy for prismatic beams. The disadvantage of the MPM is that the complete natural modes of vibration cannot be obtained directly. Specifically, the angle of rotation due to bending is not available. Pan and Lou (2009) applied the MPM into simply-supported non-prismatic Timoshenko beams whose boundary conditions can be represented exactly by the mode shapes of their corresponding Euler-Bernoulli beams.

In this paper, the MPM is further developed to deal with any boundary conditions of non-prismatic Timoshenko beams. First, the natural modes of vibration and their derivatives of the corresponding prismatic Euler-Bernoulli beams are modified to meet various boundary conditions of the Timoshenko beams. Then, as Ritz base functions, the modified vibration modes and their derivatives are utilized to transform the two coupled partial differential equations of the Timoshenko beams into a set of algebraic equations. The accuracy of the new perturbation method is validated with three numerical examples and compared with the previous methods.

2. The modified modal perturbation method

In the new perturbation method, a non-prismatic Timoshenko beam is considered as a modified system of its corresponding prismatic Euler-Bernoulli beam with the same span length and boundary conditions. In this case, the modification or perturbation from the Euler-Bernoulli beam is two-fold: (a) modification from the prismatic to non-prismatic structure, and (b) effect of rotational inertia and shear deformation. However, the natural modes of vibration of any Euler-Bernoulli beam don't satisfy the free boundary condition of a Timoshenko beam due to shear deformation. In this study, the natural modes of an Euler-Bernoulli beam are modified to enforce the free boundary condition in order to extend the application of the MPM into any boundary conditions of Timoshenko beams. Due to this modification, the new method is referred to as the modified modal perturbation method (MMPM) in the following discussions.

2.1 Non-prismatic Timoshenko beam

Consider a single-span, non-prismatic, straight Timoshenko beam of span length l. When damping is not present, the free vibration of the beam is governed by the following two coupled equations of motion

$$\frac{\partial}{\partial x}\left[\chi GA\left(\varphi-\frac{\partial y}{\partial x}\right)\right]+\rho A\frac{\partial^2 y}{\partial t^2} = 0 \tag{1}$$

$$\frac{\partial}{\partial x}\left[EI\frac{\partial \varphi}{\partial x}\right] = \chi GA\left(\varphi-\frac{\partial y}{\partial x}\right)+\rho I\frac{\partial^2 \varphi}{\partial t^2} \tag{2}$$

where E is the Young's modulus, G is the shear modulus, ρ is the mass density, χ is the shear correction factor, $A(x)$ is the cross sectional area, $I(x)$ is the second moment of area with respect to the neutral axis of the cross section, $y(x, t)$ represents the transverse displacement of the beam, $\partial y/\partial x$ denotes the slope of the beam, and φ is the angle of rotation due to bending. Eqs. (1) and (2) represent the equilibrium of forces in transverse direction and moments, respectively.

For Eqs. (1) and (2), a harmonic solution of frequency $\overline{\omega}$, transverse displacement amplitude $\overline{Y}(x)$, and rotation amplitude $\overline{\Phi}(x)$ can be written as

$$y(x,t) = \overline{Y}(x)\sin(\overline{\omega}t+\gamma) \tag{3}$$

$$\varphi(x,t) = \overline{\Phi}(x)\sin(\overline{\omega}t+\gamma) \tag{4}$$

Substituting Eqs. (3) and (4) into Eqs. (1) and (2) gives

$$-[\chi GA(\overline{\Phi}-\overline{Y}')]'+\overline{\lambda}\rho A\overline{Y} = 0 \tag{5}$$

$$(EI\overline{\Phi}')'-\chi GA(\overline{\Phi}-\overline{Y}')+\overline{\lambda}\rho I\overline{\Phi} = 0 \tag{6}$$

where γ represents an initial phase angle, $\overline{\lambda} = \overline{\omega}^2$ is the eigenvalue of the Timoshenko beam and a prime on each letter or the argument in a bracket denotes a derivative with respect to x. Eqs. (5) and (6) is a pair of characteristic equations for the non-prismatic beam. They lead to the eigenvalue and its associated shape function of the j^{th} mode of vibration, $\overline{\lambda}_j$ and ($\overline{Y}_j, \overline{\Phi}_j$), where j is an integer.

2.2 Corresponding prismatic Euler-Bernoulli beam

The corresponding prismatic Euler-Bernoulli beam of a non-prismatic Timoshenko beam is defined as the member that has the same span length and boundary conditions as the Timoshenko beam. It is represented by a constant flexural rigidity EI_0 and mass m_0 per unit length. To ensure that the Euler-Bernoulli beam is a good representation of the Timoshenko beam, EI_0 and m_0 can be determined to be the average value of the beam as follows

$$m_0 = \int_0^l \rho A dx/l \tag{7}$$

$$EI_0 = \int_0^l EI dx/l \tag{8}$$

Eqs. (5) and (6) cannot be solved analytically for general non-prismatic Timoshenko beams. However, after the rotational inertia and shear deformation effects have been neglected, the characteristic equation of its corresponding prismatic Euler-Bernoulli beam, $EI(x) = EI_0$, $\rho A(x) = \rho A_0 = m_0$, can be expressed into

