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Generalized curved beam on elastic foundation solved by transfer matrix method

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Abstract. A solution of space curved bars with generalized Winkler soil found by means of Transfer Matrix Method is presented. Distributed, concentrated loads and imposed strains are applied to the beam as well as rigid or elastic boundaries are considered at the ends. The proposed approach gives the analytical and numerical exact solution for circular beams and rings, loaded in the plane or perpendicular to it. A well-approximated solution can be found for general space curved bars with complex geometry. Elastic foundation is characterized by six parameters of stiffness in different directions: three for rectilinear springs and three for rotational springs. The beam has axial, shear, bending and torsional stiffness. Numerical examples are given in order to solve practical cases of straight and curved foundations. The presented method can be applied to a wide range of problems, including the study of tanks, shells and complex foundation systems. The particular case of box girder distortion can also be studied through the beam on elastic foundation (BEF) analogy.

Keywords: elastic foundation; curved beam; transfer matrix; generalized Winkler soil

1. Introduction

Winkler model of the beam on elastic foundation consists of a beam lying on an indefinite layer of springs, one independent from each other and characterized by a finite value of stiffness in settlement direction (Hetenyi 1946). Winkler soil has only one parameter of stiffness but other soil models have been introduced (Selvadurai 1979), by taking into account interaction between spring displacements, through tension in elastic membrane (Filonenko-Borodich model) or shear layer (Pasternak model): in these cases two-parameter models have been developed.

A generalized Winkler soil can be seen as a model in which every segment of the beam is connected to adjacent segments by “serial” springs and to soil by “parallel” springs. Serial springs between beam segments represent axial, bending, shear and torsional stiffness, while parallel springs are related to soil and its stiffness in every direction. A restricted concept of generalized Winkler soil has been defined by Kerr (1964) for rectilinear beams, in which elastic springs are considered in the translational and rotational directions with their own values of stiffness. In the present study the generalized Winkler model is seen as a one-dimensional beam curved in space, treated with its

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three components of displacement and three components of rotation, surrounded by a Winkler soil with three translational and three rotational springs.

The classical problem of a Bernoulli rectilinear beam on elastic foundation (BEF) leads to a 4^{th} order differential equation or to an equivalent system of four 1^{st} order differential equations. It can be solved by different ways and analytical solutions can be found for the most common cases of boundaries, distributed and concentrated loads (Hetenyi 1946). All efforts of recent works in literature are finalized to find finite elements formulation of this problem, by increasing as much as possible the accuracy of solution; generally the beam has to be divided into a great number of elements in order to achieve the requested precision. Eisenberger *et al.* (1985) found an exact stiffness matrix for the beam on elastic foundation, based on the 4^{th} order differential equation.

Classical solutions does not take into account shear deformation and different approaches have been proposed for Timoshenko beam models (Aydoğan 1995, Gelu Onu 2008, Ergüven and Gedikli 2003).

Other authors focalized their attention on curved beams on elastic foundation. From the early works of Volterra (1952), who considers an elastic curved beam on a Winkler soil with stiffness in the only direction of settlements, solved by a system of two differential equations (one of 2^{nd} order for bending moment and one of 1^{st} order for twisting moment), different researchers take inspiration for analytical and finite element procedures in order to solve the problem of curved beams on Winkler soil. Rodriguez (1961) proposed an analytical solution of rings on elastic foundation, in which the elastic beam lies on a soil with translational and torsional springs. He solves this problem through the statement of a 6^{th} order differential equation; it is equivalent to a system of a classical 4^{th} order differential equation with a 2^{nd} order equation. The same problem of circular beams with translational and rotational springs, have been faced by Aköz *et al.* (1996) with a mixed FE solution for elements in plane. Another FE formulation for curved beams on Winkler soil with translational and rotational springs is given by Dasgupta and Sengupta (1988) for in-plane elements in which bending, shear and torsional stiffness are considered. Banan *et al.* (1989) proposed a comprehensive study with FE analyses of some examples, based on a particular approach in which all components of displacement and rotation are taken into account both for the curved beam and for stiffness of Winkler soil. They found two solving systems of 1^{st} order differential equations by writing equilibrium, constitutive and compatibility relations and then solution is given by two exponential matrices in order to find the stiffness matrix of the element. Haktanir and Kiral (1993) solved the more general problem of circular and helicoidal structures on the generalized elastic foundation, deriving transfer and stiffness matrices. This last approach is similar to that of Banan *et al.* (1989). It considers rings on Winkler soil with loads in the plane and perpendicular to it (ring foundations) and the problem of pipes longitudinally loaded is presented with numerical examples; helicoidal structures are also studied as 3D Timoshenko beams.

A number of studies addressed instead the problem of force and displacement discontinuities on rectilinear beams (Yavari 2001). Chen (1998) solved BEF with concentrated discontinuities by the Differential Quadrature method while Colajanni *et al.* (2009) solved it by generalized functions. This last study gives an exact closed form solution for Bernoulli rectilinear BEF with all kinds of discontinuities due to loads or constraints. Kim *et al.* (2005) derived the exact static element stiffness matrices of thin-walled beam-columns on elastic foundations while Kim and Shin (2009) gave a series solution for the deflection of thin-walled Timoshenko curved beams on elastic foundation. Guo *et al.* (2002) gave a solution method for beams on non-uniform elastic foundations, by using Green's function formulation.

Arici (1985) established the reciprocal conjugate method for rectilinear beams, as a generalization of Mohr analogy, on generalized Winkler soil by considering static loads and inelastic imposed actions (distortions). He also solved analytically the problem of space curved bars and established a conjugate method in which soil parameters are related to those of conjugate beam and vice-versa. In this way Winkler parameters are related to axial, shear, bending and torsional stiffness of conjugate beam while beam stiffness is related to conjugate soil parameters. Moreover the complete six-order vectorial character of conjugate method has been underlined (Arici 1989).

Besides finite element formulations some authors gave transfer matrices of beams on elastic foundation. Although these authors derive Transfer Matrix Method (TMM) on the base of Pestel and Leckie (1963) studies, this method has been widely applied by French authors as Courbon (1972) and Lacroix (1967). Straight beams on Winkler soil have been solved through this approach by Géry and Calgaro (1973). They consider a rectilinear beam with a one-parameter elastic soil, giving the exact expressions of transfer matrices in order to solve the classical Winkler problem. Transfer matrices are derived directly from the system of four 1st order differential equations of the beam and the problem is solved for in-plane beams.

Arici and Granata (2005) expanded the Transfer Matrix Method to general space curved bars, not resting on a soil, and applied the method to curved bridges (Arici and Granata 2007), by considering axial, shear and torsional deformations. They derived transfer matrices from an energetic approach, based on the principle of stationary total potential energy with the statement of an Hamiltonian functional, deriving a mixed system of twelve 1st order differential equations. This last approach is quite different from that of Pestel and Leckie, which is only a matrix form of the 4th order differential equation of elastic straight beams; in fact it is based on an energetic functional for general 3D curved elastic beams and each quantity has its own physical and mechanical meaning.

