Structural Engineering and Mechanics, Vol. 4, No. 6 (1996) 703-715
DOI: http://dx.doi.org/10.12989/sem.1996.4.6.703

Stochastic finite element analysis of plate structures by weighted integral method

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Abstract. In stochastic analysis, the randomness of the structural parameters is taken into consideration and the response variability is obtained in addition to the conventional (mean) response. In the present paper the structural response variability of plate structure is calculated using the weighted integral method and is compared with the results obtained by different methods. The stochastic field is assumed to be normally distributed and to have the homogeneity. The decomposition of strain-displacement matrix enabled us to extend the formulation to the stochastic analysis with the quadratic elements in the weighted integral method. A new auto-correlation function is derived considering the uncertainty of plate thickness. The results obtained in the numerical examples by two different methods, i.e., weighted integral method and Monte Carlo simulation, are in a close agreement. In the case of the variable plate thickness, the obtained results are in good agreement with those of Lawrence and Monte Carlo simulation.

Key words: stochastic finite element analysis, weighted integral method; auto-correlation function; material and geometrical randomness; Monte Carlo simulation.

1. Introduction

When the structure is analyzed with finite element methods, the assumption that the structures have deterministic parameters is implicitly accepted. Thus the element stiffness matrix is formed based on the assumption that the elastic modulus, Poisson's ratio, and the thickness are constant throughout the domain of structures. However, in real structures, the material and geometrical properties have certain uncertainties, thus the assumption of spatial and/or temporal uncertainties of the structural parameters need to be adopted in the analysis. The spatial uncertainties include the material and geometrical uncertainties, and the load varying with time can be classified as one of temporal uncertainties.

The deterministic hypothesis of structural parameters can be considered as an ideal case. Traditionally, the engineers consider the uncertainty of the structures in his design problem through the increase of the safety factors almost arbitrarily. However, this approach lacks the theoretical background (Kleiber and Hein 1992) and therefore the stochastic analysis should be performed with consideration of the randomness of parameters. The stochastic analysis can be divided into two major categories. The first category is the method using a statistical approach, which includes the Monte Carlo simulation (MCS). The MCS uses the real random fields generated according to the assumed statistics of the stochastic field. Therefore, this method can be considered as an exact solution scheme, which can be attained with large number of generated

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samples. As an alternative to the MCS, the Neumann expansion method can be used in which the solution scheme of the structural matrix is improved by the use of the Cholesky's matrix decomposition, thus the computing time of analysis can be reduced (Yamazaki 1988). The second category is the nonstatistical approach which includes the numerical integration, second moment analysis, and the stochastic finite element analysis (Liu, Belytschko and Mani 1986).

Until now, the study on stochastic analysis has been heavily concentrated on the topics considering the influence of the spatial randomness of elastic modulus on the structural response variability. In the present study, in addition to the material property randomness of the elastic modulus, the spatial randomness of the plate thickness as a geometrical uncertainty are analyzed using the weighted integral method. The use of the weighted integral method is expanded to apply to the quadrilateral elements. The key to the expansion is the decomposition of the strain-displacement matrix into the constant matrices multiplied by the independent polynomials. Some numerical examples are provided for the validation of the proposed schemes.

2. Weighted integral method

The weighted integral is defined as the integration of the stochastic field function $f(x)$ multiplied by the known deterministic function $t(x)$ over the given domain. The domain of integration is the area of each element in the case of the two-dimensional finite element analysis. The integrated term is a kind of random variable that can be denoted as X.

$$X=\int_{\Omega^e} f(x)\,t(x)\,d\Omega^e \quad (x\in\Omega^e,\ \Omega^e\subset\Omega) \tag{1}$$

In the process of application into the real problem, the unknown stochastic field function $f(x)$ is replaced by the auto-correlation function (Deodatis, Wall and Shinozuka 1991) defined as

$$R_{ff}(\xi)=E[f(x_1)f(x_2)],\ \xi=x_2-x_1 \tag{2}$$

where, $E[\bullet]$ is an expectation operator and ξ is a separation vector between two points x_1 and x_2 in the structural domain. As being seen in Eq. (2), the main characteristic of the auto-correlation function is that this function is only a function of separation vector ξ. This function assumes the maximum value when the separation vector has the value of zero (0.0), and decreases exponentially with the increase of the value of separation vector. This function is included in the formulation for the covariance of response (see Eqs. (20) and (29)).