$$EI_0 Y^{(4)} - \lambda m_0 Y = 0 \tag{9}$$

where λ and $Y(x)$ respectively denote the eigenvalue and mode shape of the Euler-Bernoulli beam, and $Y^{(4)} = d^4Y/dx^4$. The general solution of Eq. (9) can then be analytically obtained as

$$Y(x) = a_1 \sin\eta x + a_2 \cos\eta x + a_3 \sinh\eta x + a_4 \cosh\eta x \tag{10}$$

where $\eta^4 = \lambda m_0/(EI_0)$, η is the eigenvalue parameter, a_1, a_2, a_3, and a_4 are four constants that can be determined from the boundary conditions. The analytic solution of Eq. (10) for various boundary conditions can be found in the classical textbooks of structural dynamics (Chopra 1995, Weaver *et al*. 1990).

3. The perturbation solution

In the new MMPM, the j^{th} eigenvalue and its associated mode of vibration for the Timoshenko beam is related to those of the corresponding Euler-Bernoulli beam by

$$\bar{\lambda}_j = \lambda_j + \Delta\lambda_j \tag{11}$$

$$\bar{Y}_j = Y_j + \sum_{k=1, k\neq j}^{n} Y_k\, p_{kj} \tag{12}$$

$$\bar{\Phi}_j = Y_j' + \sum_{k=1, k\neq j}^{n} Y_k'\, p_{kj} + \sum_{k=1}^{n} (Y_k' - H_k) q_{kj} \tag{13}$$

in which λ_j and $Y_j(x)$ are the j^{th} eigenvalue and mode shape of the Euler-Bernoulli beam, respectively, $\Delta\lambda_j$ is the perturbation of the jth eigenvalue, p_{kj} and q_{kj} are two generalized coordinates for the Ritz functions Y_k (or Y_k') and its correction term for boundary effects, and H_k is a modification shape function that is introduced to ensure that the boundary conditions of the Timoshenko beam be met in the use of the natural modes of vibration of its corresponding Euler-Bernoulli beam as Ritz bases. H_k is a polynomial function that can be determined by boundary conditions. For example, the boundary conditions of a cantilever Timoshenko beam are

$$x = 0,\ \bar{Y} = \bar{\Phi} = 0;\ x = l,\ \bar{\Phi}' = \bar{\Phi} - \bar{Y}' = 0 \tag{14}$$

The boundary conditions of its corresponding Euler-Bernoulli beam are

Table 1 Boundary conditions and H_k functions

Boundary	Beam: Euler-Bernoulli	Beam: Timoshenko	H_k function
Pinned-pinned	$x=0,\ Y=Y''=0$ $x=l,\ Y=Y''=0$	$x=0,\ \overline{Y}=\overline{\Phi}'=0$ $x=l,\ \overline{Y}=\overline{\Phi}'=0$	$H_k=0$
Fixed-fixed	$x=0,\ Y=Y'=0$ $x=l,\ Y=Y'=0$	$x=0,\ \overline{Y}=\overline{\Phi}=0$ $x=l,\ \overline{Y}=\overline{\Phi}=0$	$H_k=0$
Fixed-free	$x=0,\ Y=Y'=0$ $x=l,\ Y''=Y'''=0$	$x=0,\ \overline{Y}=\overline{\Phi}=0$ $x=l,\ \overline{\Phi}'=\overline{\Phi}-\overline{Y}'=0$	$H_k(x)=Y_k'(l)(-x^2+2lx)/l^2$
Pinned-free	$x=0,\ Y=Y''=0$ $x=l,\ Y''=Y'''=0$	$x=0,\ \overline{Y}=\overline{\Phi}'=0$ $x=l,\ \overline{\Phi}'=\overline{\Phi}-\overline{Y}'=0$	$H_k(x)=Y_k'(l)(-2x^3+3lx^2)/l^3$
Free-free	$x=0,\ Y''=Y'''=0$ $x=l,\ Y''=Y'''=0$	$x=0,\ \overline{\Phi}'=\overline{\Phi}-\overline{Y}'=0$ $x=l,\ \overline{\Phi}'=\overline{\Phi}-\overline{Y}'=0$	$H_k(x)=Y_k'(0)+$ $[Y_k'(l)-Y_k'(0)](-2x^3+3lx^2)/l^3$

$$x=0,\ Y=Y'=0;\ x=l,\ Y''=Y'''=0 \tag{15}$$

Substituting Eqs. (12) and (13) into Eq.(14) and introducing Eq. (15) lead to

$$H_k(0)=0,\ H_k(l)=Y_k'(l),\ H_k'(l)=0 \tag{16}$$

Based on the Hermite interpolation, the H_k function (lowest order of polynomial) can thus be expressed into

$$H_k(x)=Y_k'(l)(-x^2+2lx)/l^2 \tag{17}$$

A complete list of all boundary conditions and the modification shape function for various ends is summarized in Table 1.