The classical generalized Winkler soil presented only two values of stiffness: one concerning vertical settlements and one concerning bending rotations. In the present study a complete solution is presented for space curved bars surrounded by a generalized Winkler soil with six parameters of spring stiffness: three concerning displacements and three concerning rotations. The complete solution system is given and the transfer matrix of the beam is derived together with load vectors for every kind of static and geometric action. Rigid and elastic concentrated boundary conditions at the ends of bar can be applied. The followed approach permits to find the exact solution of circular beam on elastic foundation and a very good approximation for generally space curved bars on elastic soil. The advantage of this approach with respect to other numerical solutions and particularly to FEM, consists of a reduced number of equations to be solved. In fact, when the entire spatial structure is divided into segments, transfer matrices of segments can be multiplied each other; so the solution system is given by a maximum of 12 equations and it does not increase its dimension, as it occurs for finite elements. The solution of Timoshenko circular beams and rings with the generalized Winkler soil is explained through numerical examples. A generalized ring foundation of a tank is discussed and comparisons with literature data are given. The presented methodology can be used to solve problems related to tanks, shells or practical cases of foundations with complex geometry. Moreover related problems to Winkler foundations, as for example, box girder distortion of bridges, can be solved through the so-called BEF analogy (Arici *et al*. 2010).

2. Transfer matrix method for space curved bars with generalized Winkler soil

With reference to a space curved bar of total axial length L, having low initial curvature and consisting of homogeneous isotropic material with linearly elastic behaviour, let s be the curvilinear coordinate joining the centroids of cross-sections, with respect to a fixed origin $s = 0$ (Fig. 1).

Consider the Frenet local coordinate system formed by the unit vectors $\mathbf{n}(s) = \mathbf{i}_1$, $\mathbf{b}(s) = \mathbf{i}_2$, $\mathbf{t}(s) = \mathbf{i}_3$, being respectively the normal, binormal and tangent to the bar axis at the general coordinate s. Principal axes of inertia of cross section are always assumed coincident with $\mathbf{i}_1$ and $\mathbf{i}_2$. Warping due to torsion and cross-section distortion are neglected. In these hypotheses the configuration of the deformed bar is fully described by the generalized displacement column array $\mathbf{u}(s)$ having six components (three displacements $u_i(s)$ and three rotations $\varphi_i(s)$, with $i = 1, 2, 3$)

$$\mathbf{u}(s) = [u_1(s),\ u_2(s),\ u_3(s),\ \varphi_1(s),\ \varphi_2(s),\ \varphi_3(s)]^T \tag{1}$$

The bar is subjected to applied distributed external static actions (loads $p_i(s)$ and moments $m_i(s)$) collected in the column array

$$\mathbf{f}_e(s) = [p_1(s),\ p_2(s),\ p_3(s),\ m_1(s),\ m_2(s),\ m_3(s)]^T \tag{2}$$

Moreover the generalized strain array, composed of uni-axial strains and curvatures, is

$$\mathbf{q}(s) = [\varepsilon_1(s),\ \varepsilon_2(s),\ \varepsilon_3(s),\ \kappa_1(s),\ \kappa_2(s),\ \kappa_3(s)]^T \tag{3}$$

In the same way stress resultant array $\mathbf{Q}(s)$ is defined

$$\mathbf{Q}(s) = [V_1(s),\ V_2(s),\ V_3(s),\ M_1(s),\ M_2(s),\ M_3(s)]^T \tag{4}$$

in which V_1 and V_2 are shear forces in direction $\mathbf{i}_1$ and $\mathbf{i}_2$ while V_3 is axial force; M_1 and M_2 are bending moments while M_3 is twisting moment referred to centroid. The total strain array (3) can be divided into two parts, one containing elastic internal strains $\mathbf{q}_i(s)$ and one containing external inelastic imposed strains $\mathbf{q}_e(s)$ (i.e., thermal strains)

$$\mathbf{q}(s) = \mathbf{q}_i(s) + \mathbf{q}_e(s) \tag{5}$$

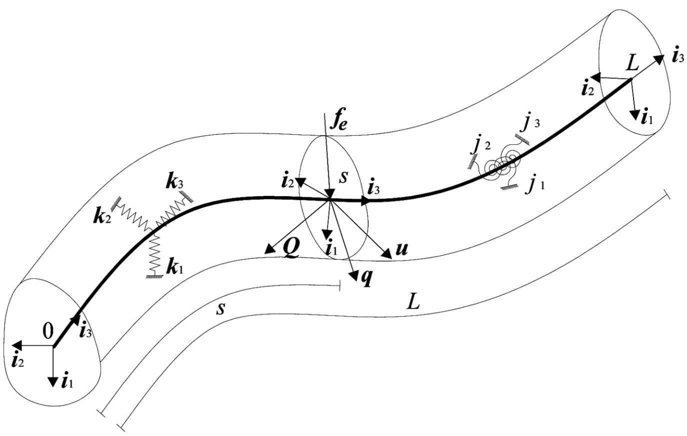

Fig. 1 Space curved bar

Morever let $\boldsymbol{\Phi}(s)$ be the non singular and symmetric flexibility matrix of the elastic bar, inverse of elastic stiffness matrix $\mathbf{E}(s)$

$$\boldsymbol{\Phi}(s) = \mathbf{E}^{-1}(s) = \begin{bmatrix} \frac{\chi_1}{GA} & \frac{\chi_{12}}{GA} & 0 & 0 & 0 & \frac{\xi_2^C}{GJ} \\ \frac{\chi_{12}}{GA} & \frac{\chi_2}{GA} & 0 & 0 & 0 & \frac{\xi_1^C}{GJ} \\ 0 & 0 & \frac{1}{EA} & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{EJ_1} & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{EJ_2} & 0 \\ \frac{\xi_2^C}{GJ} & \frac{\xi_1^C}{GJ} & 0 & 0 & 0 & \frac{1}{GJ} \end{bmatrix} \tag{6}$$

in which χ_1, χ_2, χ_{12} are shear factors; ξ_1^C, ξ_2^C are shear centre coordinates with respect to cross section centroid; G and E are shear and Young moduli; A the cross-section area; J_i (for i = 1, 2) the principal inertia moments and J the torsional constant. Displacements and internal forces at coordinate s are always related to the centroid. If cross section has one axis of symmetry, it results $\chi_{12} = 0$ and one of the distances ξ_i^C between shear centre and centroid is zero. When cross section has two axes of symmetry, shear centre coincides with centroid and $\boldsymbol{\Phi}(s)$ becomes a diagonal matrix. Elastic constitutive equations can be written

$$\mathbf{q}_i(s) = \mathbf{E}^{-1}(s)\,\mathbf{Q}(s) \tag{7}$$

If the space curved bar is surrounded by an elastic Winkler medium, let $\mathbf{R}(s)$ be the matrix characterizing the generalized Winkler spring stiffnesses

$$\mathbf{R}(s) = \begin{bmatrix} k_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & k_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & j_1 & 0 & 0 \\ 0 & 0 & 0 & 0 & j_2 & 0 \\ 0 & 0 & 0 & 0 & 0 & j_3 \end{bmatrix} \tag{8}$$