3. Formulation for stochastic finite element anslysis

The 8 node plate elemenmt with nonconforming modes (Choi and Kim 1989) is used in the analysis. The displacement field including the nonconforming modes can be written as Eq. (3)

$$u=\sum_i N_i u_i+\sum_j \bar{N}_j \bar{u}_j \tag{3}$$

With this displacement field, the load-displacement relation can be derived as the following partitioned matrix equation, where 'c' and 'n' denote the contribution from the conforming and non-conforming modes, respectively.

$$\begin{bmatrix} K_{cc} & K_{cn} \\ K_{cn}^T & K_{nn} \end{bmatrix} \begin{bmatrix} u_c \\ u_n \end{bmatrix} = \begin{bmatrix} R \\ 0 \end{bmatrix} \tag{4}$$

The additional degree of freedom $\bar{u}_j$'s in Eq. (3) are condensed out by the static condensation and then the final form of the stiffness takes the form as follows

$$K' = K_{cc} - K_{cn} K_{nn}^{-1} K_{cn}^T \tag{5}$$

For the stochastic analysis, the elastic modulus E and the thickness of the plate t are assumed as a function of $\boldsymbol{x}$, i.e., $E = E(\boldsymbol{x})$, $t = t(\boldsymbol{x})$. Where, $\boldsymbol{x}$ is the position vector of a point in the structural domain. The element stiffness matrix can be written as

$$K^e = \int_{\Omega^e} B^T D_e B d\Omega^e \tag{6}$$

The first stochastic field to be considered is the spatial randomness of the elastic modulus E. The field of elastic modulus is assumed to have the following form (Deodatis, Wall and Shinozuka 1991).

$$E(x) = E_o[1 + f(x)] \tag{7}$$

where, the mean value and the stochastic field of elastic modulus are denoted by E_o and $f(x)$, respectively. The mean value of the stochastic field function is zero (0.0). The stochastic function is assumed to be homogeneous and to have the Gaussian distribution. The Gaussian distribution can be defined by the two terms; i.e., the mean and the standard deviation. With Eq. (7), the constitutive matrix can be written as follows

$$D_e = D_o[1 + f(x)] \tag{8}$$

Separating the B matrix in Eq. (6) into B_b and B_s, the stiffiness matrix can be written as

$$\begin{aligned} K^e &= \int_{\Omega^e} B^T D_e B d\Omega^e \\ &= \int_{\Omega^e} B_b^T D_e B_b d\Omega^e + \int_{\Omega^e} B_s^T D_e B_s d\Omega^c \\ &= K_b^e + K_s^e \end{aligned} \tag{9}$$

where, the subscripts b and s mean the bending and shear contributions, respectively. With the substitution of Eq. (8) into the element stiffness Eq. (6). The stiffness matrix can also be separated into two parts; i.e., the mean stiffness and deviatoric stiffness, as

$$\begin{aligned} K^e &= K_o^e + \Delta K^e \\ K_o^e &= K_{bo}^e + K_{so}^e \\ \Delta K^e &= \Delta K_b^e + \Delta K_s^e \end{aligned} \tag{10}$$

The strain-displacement matrix B used in calculating the deviatoric stiffness ΔK^e can be written as the summation of a constant matrix B_i multiplied by the independent polynomials p_i.

$$B_b = B_{b_1} p_{b_1} + B_{b_2} p_{b_2} + \cdots + B_{b_{Nb}} p_{b_{Nb}}$$
$$B_s = B_{s_1} p_{s_1} + B_{s_2} p_{s_2} + \cdots + B_{s_{Ns}} p_{s_{Ns}} \tag{11}$$

where, N_b and N_s are the number of independent polynomials in B_b and B_s respectively. Only with this expansion of B, the formulation of stochastic analysis using the weighted integral becomes possible for the quadrilateral element where the strain-displacement matrix contains the function of polynomials, unlike the constant strain triangle (CST) where the strain-displacement matrix contains constant terms only.