For the j^{th} mode, substituting Eqs. (11) - (13) into Eqs. (5) and (6) gives

$$-\left[\chi GA\sum_{k=1}^{n}(Y_k'-H_k)q_{kj}\right]'+(\lambda_j+\Delta\lambda_j)\rho A\left(Y_j+\sum_{k=1,k\neq j}^{n}Y_k\,p_{kj}\right)=0 \tag{18}$$

$$\left[EI\left(Y_j''+\sum_{k=1,k\neq j}^{n}Y_k''\,p_{kj}+\sum_{k=1}^{n}(Y_k''-H_k')q_{kj}\right)\right]'-\left[\chi GA\sum_{k=1}^{n}(Y_k'-H_k)q_{kj}\right]$$
$$+(\lambda_j+\Delta\lambda_j)\rho I\left[Y_j'+\sum_{k=1,k\neq j}^{n}Y_k'\,p_{kj}+\sum_{k=1}^{n}(Y_k'-H_k)q_{kj}\right]=0 \tag{19}$$

After multiplying Eq. (18) by $Y_i(x)$ and Eq. (19) by $-Y_i'(x)$, integrating each resulting equation from $x=0$ to $x=l$, and moving the summation signs out of their respective integrands, the following equations can be obtained

$$\sum_{k=1}^{n} q_{kj} S_{ik} + (\lambda_j + \Delta\lambda_j)\left[M_{ij} + \sum_{k=1, k\neq j}^{n} p_{kj} M_{ik}\right] = 0 \tag{20}$$

$$K_{ij} + \sum_{k=1, k\neq j}^{n} p_{kj} K_{ik} + \sum_{k=1}^{n} q_{kj}(\beta_{ik} + T_{ik})$$

$$+ (\lambda_j + \Delta\lambda_j)\left[R_{ij} + \sum_{k=1, k\neq j}^{n} p_{kj} R_{ik} + \sum_{k=1}^{n} q_{kj}\gamma_{ik}\right] = 0 \tag{21}$$

in which $M_{ik}, K_{ik}, S_{ik}, T_{ik}, R_{ik}, \beta_{ik}$, and γ_{ik} can be calculated by

$$M_{ik} = \int_0^l mY_iY_k dx\,,\ \ K_{ik} = -\int_0^l Y_k(EIY_k'')'' dx\,,\ \ S_{ik} = -\int_0^l Y_i[\chi GA(Y_k' - H_k)]' dx\,,$$

$$T_{ik} = -\int_0^l \chi GAY_i'(Y_k' - H_k)dx\,,\ \ R_{ik} = -\int_0^l \rho IY_i'Y_k' dx\,,\ \ \beta_{ik} = -\int_0^l Y_i'[EI(Y_k'' - H_k')]' dx$$

$$\gamma_{ik} = -\int_0^l \rho IY_i'(Y_k' - H_k)dx$$

When every end of the single-span beam is free, pinned or fixed, three of them $(K_{ik}, S_{ik}, \beta_{ik})$ can be further simplified into

$$K_{ik} = \int_0^l EIY_i''Y_k'' dx\,,\ \ S_{ik} = \int_0^l \chi GAY_i'(Y_k' - H_k)dx\,,\ \ \beta_{ik} = \int_0^l EIY_i''(Y_k'' - H_k')dx$$

After $p_{jj} = \Delta\lambda_j + \lambda_j$ has been introduced and i is taken from 1 to n, Eqs. (20) and (21) can be written in matrix form

$$\left(\begin{bmatrix}\mathbf{C}_{11} & \mathbf{C}_{12}\\ \mathbf{C}_{21} & \mathbf{C}_{22}\end{bmatrix} + \lambda_j p_{jj}\begin{bmatrix}\mathbf{D}_{11} & 0\\ \mathbf{D}_{21} & \mathbf{D}_{22}\end{bmatrix}\right)\begin{Bmatrix}\mathbf{p}_j\\ \mathbf{q}_j\end{Bmatrix} = \begin{Bmatrix}\mathbf{F}_j\\ \mathbf{G}_j\end{Bmatrix} \tag{22}$$

where $\mathbf{C}_{11}, \mathbf{C}_{12}, \mathbf{C}_{21}, \mathbf{C}_{22}, \mathbf{D}_{11}, \mathbf{D}_{21}$,and $\mathbf{D}_{22}$ are all $n\times n$ square matrices, $\mathbf{p}_j, \mathbf{q}_j, \mathbf{F}_j$, and $\mathbf{G}_j$ are $n\times 1$ vectors. Except for $\mathbf{C}_{21}, \mathbf{D}_{11}$ and $\mathbf{D}_{21}$, all $n\times n$ square matrices can be expressed into: $\mathbf{C}_{11} = \lambda_j[M_{ik}]$, $\mathbf{C}_{12} = [S_{ik}]$, $\mathbf{C}_{22} = [\beta_{ik}] + \lambda_j[\gamma_{ik}] + [T_{ik}]$, and $\mathbf{D}_{22} = [\gamma_{ik}]$. The exceptions are that the j^{th} column of $[K_{ik}]$ is zero in $\mathbf{C}_{21} = [K_{ik}]+\lambda_j[R_{ik}]$, and that the j^{th} column of both $[M_{ik}]$ and $[R_{ik}]$ are zero in $\mathbf{D}_{11} = [M_{ik}]$ and $\mathbf{D}_{21} = [R_{ik}]$. The $n\times 1$ vectors are given below.