in which k_1, k_2, k_3, j_1, j_2, j_3 are translational and rotational Winkler coefficients at coordinate s (Fig. 1). Constitutive equations of Winkler generalized soil are

$$\mathbf{f}_i(s) = -\mathbf{R}(s)\,\mathbf{u}(s) \tag{9}$$

being $\mathbf{f}_i(s)$ the reactive forces acting on the bar. They can be collected in a unique array together with external forces applied to the bar, obtaining array $\mathbf{f}(s)$ of total distributed forces

$$\mathbf{f}(s) = \mathbf{f}_i(s) + \mathbf{f}_e(s) \tag{10}$$

By introducing the square gradient matrix $\mathbf{B}(s)$ of displacements

$$\mathbf{B}(s) = \begin{bmatrix} \mathbf{B}_0 & \mathbf{B}_1 \\ \mathbf{0} & \mathbf{B}_0 \end{bmatrix} \tag{11}$$

and by naming τ_0 and k_0 the initial tortuosity and the initial curvature ($1/R$) of the bar axis (Arici 1989), the sub-matrices of $\mathbf{B}(s)$ can be written as follows

$$\mathbf{B}_0(s) = -\mathbf{B}_0^T(s) = \begin{bmatrix} 0 & -\tau_0 & k_0 \\ \tau_0 & 0 & 0 \\ -k_0 & 0 & 0 \end{bmatrix}, \quad \mathbf{B}_1(s) = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{12a,b}$$

For straight bars $\tau_0 = k_0 = 0$ and $\mathbf{B}_0(s)$ is a null matrix, while for bars with in-plane curvature, $k_0 = 1/R$ and $\tau_0 = 0$. A finite value of tortuosity $\tau_0 \neq 0$ can be considered for the study of helix or complex curved bars with helicoidal shape (Haktanir *et al*. 1996, Arici and Granata 2005).

Field compatibility equations of the bar are

$$\mathbf{u}'(s) = -\mathbf{B}\mathbf{u}(s) + \mathbf{E}^{-1}(s)\, \mathbf{Q}(s) + \mathbf{q}_e(s) \tag{13}$$

where symbol ( )' represents the total derivative with respect to the curvilinear coordinate s.

Equilibrium equations instead can be written through the transpose of matrix $\mathbf{B}(s)$, obtaining the relation

$$\boldsymbol{Q}'(s) = \boldsymbol{B}^T(s)\, \boldsymbol{Q}(s) - \boldsymbol{R}(s)\, \boldsymbol{u}(s) - \boldsymbol{f}_e(s) \tag{14}$$

In this way the classical problem of an elastic beam lying on a generalized Winkler soil has been stated by considering 3-D curved bars and different elastic soil properties in each direction.

Let $\mathbf{J}$ be now the symplectic operator and $\mathbf{A}(s)$ the symmetric matrix collecting material, soil and gradient matrices $\mathbf{E}^{-1}(s)$, $\mathbf{R}(s)$ and $\mathbf{B}(s)$ of the bar

$$\mathbf{J} = \begin{bmatrix} 0 & \mathbf{I}_6 \\ -\mathbf{I}_6 & 0 \end{bmatrix}, \quad \mathbf{A}(s) = \begin{bmatrix} -\mathbf{R}(s) & -\mathbf{B}^T(s) \\ -\mathbf{B}(s) & \mathbf{E}^{-1}(s) \end{bmatrix} \tag{15a,b}$$

in which $\mathbf{I}_6$ is the 6×6 diagonal unitary matrix. Generally, along abscissa s, matrix $\mathbf{A}(s)$ has variable values. By defining the mixed state array

$$\mathbf{z}(s) = [\mathbf{u}^T(s), \mathbf{Q}^T(s)]^{\mathrm{T}} \tag{16}$$

which collects generalized displacements and internal forces, and the external action array

$$\mathbf{d}_e(s) = [\mathbf{f}_e{}^T(s), \mathbf{q}_e{}^T(s)]^T \tag{17}$$

which collects all external distributed actions (loads and imposed strains), the governing system of the elasto-static problem, containing compatibility and equilibrium equations, can be written as a canonical Hamiltonian system of twelve 1st order differential equations

$$\mathbf{z}' = \mathbf{J}(\mathbf{A}\mathbf{z} + \mathbf{d}_e) \tag{18}$$

Eq. (18) is equivalent to Eqs. (13) and (14), but in order to solve a linear differential equation system with constant coefficients, matrix **A** has to be with constant values in the circular or straight segment 0-s. When geometric, material and soil properties vary along the bar, it is possible to divide the entire structure of length L in different segments, each one with constant properties. In this way matrices $\mathbf{B}(s)$, $\mathbf{E}^{-1}(s)$ and $\mathbf{R}(s)$ become constant and independent from s, i.e., contains constant geometric, material and soil properties in the considered segment.

The derivation of the Hamiltonian system from energetic considerations and the integration of solving system are not reported here but they can be found in Arici and Granata (2005). Solution of system (18) is given by the following relation

$$\mathbf{z}(s) = \begin{Bmatrix} \mathbf{u}(s) \\ \mathbf{Q}(s) \end{Bmatrix} = \mathbf{C}(s)\begin{Bmatrix} \mathbf{u}(0) \\ \mathbf{Q}(0) \end{Bmatrix} + \begin{Bmatrix} \mathbf{N}_u(s) \\ \mathbf{N}_Q(s) \end{Bmatrix} = \mathbf{C}(s)\mathbf{z}(0) + \mathbf{N}(s) \tag{19}$$

in which $\mathbf{C}(s)$ is the *fundamental matrix* of the homogeneous system of differential equations. It also can be seen as the *Transfer Matrix* of the elastic structure at coordinate s and it can be found as an exponential matrix, through the following relation

$$\mathbf{C}(s) = \exp(\mathbf{JA}\, s) \tag{20}$$

in which **A** has constant values.

Transfer matrix has the following properties: $\boldsymbol{C}(0) = \boldsymbol{I}_{12}$, $\mathbf{C}(-s) = \mathbf{C}^{-1}(s)$, $\mathbf{C}(s-\eta) = \mathbf{C}(s)\mathbf{C}(-\eta)$.

In Eq. (19) the column array $\mathbf{N}(s)$ contains all terms directly related to external actions

$$\mathbf{N}(s) = \int_0^s \mathbf{C}(s-\eta)\mathbf{J}\mathbf{d}_e(\eta)d\eta \tag{21}$$

Transfer matrix **C** can be found in closed form by relation (20) with a common software package of mathematical computation with symbolic calculus. It can be found also as a numerical matrix when values are given to all terms of **A**.

In the case of a curved bar which lies on a plane and is loaded in the same plane (arches), the active degrees of freedom and the related internal forces are only the six odd components of state array $\mathbf{z}(s)$. When loads are applied instead perpendicularly to the plane (curved beams), only the six even components are activated. Transfer matrices and state arrays of 2-D problems can be obtained by deleting even or odd rows and columns of the general ones, defined above.