With the substitution of Eq. (11) into the last equation in Eq. (10), two deviatoric stiffness equations for the bending and shear stiffness which include the random variable X_{ij}^c can be obtained.

$$\begin{aligned} \Delta K_b^e &= \int_{\Omega^e} f(x) B_b^T D_o^b B_b d\Omega^e \\ &= B_{b_1}^T D_o^b B_{b_1} X_{11}^b + (B_{b_1}^T D_o^b B_{b_2} + B_{b_2}^T D_o^b B_{b_1}) X_{12}^b + \cdots + B_{b_{Nb}}^T D_o^s B_{b_{Nb}} X_{NbNb}^b \end{aligned} \tag{12a}$$

$$\begin{aligned} \Delta K_s^e &= \int_{\Omega^e} f(x) B_s^T D_o^s B_s d\Omega^e \\ &= B_{s_1}^T D_o^s B_{s_1} X_{11}^s + (B_{s_1}^T D_o^s B_{s_2} + B_{s_2}^T D_o^s B_{s_1}) X_{12}^s + \cdots + B_{s_{Ns}}^T D_o^s B_{s_{Ns}} X_{NsNs}^s \end{aligned} \tag{12b}$$

$$\text{where, } X_{ij}^c = \int_{\Omega^e} f(x) p_{ci} p_{cj} d\Omega^e$$

$c=b$: bending, s shear
$i,\ j=1,\ 2,\ \cdots,\ N_b$ or N_s

Since the element stiffness matrix K^e contains the deviatoric stiffness, it is a function of random variable X_{ij}^c and thus the overall structural stiffness matrix K is also a function of random variable X_{ij}^c.

$$\Delta K^e = \text{function of } X_{ij}^c,\ c=b \text{ or } s$$
$$\text{thus } K^e \ \&\ K = \text{function of } X_{ij}^c,\ i,j=1,\ 2,\ \cdots,\ N_b \text{ or } N_s \tag{13}$$

As the stiffness matrix in the equilibrium equation is a function of random variable X, the displacement vector U, which can be obtained from the inversion of stiffness matrix multiplied by the force vector, can also be assumed as a function of random variable X.

4. Stochastic analysis of displacement

In this chapter the random variable is denoted as X_{WI}^e to indicate the weighted integral method. The superscript 'e' means the random variable for each element. Consider now the first-order Taylor series expansion of the displacement vector U about the mean value of the random variable X_{WI}^{eo}.

$$U \approx U_o + \sum_{e=1}^{N_e} \sum_{WI=1}^{N_{WI}} (X_{WI}^e - X_{WI}^{eo}) \left(\frac{\partial U}{\partial X_{WI}^e} \right)_E \tag{14}$$

where, 'o' means the mean value and $(\bullet)_E$ denotes the calculation at the mean value.

In the above equation the last partial derivative term can be calculated by partially differentiating the equilibrium equation $KU=F$ with respect to X_{WI}^e and then evaluating the value at the mean (Deodatis, Wall and Shinozuka 1991, Choi and Noh 1993). That is

$$\left(\frac{\partial K}{\partial X_{WI}^e}U+K\frac{\partial U}{\partial X_{WI}^e}\right)_E=0$$

$$\left(\frac{\partial U}{\partial X_{WI}^e}\right)_E=-K_o^{-1}\left(\frac{\partial K}{\partial X_{WI}^e}\right)_E U_o \tag{15}$$

Thus substituting Eq. (15) into Eq. (14), the following result is obtained.

$$U\approx U_o-\sum_{e=1}^{N_e}\sum_{WI=1}^{N_{WI}} X_{WI}^e K_o^{-1}\left(\frac{\partial K}{\partial X_{WI}^e}\right)_E U_o \tag{16}$$