$$\mathbf{F}_j = -\lambda_j\{M_{1j}\ M_{2j}\ \ldots\ M_{nj}\}^T$$

$$\mathbf{G}_j = -\{K_{1j} + \lambda_j R_{1j}\ K_{2j} + \lambda_j R_{2j}\ \ldots\ K_{nj} + \lambda_j R_{nj}\}^T$$

$$\mathbf{p}_j = \{p_{1j}\ p_{2j}\ \ldots\ p_{(j-1)j}\ \Delta\lambda_j/\lambda_j\ p_{(j+1)j}\ \ldots\ p_{nj}\}^T$$

$$\mathbf{q}_j = \{q_{1j}\ q_{2j}\ \ldots\ q_{nj}\}^T$$

The coupled partial differential Eqs. (5) and (6) with unknown functions $\bar{Y}_j(x)$ and $\bar{\Phi}_j(x)$ and unknown eigenvalue $\bar{\lambda}_j$ have been transformed into a set of $2n$ nonlinear algebraic equations with $2n$ unknown values $\mathbf{p}_j$ and $\mathbf{q}_j$ in Eq. (22). After $\mathbf{p}_j$ and $\mathbf{q}_j$ are determined, the j^{th} eigenvalue and associated natural mode of vibration of a non-prismatic Timoshenko beam can be obtained from Eqs. (11)-(13). In general, solving the nonlinear algebraic equations is easier than solving the coupled partial differential equations. The main efforts in the application of the MMPM are calculations of the integrals of $M_{ik}, K_{ik}, S_{ik}, T_{ik}, R_{ik}, \beta_{ik}$, and γ_{ik} as well as the solution of the nonlinear algebraic equations. The integrals can be obtained analytically for simple functions or numerically for complicated functions. For a set of $2n$ nonlinear algebraic equations, an iterative technique can be used to obtain its solution such as the Newton-Raphson method that can be found in the classical textbooks of finite element method or numerical analysis.

4. Solution procedure with an illustrative example

To illustrate the above-discussed perturbation solution step-by-step, a prismatic simply-supported Timoshenko beam with length l, flexural rigidity EI_0, cross-sectional area A_0 and mass m_0 per unit length is taken as an example.

4.1 Properties of the corresponding beam

For this simply-supported example, the flexural rigidity and mass per unit length of the corresponding Euler-Bernoulli beam are the same as those of the Timoshenko beam. With pin and roller supports at the two ends of the beam, the modification shape function H_k is equal to 0 and the eigenvalues and natural modes of vibration for the corresponding beam are

$$\lambda_j = \frac{j^4\pi^4}{l^4}\frac{EI_0}{m_0} \text{ and } Y_j = \sin\frac{j\pi x}{l} \quad (j = 1, 2, 3\ldots) \tag{23}$$

The orthogonality conditions for the natural modes of the corresponding beam can be expressed into

$$\int_0^l m_0 Y_i Y_k dx = M_{ii}\delta_{ik} = \frac{m_0 l}{2}\delta_{ik} \tag{24}$$

$$\int_0^l EI_0 Y_i Y_k^{(4)} dx = \lambda_i M_{ii}\delta_{ik} \tag{25}$$

in which δ_{ik} is the Kronecker's Delta function.

4.2 Integrations from the characteristics of the corresponding beam

By introducing both orthogonality conditions given in Eqs. (24) and (25), $M_{ik}, K_{ik}, S_{ik}, T_{ik}, R_{ik}, \beta_{ik}$, and γ_{ik} can be analytically integrated out to become

$$M_{ik} = M_{ii}\delta_{ik},\quad K_{ik} = \beta_{ik} = \lambda_i M_{ii}\delta_{ik},\quad S_{ik} = T_{ik} = \left(\frac{j\pi}{l}\right)^2\frac{\chi GA_0 l}{2}\delta_{ik},\quad R_{ik} = \gamma_{ik} = -\left(\frac{j\pi}{l}\right)^2\frac{\rho I_0 l}{2}\delta_{ik}$$