Through this approach, by combining transfer matrix terms, it is also possible to obtain the stiffness matrix of a 3-D curved bar for a more traditional analysis (Arici and Granata 2005), useful for the statement of a finite element procedure.

3. Analysis of complex structures with TMM

When the space curved bar (total axial length L) is made up of several elements of length l_i ($l_i < L$), each of them having different but constant geometric and elastic properties, the structure can be divided into <n> segments (for i = 1..n). For the i-th segment it is possible, through Eq. (19), to obtain the state array $\mathbf{z}_i(s)$ in the generic section of coordinate s, with reference to array $\mathbf{z}_i(0)$ in the initial section of the segment (Fig. 2).

Moreover, when a node I, between segments <i> and <$i+1$>, presents imposed kinematical

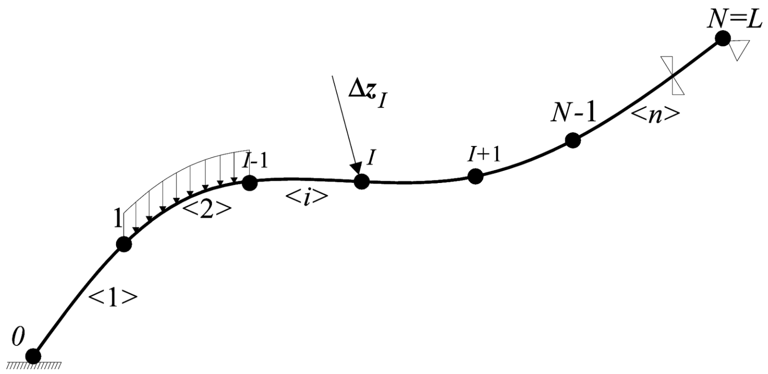

Fig. 2 Complex structure divided into segments with general boundaries and actions applied

discontinuities $\Delta\mathbf{u}_I$ (concentrated strains) or static discontinuities $\Delta\mathbf{Q}_I$ (concentrated forces), they can be taken into account by considering compatibility and equilibrium on the left and right hands of node I. We must have

$$\mathbf{z}_{i+1}(0) = \mathbf{z}_i(l_i) + \Delta\mathbf{z}_I, \quad \Delta\mathbf{z}_I = \begin{Bmatrix} \Delta\mathbf{u}_I \\ \Delta\mathbf{Q}_I \end{Bmatrix} \tag{22}$$

which is of course valid when $\Delta\mathbf{z}_I = \mathbf{0}$.

In order to compact solution (19), it can be convenient to add a unitary component to the state array $\mathbf{z}_i(s)$ and one row and one column to transfer matrices, bringing their dimension to 12 + 1. It is thus possible to define the new expanded state array $\mathbf{S}_i(s)$, obtaining the compact form of Eq. (19)

$$\boldsymbol{S}_i(s) = \boldsymbol{F}_i(s)\, \boldsymbol{S}_i(0) \tag{23}$$

in which new expanded state array $\mathbf{S}_i(s)$ and field transfer matrix $\mathbf{F}_i$ of the i-th segment are defined

$$\mathbf{S}_i(s) = \begin{bmatrix} \mathbf{z}_i(s) \\ 1 \end{bmatrix}, \quad \mathbf{F}_i(s) = \begin{bmatrix} \mathbf{C}(s) & \mathbf{N}(s) \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{24a,b}$$

It is also convenient to introduce a nodal point matrix $\mathbf{P}_I$ for the imposed discontinuities in node I

$$\mathbf{P}_I = \begin{bmatrix} \mathbf{I}_{12} & \Delta\mathbf{z}_I \\ \mathbf{0}^T & 1 \end{bmatrix} \tag{25}$$

being $\mathbf{I}_{12}$ the 12 × 12 unitary diagonal matrix. In this way, for the element <i> the state array $\mathbf{S}_i(s)$ can be expressed by the recursive formula

$$\mathbf{S}_i(s) = \mathbf{F}_i(s) \cdot \mathbf{P}_{I-1} \cdot \mathbf{F}_{i-1}(l_{i-1}) \cdot \ldots\ldots\ldots\ldots \cdot \boldsymbol{P}_1 \cdot \mathbf{F}_1(l_1) \cdot \mathbf{S}_1(0) \tag{26}$$

Proceeding up to the final right node N, coinciding with the end of the bar, the linear system of equations on the six unknown quantities is finally obtained

$$\mathbf{S}_n(l_n) = \mathbf{F}_n(l_n) \cdot \mathbf{P}_{N-1} \cdot \mathbf{F}_{n-1}(l_{n-1}) \cdot \mathbf{P}_I \cdot \mathbf{F}_i(l_i) \ldots\ldots\ldots\ldots \cdot \boldsymbol{P}_1 \cdot \mathbf{F}_1(l_1) \cdot \mathbf{S}_1(0) = \mathbf{F}_{\text{tot}}\, \mathbf{S}_1(0) \tag{27}$$

and it can be solved by imposing the six known boundary conditions.

Note that in system (27), some terms of the end state array $\mathbf{S}_n(l_n)$ are known and other unknown as well as for the beginning state array $\mathbf{S}_1(0)$, depending on the boundary conditions at the ends. So, a partition of state arrays needs to separate known elements from unknowns and total matrix $\mathbf{F}_{\text{tot}}$ has to be reorganized by changing rows and columns. When unknown quantities at the beginning and at the end of the bar have been found, the system stated by relation (27) is solved. In this way state array $\mathbf{S}_1(0)$ is completely known and by means of Eq. (26) it is possible to calculate the column state array in each element, i.e. arrays of displacements $\mathbf{u}_i(s)$ and forces $\mathbf{Q}_i(s)$ for the whole structure. Any kind of space curved bar can be subdivided into an appropriate number of small straight or curved elements having constant geometric and mechanical properties.

As it can be seen from Eq. (27), by increasing the number of segments, the solving system does not increase its dimension, because the total transfer matrix $\mathbf{F}_{\text{tot}}$ of the entire structure is always given by multiplication of matrices of the same order. As a consequence, computational burden is limited to simple matrices multiplication, unlike Finite Element Method in which the dimension of solving stiffness matrix depends on the number of elements used to subdivide the structure, increasing with them. Although transfer matrices for circular elements can be defined, it is possible to approximate a general complex curve in space by a sequence of a great number of straight segments, whose transfer matrix definition is very simple. The simple multiplication of transfer matrices in Eq. (27) maintains the 12$^{\text{th}}$ order of the linear system, without excessive increasing in computational complexity. Moreover point rotation matrices for sudden changes in structure axis direction can be introduced apart from nodal point matrices, when straight segments present relative rotations between their axes for geometrical reasons.

4. Numerical examples

In this section numerical examples are given and discussed for different practical cases of beams on elastic foundation, by comparing them with literature data, in order to demonstrate the efficiency of the proposed method.

Example 1

The first example is derived from Colajanni *et al.* (2009); it consists of a classical rectilinear beam on Winkler soil which represents a foundation subjected to distributed and concentrated loads.