With the above equation, the first-order approximation of the mean vector and the covariance matrix of displacement vector U can be easily found as

$$E[U]=U_o \tag{17}$$

$$\begin{aligned}&Cov[U,\ U]\\&E[(U-U_o)(U-U_o)^T]\\&\sum_{e_1}^{N_e}\sum_{e_2}^{N_e}\sum_{WI_1}^{N_{WI}}\sum_{WI_2}^{N_{WI}}\left(K_o^{-1}\frac{\partial K}{\partial X_{WI_1}^{e_1}}U_oU_o^T\left(\frac{\partial K}{\partial X_{WI_2}^{e_2}}\right)^T K_o^{-T}\right)E[X_{WI_1}^{e_1}X_{WI_2}^{e_2}]\end{aligned} \tag{18}$$

where, $E[\bullet]$ is the mean or expectation operator.

The above equation can be simplified by using the definition of the stiffness matrix that is divided into constant matrices and random variables (Eq. (12a) and (12b)).

$$Cov[U,U]=\sum_{e_1}^{N_e}\sum_{e_2}^{N_e}K_o^{-1}E[\Delta K^{e_1}U_oU_o^T\Delta K^{e_2}]K_o^{-T} \tag{19}$$

Now, terms in the expectation operator can be arranged utilizing the definition of the auto-correlation function as follow

$$\begin{aligned}&E[\Delta K^{e_1}U_oU_o^T\Delta K^{e_2}]\\&=\int_{\Omega_1^e}\int_{\Omega_2^e}E[f(x_1)f(x_2)]B_{e_1}^TD_oB_{e_1}U_oU_o^TB_{e_2}^TD_oB_{e_2}d\Omega_2^e d\Omega_1^e\\&=\int_{\Omega_1^e}\int_{\Omega_2^e}R_{ff}(\xi_1,\ \xi_2)B_{e_1}^TD_oB_{e_1}U_oU_o^TB_{e_2}^TD_oB_{e_2}d\Omega_2^e d\Omega_1^e\end{aligned} \tag{20}$$

The function used is

$$R_{ff}(\xi_1,\ \xi_2)=\sigma_{ff}^2\bullet\exp\left\{-\frac{|\xi_1|+|\xi_2|}{d}\right\} \tag{21}$$

where, d is the correlation distance, ξ_i's are components of the separation vector ξ in the plane Cartesian coordinate system, and σ_{ff} is the coefficient of variation (COV) of the stochastic field.

5. Response variability due to randomness of thickness

One of the parameters that have spatial uncertainties for plate structures is the plate thickness. The plate thickness may have different values at one point to another in the structural domain even though the nominal thickness of the plate is assumed to be the same. Thus the plate thickness can be assumed to have stochastic properties like the elastic modulus. However, the influence of this parameter on the structural response may be different from that of elastic modulus since the stiffness matrix is third order function of the thickness.

$$K^e = K^e(t(x)^3) \tag{22}$$

while the stiffness of the plate is first-order function of the elastic modulus.

The stress-strain matrix D_e can be written as

$$D_e = \frac{t(x)^3}{12}\begin{bmatrix} 2\mu+\bar{\lambda} & \bar{\lambda} & 0 \\ \bar{\lambda} & 2\mu+\bar{\lambda} & 0 \\ 0 & 0 & \mu \end{bmatrix} \tag{23}$$

with the Lamé parameters which are expressed as follows.

$$\mu = G = \frac{E}{2(1+\nu)},\ \bar{\lambda} = \frac{2\lambda\mu}{\lambda+2\mu},\ \lambda = \frac{2G\nu}{1-2\nu} \tag{24}$$

The spatial variation of the plate thickness can be modeled as a stochastic field and takes the following form which is the same as that of elastic modulus.