As a result, $\mathbf{C}_{11}, \mathbf{C}_{12}, \mathbf{C}_{21}, \mathbf{C}_{22}, \mathbf{D}_{11}, \mathbf{D}_{21}$ and $\mathbf{D}_{22}$ are all diagonal square matrices. For vectors $\mathbf{F}_j$ and $\mathbf{G}_j$, all but their j^{th} element are zero. The j^{th} element F_{jj} and G_{jj} can be determined by

$$F_{jj} = -\lambda_j M_{jj}, \quad G_{jj} = -(K_{jj} + \lambda_j R_{jj})$$

4.3 Algebraic equations and solutions for the Timoshenko beam

It can be observed from Eq. (22) that all but the j^{th} element of vectors $\mathbf{p}_j$ and $\mathbf{q}_j$ are zero. The j^{th} element $p_{jj} = \Delta\lambda_j/\lambda_j$ and q_{jj} can be obtained from

$$\left(\begin{bmatrix} \lambda_j M_{jj} & S_{jj} \\ \lambda_j R_{jj} & \beta_{jj} + \lambda_j \gamma_{jj} + S_{jj} \end{bmatrix} + \Delta\lambda_j \begin{bmatrix} 0 & 0 \\ 0 & \gamma_{jj} \end{bmatrix}\right) \begin{Bmatrix} \Delta\lambda_j/\lambda_j \\ q_{jj} \end{Bmatrix} = \begin{Bmatrix} -\lambda_j M_{jj} \\ -K_{jj} - \lambda_j R_{jj} \end{Bmatrix} \tag{26}$$

By introducing Eq. (11), Eq. (26) becomes

$$\frac{\rho I_0}{\chi G A_0} \bar{\lambda}_j^2 - \left[\left(\frac{j\pi}{l}\right)^2 \left(\frac{EI_0}{\chi G A_0} + \frac{\rho I_0}{m_0}\right) + 1\right] \bar{\lambda}_j + \lambda_j = 0 \tag{27}$$

$$q_{jj} = -\left(\frac{l}{j\pi}\right)^2 \frac{m_0}{\chi G A_0} \bar{\lambda}_j \tag{28}$$

or

$$\bar{\lambda}_j = \frac{\vartheta \pm \sqrt{\vartheta^2 - 4\xi\lambda_j}}{2\xi}, \quad \bar{Y}_j = \sin\frac{j\pi x}{l}, \quad \bar{\Phi}_j = \left(\frac{j\pi}{l} - \frac{l}{j\pi} \frac{m_0 \bar{\lambda}_j}{\chi G A_0}\right) \cos\frac{j\pi x}{l}$$

where $\xi = \dfrac{\rho I_0}{\chi G A_0}$ and $\vartheta = \left(\dfrac{j\pi}{l}\right)^2 \left(\dfrac{EI_0}{\chi G A_0} + \dfrac{\rho I_0}{m_0}\right) + 1$. In fact, Eq. (27) represents the analytical frequency equation of a simply-supported Timoshenko beam. It is therefore clear that the MMPM results in an exact eigensolution of the simply-supported Timoshenko beam and satisfactorily accounts for the shear deformation effect of the Timoshenko beam.

5. Verification with numerical results

To investigate the validity and accuracy of the MMPM, single-span beams with various support conditions are considered below. Although applicable to more complex structures, the new MMPM is numerically tested with homogeneous and isotropic beams of a rectangular cross section with constant width b. The height of the cross section linearly varies with distance as given by

$$h(x) = h_1(1 + \alpha x/l), \quad (0 \le x \le l) \tag{29}$$

where α is the tapering parameter, l is the span length of beam, and h_1 denotes the height at $x = 0$. In the new MMPM, the corresponding Euler-Bernoulli beam has the following geometric parameters: $h_0 = h_1(1 + 0.5\alpha)$, $I_0 = bh_0^3/12$, and $A_0 = bh_0$.

For the sake of comparison, two dimensionless parameters representing the natural frequency of the Timoshenko beam and the slenderness ratio are respectively introduced as follows

$$\Omega_i = \overline{\omega}_i l^2 \sqrt{\rho A_1 / E I_1} \tag{30}$$

$$r = \sqrt{I_1 / A_1 l^2} \tag{31}$$

where A_1 and I_1 are the cross sectional area and the area moment of inertia at x=0.

5.1 Simply-supported tapered Timoshenko beams

In theory, an infinite number of modes $Y_k\,(k = 1, 2, \dots, \infty)$ are required to converge to the exact solutions of $\overline{Y}_j$ and $\overline{\Phi}_j$ in Eqs. (12) and (13) for non-prismatic Timoshenko beams. However, relatively few modes can provide sufficiently accurate results in engineering applications. To represent the j^{th} vibration mode of a non-prismatic Timoshenko beam, the number of modes, n, of its corresponding prismatic Euler-Bernoulli beam must be

$$n = j + \Delta n \tag{32}$$

As the additional number of modes, Δn, increases, the result gradually converges to the exact solution. The relative error e of the j^{th} natural frequency obtained by using n modes of vibration can be expressed into

$$e = \frac{\Omega_j - \Omega_j^*}{\Omega_j^*} \times 100\% \tag{33}$$

in which Ω_j and Ω_j^* are the approximate and exact dimensionless frequencies, respectively.

The convergence rate of the MMPM mainly depends upon the tapering parameter and slenderness ratio r. To investigate the accuracy of the MMPM, a simply-supported, tapered Timoshenko beam as shown in Fig. 1 is analyzed. For comparison, the natural frequencies obtained when $\Delta n = 50$ are considered as exact solutions.