Geometric characteristics and load data are shown in Fig. 3. The elastic modulus is E = 30 GPa while the Winkler constant is k = 150 MPa. Calculus was repeated two times: without and with shear deformability (Bernoulli and Timoshenko beams) and results are shown in Figs. 4 and 5.

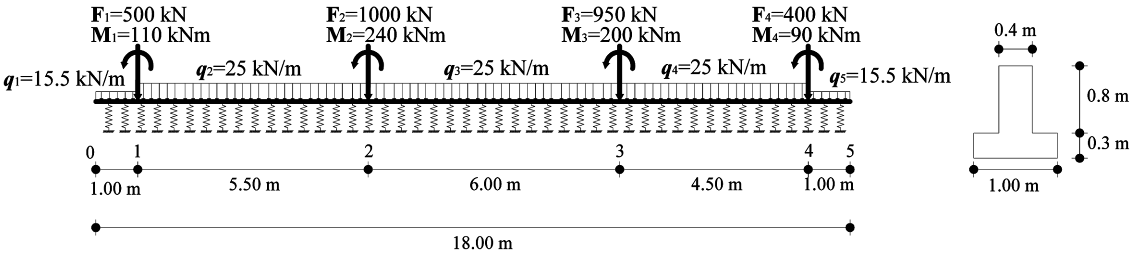

Fig. 3 Geometric characteristics of rectilinear beam of example 1

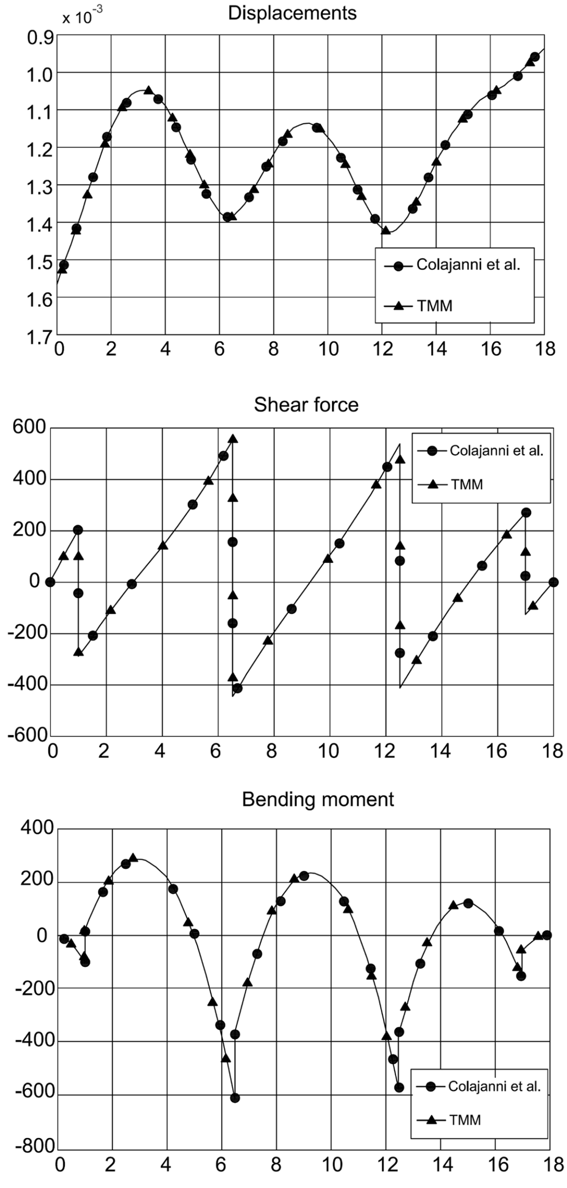

Fig. 4 Displacement [m], Shear force [kN] and bending moment [kNm] diagrams of example 1. Bernoulli beam

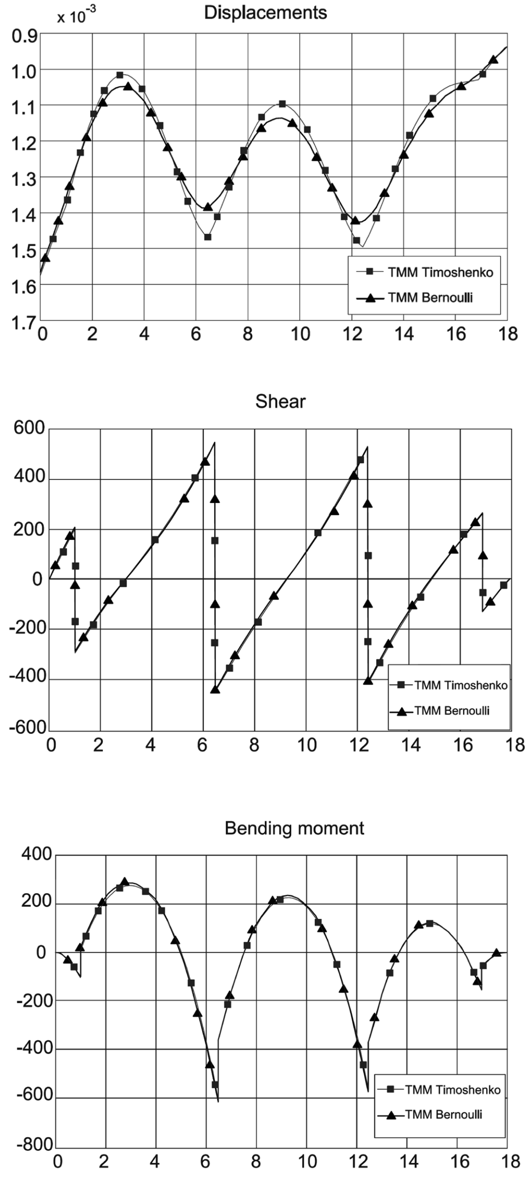

Fig. 5 Comparison between TMM with and without shear stiffness. Displacements [m], shear [kN] and bending moment [kNm] diagrams of example 1

Solution has been found through the proposed method, by dividing the total length of the beam into five segments, each characterized by the same cross section properties, with different lengths and distributed load values. Four nodal point matrices have been introduced in order to consider concentrated loads F_I and M_I.

Graphs of Fig. 4 show results in terms of vertical displacements, shear force and bending moment diagrams. They reveal the very good agreement with results obtained by Colajanni *et al.* (2009) for the solution of Bernoulli beam.

About shear deformability, the calculus of Timoshenko beam (Fig. 5) shows that differences can be appreciated in displacements with respect to Bernoulli solution, but no significant effects can be seen in internal force diagrams.

Example 2

The second example is derived from Volterra (1952). Aköz *et al.* (1996), Haktanir and Kiral (1993), Dasgupta and Sengupta (1988), solved the same problem with different methods. It consists of a whole ring on Winkler soil subjected to four vertical loads, related to four columns of an upper tank. A comparison is given with literature data and with solutions obtained by a SAP2000 model in which the ring is divided into 64 and 360 straight elements. Solution found by SAP2000 has been chosen as an example of a common software package which could be used by an engineer in his analysis, even if this solution presents a less numerical precision when only 64 elements are used. In order to converge to the exact solution, the number of finite elements has to be increased much more, so the model with 360 elements (one for each degree) gives results with a better precision.