$$t(x) = t_o(1+f(x)) \tag{25}$$

where the stochastic field function $f(x)$ of the plate thickness has the same characteristics of the function assumed for the spatial randomness of the elastic modulue E. With the substitution of Eq. (25) into Eq. (23), D_e can be divided into two parts as;

$$D_e = D_o + \Delta D$$
$$\text{where } \Delta D = (3f+3f^2+f^3)D_o \tag{26}$$

Substituting the above equation into the element stiffness matrix equation Eq. (6), the following equation can be obtained

$$K^e = \int_{\Omega^e} B^T D_o B d\Omega^e + \int_{\Omega^e} (3f+3f^2+f^3) B^T D_o B d\Omega^e$$
$$= K_o^e + \Delta K^e \tag{27}$$

where D_o is obtained from D_e with the mean value of the thickness t_o, and the matrices K_o^e and ΔK^e are the mean stiffness and the deviatoric stiffness, respectively. With the stiffness matrix in Eq. (27), the same procedures as used for the case of the randomness of elastic modulus can be utilized for the stochastic analysis of displacement with the randomness of the plate thickness (Eqs. (10)-(19)). The different influence of these two random parameters, i.e., the elastic modulus and the plate thickness, on the variability of structural response can be seen in the deviatoric stiffness equation. In the case of spatial randomness of the plate thickness, the terms in Eq. (20) are slightly different from those of the spatial randomness of elastic modulus, i.e.,

$$
\begin{aligned}
& E[\Delta K^{e1} U_o U_o^T \Delta K^{e2}] \\
&= E\left[\int_{\Omega_1^e}\int_{\Omega_2^e}(3f_1+3f_1^2+f_1^3)(3f_2+3f_2^2+f_2^3)B_1^T D_o B_1 U_o U_o^T B_2^T D_o B_2 d\Omega_2^e d\Omega_1^e\right] \\
&= \int_{\Omega_1^e}\int_{\Omega_2^e} E\left[(3f_1+3f_1^2+f_1^3)(3f_2+3f_2^2+f_2^3)\right] B_1^T D_o B_1 U_o U_o^T B_2^T D_o B_2 d\Omega_2^e d\Omega_1^e
\end{aligned} \tag{28}
$$

Denoting the term in the mean operator in the above equation as $\bar{R}_{ff}(\xi)$, the equation is rewritten as

$$E[\Delta K^{e1} U_o U_o^T \Delta K^{e2}] = \int_{\Omega_1^e}\int_{\Omega_2^e} \bar{R}_{ff}(\xi) B_1^T D_o B_1 U_o U_o^T B_2^T D_o B_2 d\Omega_2^e d\Omega_1^e \tag{29}$$

where, the newly derived auto-correlation function $\bar{R}_{ff}(\xi)$ can be rearranged as Eq. (30) according to the general formula for the expectation of random variables.

$$\bar{R}_{ff}(\xi) = (9+18\sigma_{ff}^2+9\sigma_{ff}^4) R_{ff}(\xi) + 18R_{ff}^2(\xi) + 6R_{ff}^3(\xi) + 9\sigma_{ff}^4 \tag{30}$$

$$R_{ff}(\xi) = \text{auto-correlation function of } f(x)$$

$$\sigma_{ff} = \text{coefficient of variation of random field}$$

This auto-correlation function represents the characteristics of the random field of plate thickness. This new auto-correlation function is needed in calculating the covariance of the structural response where the stiffness of the plate is a third-order function of thickness.

Through the stochastic finite element analysis, the structural response variability, using the field function assumed to have the stochastic properties, can be obtained. The terms must be obtained by the analysis are the first-moment (mean) and the second-moment (covariance) of the structural response. The variance of response reveals the effects of the spatial uncertainties of the uncertain structural parameters on the structural response. The variability of the response can be denoted by the COV which is expressed as the following equation

$$\alpha_x(COV) = \left[\frac{Var(x)}{E[x]^2}\right]^{1/2} = \frac{\sigma_x}{|x_o|} \tag{31}$$

It is the indication of the relative value of variance of the response to the mean value.