The relative errors of the fundamental frequency are presented in Fig 2(a) for various α values and in Fig 2(b) for various r, respectively. The relative errors of different natural frequencies are shown in Fig. 3 for various number of modes Δn. It can be observed from Fig. 2 that α significantly influences the convergence rate but the effect of slenderness ratio r is negligible. The more uniform (larger α) the non-prismatic beam, the more rapidly (smaller Δn) a converged and

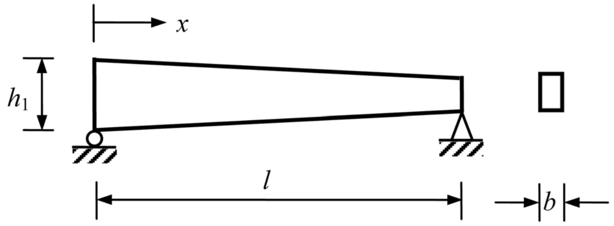
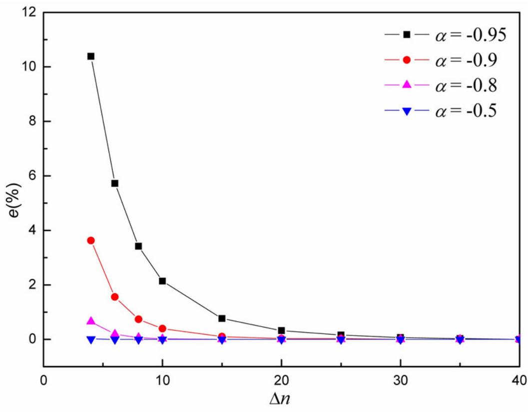
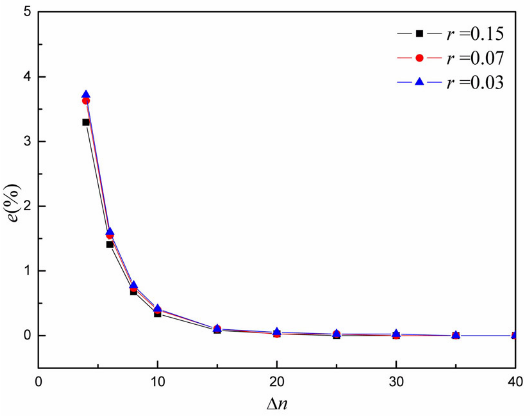


Fig 1. A simply-supported tapered beam



(a) r = 0.0707



(b) α= -0.9

Fig 2. Relative error of the fundamental frequency: E/G = 2.6, χ = 5/6

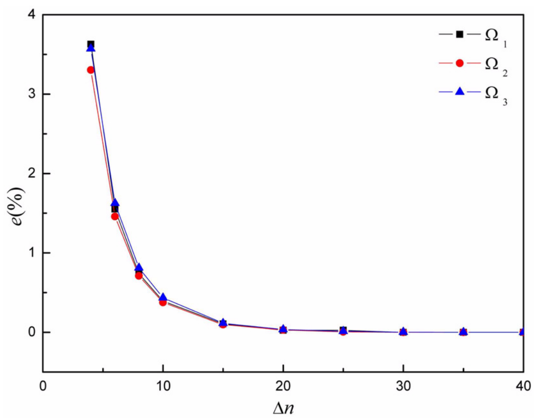


Fig 3. Relative error of first three frequencies: E/G = 2.6, χ = 5/6, r = 0.0707 and α = -0.9

accurate result can be achieved. When the ratio between the heights of two end sections increases from 0.05 to 20, the relative error is within 5% when Δn is larger than 8. This statement is also true

Table 2 The first three natural frequencies (dimensionless) of simply-supported beams

α		0				0.1				0.15			
r	Ω	Present study	Auciello and Ercolano (2004)	Auciello (1993)	Gutier-rez *et al.* (1991)	Present study	Auciello and Ercol-ano (2004)	Auciello (1993)	Gutier-rez *et al.* (1991)	Present study	Auciello and Ercol-ano (2004)	Auciello (1993)	Gutier-rez *et al.* (1991)
	Ω_1	9.695	9.695	9.676	9.748	10.154	10.154	10.134	10.235	10.377	10.377	10.360	10.458
0.03	Ω_2	36.927	36.957	36.681		38.518	38.546	38.226		39.293	39.318	39.037	
	Ω_3	77.484	77.954	76.526		80.388	80.934	79.423		81.783	82.411	80.815	
	Ω_1	9.411	9.411	9.393	9.446	9.829	9.829	9.811	9.850	10.031	10.031	10.010	10.054
0.05	Ω_2	33.549	33.572	33.373		34.746	34.768	34.573		35.320	35.34	35.147	
	Ω_3	65.647	65.971	65.084		67.453	67.817	66.894		68.304	68.713	67.747	
	Ω_1	9.036	9.036	9.021	9.057	9.405	9.405	9.390	9.428	9.582	9.5825	9.566	9.604
0.07	Ω_2	30.019	30.037	29.899		30.890	30.907	30.770		31.302	31.318	31.183	
	Ω_3	55.489	55.714	55.104		56.622	56.868	56.237		57.149	57.419	56.763	

for the 2nd and 3rd frequencies as shown in Fig. 3. Therefore, $n = j + 8$ in the MMPM is satisfactory in engineering applications.