Aköz *et al.* (1996) solved the problem by using a mixed finite element solution with 20 circular elements. Haktanir and Kiral (1993) solved instead the same example with a stiffness matrix method on a curved beam, including axial and shear deformation. Dasgupta and Sengupta (1988) divided the ring into 20 and 40 horizontally curved isoparametric finite elements, including shear deformation. Results obtained in this study by TMM have been compared with these literature data by considering the equivalence between SI units and British ones: R = 7.62 m, rectangular cross

Table 1 Comparison of numerical results for example 2. Vertical displacement and shear force

θ [°]	u_2 [m]						V_2 [kN]					
	TMM	Haktanir	Dasgupta	Aköz	SAP2000 (64 el.)	SAP2000 (360 el.)	TMM	Haktanir	Dasgupta	Aköz	SAP2000 (64 el.)	SAP2000 (360 el.)
0	0.008148	0.008150	0.008077	0.008060	0.008156	0.008150	333.6	333.6	-	333.6	302.1	328.0
11.25	0.007092	0.007096	0.007071	-	0.007103	0.007020	213.4	213.5	-	-	186.6	211.2
22.5	0.005208	0.005208	0.005212	0.005198	0.005217	0.005210	117.8	117.9	-	119.4	98.5	118.1
33.75	0.003650	0.003642	0.003658	-	0.003653	0.003652	50.3	50.3	-	-	36.7	51.7
45	0.003070	0.003050	0.003109	0.003116	0.003062	0.003063	0.0	0.0	-	0.0	0.0	0.0

Table 2 Comparison of numerical results for example 2. Bending moment and torsion

θ [°]	M_1 [kNm]						M_3 [kNm]					
	TMM	Haktanir	Dasgupta	Aköz	SAP2000 (64 el.)	SAP2000 (360 el.)	TMM	Haktanir	Dasgupta	Aköz	SAP2000 (64 el.)	SAP2000 (360 el.)
0	-592.80	-593.48	-589.73	-596.49	-588.32	-592.99	0.00	0.00	0.00	0.00	0.00	0.00
11.25	-176.20	-177.33	-169.46	-	-170.07	-177.07	72.46	72.95	71.85	-	75.56	73.75
22.5	84.17	83.04	88.12	84.21	89.27	82.91	79.20	79.89	81.34	80.30	81.34	79.92
33.75	218.70	219.08	216.91	-	222.86	218.86	48.20	48.40	48.81	-	37.56	49.44
45	260.51	260.53	264.36	264.86	261.61	260.62	0.00	0.00	0.00	0.00	0.00	0.00

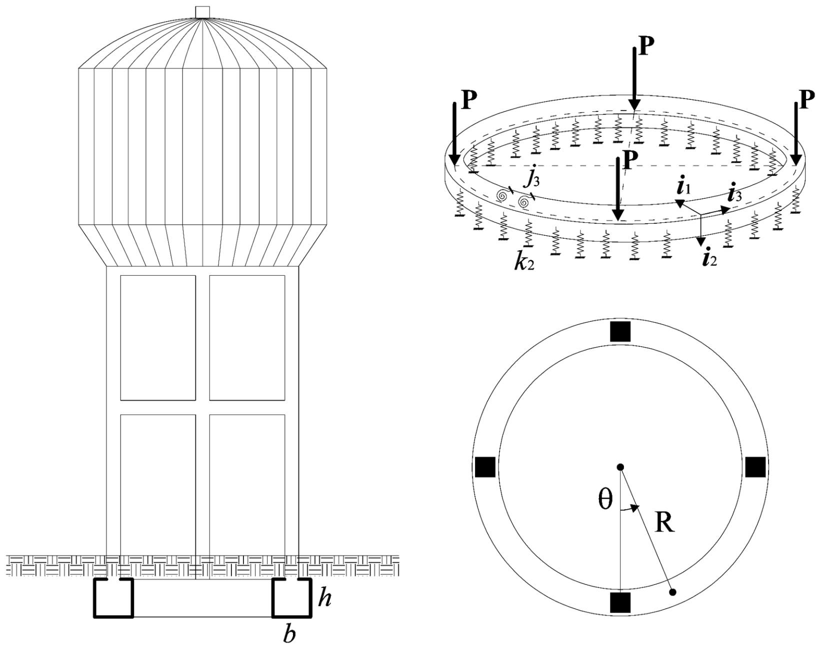

Fig. 6 Characteristics of tank and ring foundation

section of foundation with $b = h = 0.762$ m, elastic moduli $E = 20{,}683$ MPa, $G = 8{,}618$ MPa, soil parameters $k_2 = 10.34$ MPa, $j_3 = 5.00$ kNm/m. Geometric characteristics are shown in Fig. 6; vertical load is $P = 667.2$ kN. Tables 1 and 2 show results of this study in comparison with those given by the above mentioned authors. The θ angle is in the anticlockwise direction and it is depicted in Fig. 6.

TMM solution has been found by dividing the ring into four circular segments, each of 90° (four equal transfer matrices), and by introducing four nodal point matrices. This division is made only in order to apply concentrated loads through point matrices. It is possible to use only one segment for the whole ring if the array $\mathbf{N}(s)$ of Eq. (21) is calculated for concentrated loads by introducing the delta of Dirac function in the integration of load terms $\mathbf{d}_e(s)$. On a mathematical point of view, in order to take into account discontinuities, the introduction of the generalized functions, as the unit step function and the delta of Dirac, could be considered a more elegant solution, but on a computational point of view it is more convenient to divide the entire length into segments and to introduce the nodal point matrices with the values of discontinuities due to concentrated loads.

Results of TMM model are in perfect agreement with those given by the other authors. It is evident that, with respect to FEM models, the same numerical precision can be achieved through TMM without the computational complexity needed by FEM meshes and without increasing the number of elements, being necessary only a few TMM elements in order to insert nodal point matrices for concentrated loads.