Table 2 compares the proposed MMPM with other existing methods for the calculation of first three natural frequencies of the tapered beam in Fig. 1 with $E/G = 2.6$ and $\chi = 5/6$. Based on the previous results, 11 modes of vibration of the corresponding prismatic beam were included in the solution of Eq. (22). The same number of modes was used in the Rayleigh-Ritz method (Gutierrez *et al*. 1991, Auciello and Ercolano 2004) and in the Lagrangian method (Auciello 1993). It is a well-known fact that the Lagrangian method gives the lower bound of natural frequencies. However, like the Rayleigh-Ritz method, the proposed MMPM gives the upper bound of natural frequencies. It can be seen from Table 2 that the natural frequencies obtained by the MMPM are always higher than those by the Lagrangian method but either equal to or lower than those by the Rayleigh-Ritz method. Therefore, the MMPM is most likely more accurate than both the Rayleigh-Ritz and Lagrangian methods.

5.2 Cantilever tapered Timoshenko beams

The main difference between the new MMPM and the previous MPM is the introduction of a H function in Eq.(13) to ensure that the free-end boundary is met. In order to investigate the effect of the H_k function, the first three natural frequencies of a cantilever Timoshenko beam with a linearly varying height and $E/G = 2.6$, $\chi = 5/6$, $r = 0.0707$ and $\alpha = -0.9$, as shown in Fig. 4, are calculated by the MPM and MMPM, respectively. For comparison, the natural frequencies obtained when $\Delta n = 50$ are considered as exact solutions. The relative errors of different natural frequencies are shown in Fig. 5 for various number of modes Δn. It can be seen that the natural frequencies obtained by the MMPM are always either equal to or lower than those by the MPM. Therefore, the MMPM is more

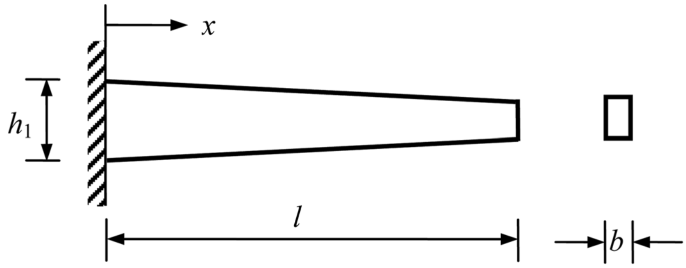


Fig 4. A cantilever tapered Timoshenko beam

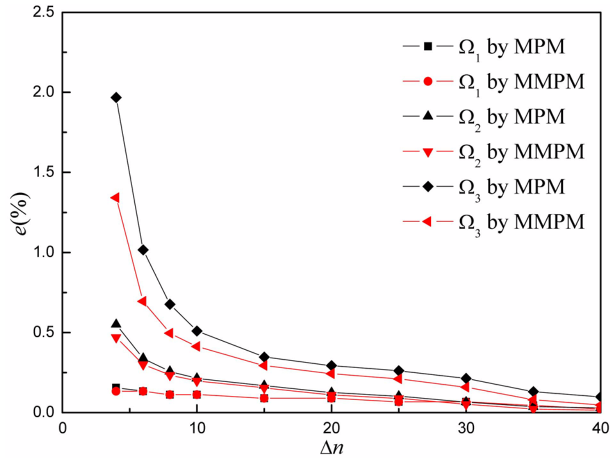


Fig 5. Relative error of the first three frequencies by MPM and MMPM

accurate than MPM because both the MPM and MMPM give the upper bound of natural frequencies. The improvement of accuracy is more significant for higher modes of vibration. This demonstrates the importance of a modified shape function H_k for high modes of vibration.

Table 3 compares the first three frequencies for various r values, which are calculated by the MMPM, MPM, finite element method (FEM by Rossi *et al*. 1990), Rayleigh-Ritz method (Auciello and Ercolano 2004), and Lagrangian method (Auciello 2000). The parameters of the cantilevered beam in Fig. 4 are $\alpha = -0.2$, $E/G = 2.6$, and $\chi = 5/6$. Rossi *et al*. (1990) divided the beam into numerous prismatic stepped elements whose masses are equal to those of the non-prismatic beam element. Therefore, the FEM method provides the upper bound of the exact solutions. On one hand, the three natural frequencies obtained by the MMPM are always higher than the lower-bound solutions by the Lagrangian method, which is qualitatively correct. On the other hand, the MMPM, Rayleigh-Ritz method, and FEM all provide the upper bound of exact solutions. As such, for the case of $\Delta n = 50$, the MMPM is slightly less accurate than the Rayleigh-Ritz method for the third natural frequency and most likely more accurate than the FEM in this example. However, the MMPM is more accurate than the MPM. In engineering applications, the accuracy of natural frequencies is satisfied for $\Delta n = 8$.