Example 3

The third example is derived from the previous one, when the upper tank is subjected to horizontal actions (wind, earthquakes, etc..). It is a generalization of the ring foundation which can

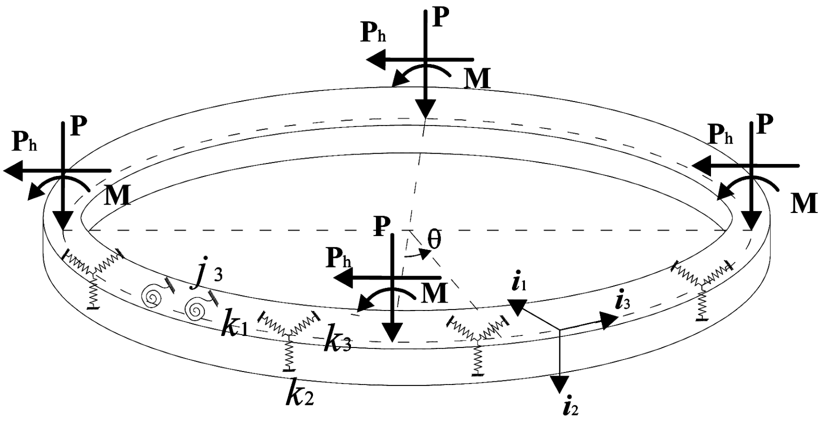

Fig. 7 Loads on the ring for horizontal actions on the tank

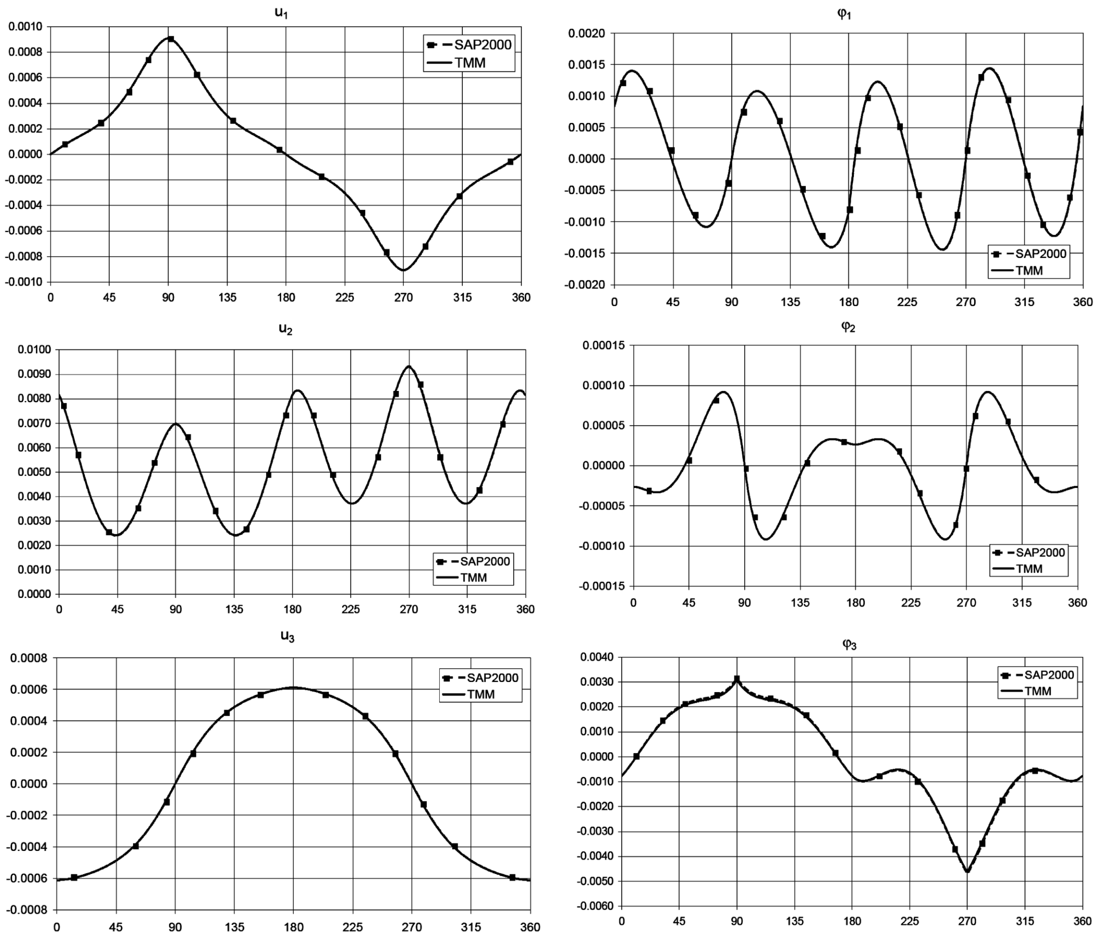

Fig. 8(a) Displacements [m] and rotations [rad] of example 3

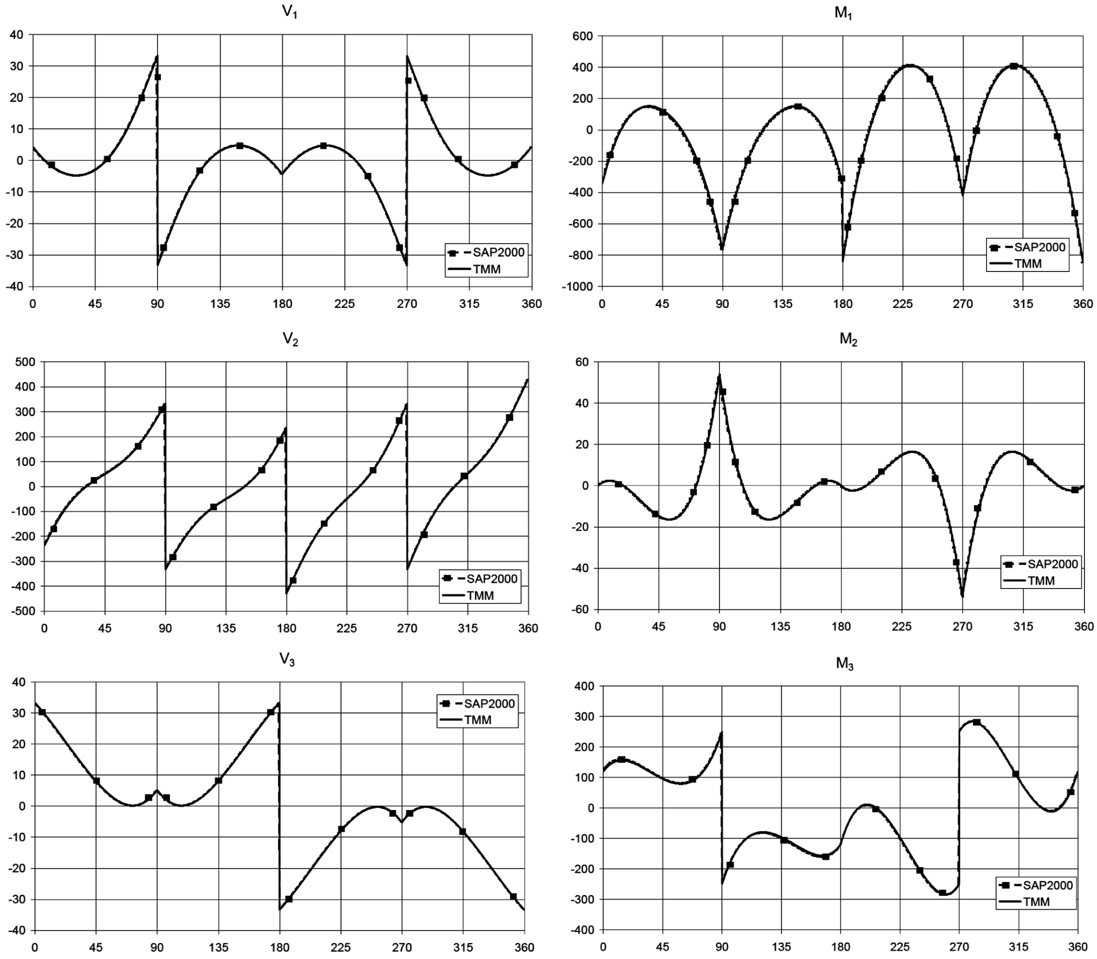

Fig. 8(b) Internal forces [kN] and moments [kNm] of example 3

take into account horizontal forces and bending moments at the base of columns. Moreover a generalized Winkler soil is considered, by introducing different soil stiffness along the three directions k_1, k_2, k_3 and with respect to twisting moment j_3.