Table 3 The first three frequencies (dimensionless) of non-prismatic cantilevered beams

r	Ω	Present study		MPM	Rossi et al. (1990)	Auciello and Ercolano (2004)	Auciello (2000)
		Δn=8	Δn=50	Δn=8			
	Ω_1	3.596	3.587	3.596	3.59	3.587	3.584
0.02	Ω_2	20.192	20.132	20.192	20.17	20.180	19.984
	Ω_3	53.558	53.336	53.572	53.48	53.488	52.445
	Ω_1	3.559	3.553	3.559	3.56	3.558	3.552
0.04	Ω_2	19.058	18.975	19.059	19.01	19.018	18.855
	Ω_3	47.613	47.417	47.653	47.43	47.398	46.693
	Ω_1	3.425	3.421	3.425	3.42	3.422	3.415
0.08	Ω_2	15.938	15.797	15.942	15.84	15.840	15.744
	Ω_3	35.625	35.462	35.699	35.35	35.271	34.960

Table 4 The first four frequencies (dimensionless) of fixed-fixed beams

Ω	Present study		Tong *et al.* (1995)			Cleghorn and Tabarrok (1992)
	Δn=8	Δn=50	Scheme (a)	Scheme (b)	Scheme (c)	
Ω_1	15.564	15.490	15.689	15.588	15.639	15.647
Ω_2	41.236	41.061	41.043	40.803	40.923	41.149
Ω_3	75.718	75.310	75.741	75.345	75.543	76.493
Ω_4	118.007	117.323	117.236	116.692	116.964	119.591

5.3 Fixed-fixed tapered Timoshenko beams

A fixed-fixed tapered Timoshenko beam is considered as shown in Fig. 6. It is characterized by the following parameters: $\alpha = -0.5$, $b = 2.54$ cm, $h_1 = 2.54$ cm, $l = 25.4$ cm, $\chi = 0.667$, $E = 210$ GPa, and $G = 80$ GPa. This example has been studied by Cleghorn and Tabarrok (1992) and by Tong *et al.* (1995). Table 4 compares the results obtained from various methods. Cleghorn and Tabarrok (1992) used the static deflection shapes to construct the dynamic stiffness that renders an exact stiffness matrix and an underestimated mass matrix. Therefore, the dynamic stiffness method also gives the upper bound of exact solutions. Tong *et al.* (1995) approximated a non-prismatic beam with a stepped beam. Since it changes both stiffness and mass distributions simultaneously, the so-called step-reduction method may or may not provide an upper bound of exact solutions. In Table 4, the natural frequencies were obtained by discretizing the beam into 100 step-reduction elements with three schemes: (a) lowest section, (b) highest section, and (c) average section. It is observed from Table 4 that, the MMPM is more accurate than the dynamic stiffness method for all the four natural frequencies when $\Delta n = 50$, and for all but the second natural frequency when $\Delta n = 8$. In general the MMPM is in agreement with the step-reduction method.

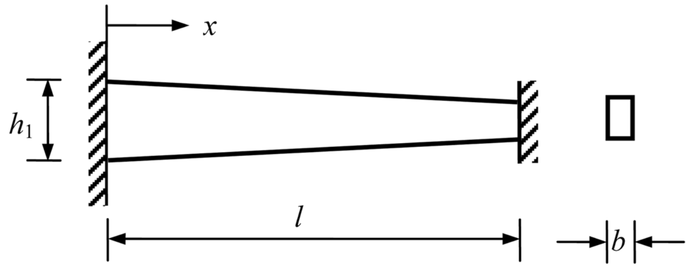


Fig 6. A clamped-clamped tapered Timoshenko beam

6. Conclusions

This paper introduces a modified modal perturbation method (MMPM) for the evaluation of natural frequencies and mode shapes of non-prismatic Timoshenko beams. This method treats a Timoshenko beam as a perturbed system of its corresponding prismatic Euler-Bernoulli beam with the same span length and support conditions. In this case, the perturbation factors include both the effects of rotational inertia and shear deformation and the effect of non-prismatic cross section. Based on extensive analyses and numerical results, the following conclusions can be drawn:

1. The new MMPM can decouple a pair of differential equations for a non-prismatic Timoshenko beam into a set of algebraic equations by using the vibration modes of the corresponding Euler-Bernoulli beam as Ritz functions. In doing so, a semi-analytical solution can be derived for the Timoshenko beam.
2. In the case of a simply-supported prismatic Timoshenko beam, the new MMPM method results in the exact solution. In the case of a cantilevered prismatic Timoshenko beam, it improves the accuracy of the results from the previous MPM due to the introduction of a modification shape function to account for free-end conditions.
3. For all support conditions investigated, the new MMPM method is always superior to the previous MPM, providing comparable results of non-prismatic Timoshenko beams with other numerical methods.
4. As the number of vibration modes included in perturbation analysis increases, the end results converge to the exact solutions. The tapering parameter α has a significant effect on the convergence rate, but the slenderness factor is negligible from a practical point of views. If the j^{th} mode characteristics of a Timoshenko beam are interested, including $(8+j)$ number of vibration modes from its corresponding Euler-Bernoulli beam is sufficient in engineering applications.

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