The new load condition is shown in Fig. 7, while results are shown in Figs. 8(a) and 8(b) in form of complete displacement and internal force diagrams. A comparison of TMM solution with SAP2000 model (360 elements) is reported in order to evaluate the goodness and efficiency of the proposed method. New data are: P_h = 66.72 kN; M = 500.40 kNm; k_1 = 10.34 MPa; k_3 = 6.89 MPa.

Note that vertical displacement u_1, axial force V_3 and bending moment M_2 are anti-symmetric diagrams, while axial displacement u_3, rotation φ_2 and shear V_1 are symmetric diagrams. The other displacements and internal forces instead have not symmetries in their diagrams. This situation is due to the particular load configuration in which, for the ring, the horizontal forces are two in the tangential direction and two in the radial one, while concentrated couples give discontinuities alternatively in bending moment or in torsion diagrams. So, the entire 0-360^o range is significant and it is shown in figures.

5. Conclusions

In this paper a study which gives the solution of space curved bars surrounded by a generalized Winkler medium has been presented through the Transfer Matrix Method. Numerical examples have been explained in order to solve practical cases of straight and curved foundations. The presented method can be applied to a wide range of problems, including the study of tanks, shells and complex foundation systems or applied to physical problems treated by the BEF analogy.

The problem presented by Rodriguez (1961), consisting of a ring with sinusoidal load and torsional spring stiffness and related to certain types of interconnect structures for rocket vehicles, can be seen as a particular case of the ring on generalized Winkler medium, solved herein.

The problem presented by Aydoğan (1995) and Ergüven and Gedikli (2003), consisting of a long beam with free ends on elastic foundation and solved by these authors with and without shear effect, can be treated in the same way by TMM in order to compare solutions of Bernoulli and Timoshenko beams and the accuracy of methods proposed by different researchers.

The problem presented by Banan *et al.* (1989), consisting of a pipe longitudinally loaded and surrounded by a Winkler medium, is a classical example which is of interest for loaded underground pipes. It can be solved and reproduced through the TMM approach presented here.

The proposed procedure has been used by the authors to solve also the problem of box girder sectional distortion in concrete bridges, by implementing the so-called BEF analogy (Arici *et al.* 2010). Besides this last application, note that general solution obtained by TMM can be always applied to particular cases of Winkler beams derived by BEF analogy, in which a physical problem is represented by a classical 4^{th} order differential equation in analogy with the beam on Winkler soil (Kristek 1979). The solution presented here is valid for each of these physical problems independently from the meaning of differential equation coefficients.

The advantage of the proposed approach with respect to other numerical solutions and particularly to FEM, consists of a reduced number of equations to be solved and of a reduced computational complexity. In fact, when the entire structure is divided into a number of segments, transfer matrices of these segments can be multiplied each other; so the solution of a 3D system is always given by a maximum of 12 equations and it does not increase its dimension with the number of segments, as it occurs for finite elements. Moreover the solution of Timoshenko beams on Winkler soil can be simply found. The presented examples show the efficiency of the proposed method and the high level of accuracy of the solution. The formulation of Transfer Matrix Method for curved beams on generalized Winkler soil, has been already developed by the authors through the energetic Hamiltonian principle and the construction of transfer matrices made by simple exponential matrix operations. This advancement in the definition and use of TMM allows researchers and engineers to have a suitable tool in order to solve complex structures without the high computational complexity of other methods.

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Notations

A	: cross section area;
$\mathbf{A}(s)$	: matrix collecting material and gradient matrices;
$\mathbf{B}$, $\mathbf{B}_0$, $\mathbf{B}_1$	: gradient matrix of displacements and its sub-matrices;
$\mathbf{C}(s)$	: Transfer Matrix at abscissa s;
E	: Young modulus;
$\mathbf{E}(s)$	: stiffness matrix of beam;
$\mathbf{f}(s)$	: static action array;
$\mathbf{f}_e(s)$	: external static actions;
$\mathbf{F}_i(s)$	: expanded transfer matrix which contains distributed load terms;
G	: shear modulus;
s	: curvilinear coordinate;
$\mathbf{i}_j$	: j-th unit vector of Frenet system;
$\mathbf{I}_h$	: diagonal unitary matrix of order h;
j_i	: i-th rotational Winkler coefficient;
J_1,J_2	: principal inertia moments;
J_3	: torsional constant;
$\mathbf{J}$	: symplectic operator;
k_i	: i-th translactional Winkler coefficient;
k_0	: geometric curvature of the bar;
$m_i(s)$	: i-th distributed moment component of $\mathbf{f}_e(s)$;
$M_i(s)$	: i-th moment component of $\mathbf{Q}(s)$;
$\mathbf{N}(s)$	: load vector at abscissa s;
$p_i(s)$	: i-th distributed load component of $\mathbf{f}_e(s)$;
$\mathbf{P}_I$	: nodal point matrix containing discontinuities.
$\mathbf{q}(s)$	: total strain array;
$\mathbf{q}_i(s)$	: internal strains;
$\mathbf{q}_e(s)$	: inelastic imposed strains (external geometric actions);
$\mathbf{Q}(s)$	: stress resultant array;
R	: radius of curvature;
$\mathbf{R}(s)$	: constitutive matrix of Winkler soil;
$\mathbf{S}_i(s)$	: expanded state array;
$u_i(s)$	: i-th displacement component of $\mathbf{u}(s)$;
$\mathbf{u}(s)$	: generalized displacement array;
$V_i(s)$	: i-th force component of $\mathbf{Q}(s)$;
$\mathbf{z}(s)$	: mixed state array;
$\Delta\mathbf{Q}_I$	: imposed static discontinuities at node I;
$\Delta\mathbf{u}_I$	: imposed kinematical discontinuities at node I;
$\Delta\mathbf{z}_I$	: mixed array of imposed discontinuities at node I;
$\varepsilon_i(s)$	: i-th uni-axial strain component of $\mathbf{q}(s)$;
$\kappa_i(s)$	: i-th curvature component of $\mathbf{q}(s)$;
ξ_1^C, ξ_2^C	: shear centre coordinates with respect to cross section centroid;
τ_0	: geometrical tortuosity of the bar;
χ_1, χ_2, χ_{12}	: shear factors;
$\varphi_i(s)$	: i-th rotation component of $\mathbf{u}(s)$;
$\mathbf{\Phi}(s)$	: flexibility matrix of beam.