6. Numerical example

6.1. Results for material randomness

The first example problem is a square plate with two different boundary conditions; i.e., a simple boundary and a clamped boundary. A unit distributed load is applied in the downward direction. The length of the side is 20 and only a quarter of the plate is actually needed to be modeled due to the symmetry (Fig. 1). To be strict, the variations of the elastic modulus and the plate thickness may disturb the symmetry of the structure. However, since the coefficient of variation of the random field is small, the asymmetries introduced by the stochastic field

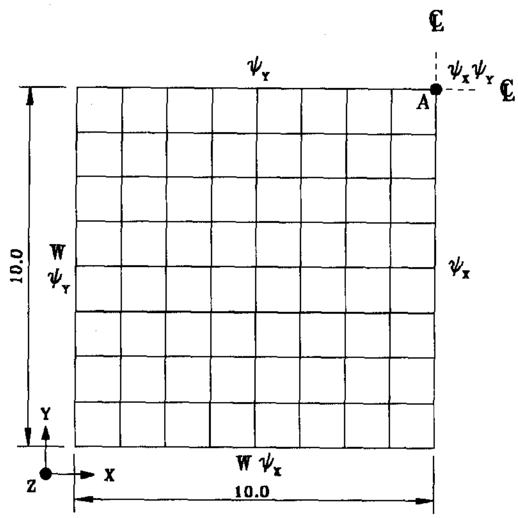


Fig. 1 Modeling of example problem (case of simple support).

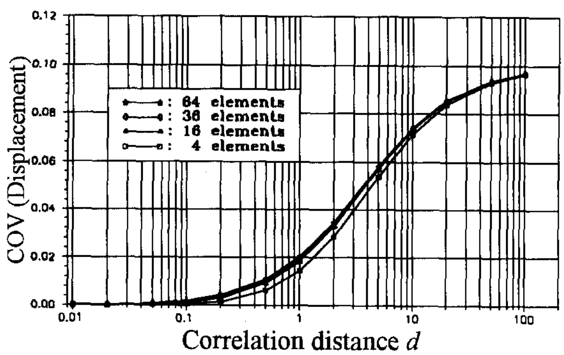
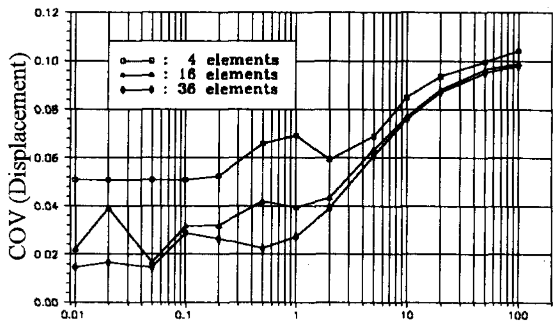


Fig. 2 COV *vs.* d (weighted integral method).

are small enough to be neglected. The value of variance of the vertical displacement is sought at the center point A. The structural parameters used are: the elastic modulus E=10.29E+03, Poisson's ratio $\nu=0.3$, and the mean thickness of the plate $t_o=1.0$. The coefficient of variation of the random field (σ_{ff}) assumed to be 0.1. All the parameters are used without units so that any units can be specified as long as they are used consistently. For the integration of auto-correlation function in Eq. (20) and (29), a 10×10 gauss quadrature is used.

The result of the weighted integral method is compared with that of the MCS. The sample generation technique used is a statistical preconditioning (Yamazaki and Shinozuka 1990), with which fairly good statistics can be obtained with a relatively small number of the generated sample fields.

The first result is for the spatial randomness of the elastic modulus. Fig. 2 and 3 show the changes of COV of the displacement due to the different values of correlation distance d of the auto-correlation function at the center point A in Fig. 1. In the MCS *nelem*×*nf*×4 samples are used, where *nelem* is the total number of elements used in the finite element model, *nf*



Correlation distance d

Fig. 3 COV *vs.* d (MCS).

is the number of cosine terms involved in generating random fields. As the value of d approaches to infinity, the COV value converges to the value of coefficient of variation of the random field assumed to be 0.1.

The two extreme values of convergence (i.e., at $d=0$ and $d=\infty$) can be explained as follows. When $d=0$, there is no correlation between any set of two points in the domain, and each stochastic field $f(x)$ can be thought as a white noise process. In this case the variance of response is zero (0.0). The small value of d related to the stochastic field of short wave length, i.e., a number of waves can be contained in the small region of the field. In this case, a randomly generated constant which is used to represent the stochastic field function can not represent the field appropriately. Thus, in the case of MCS the irregular results appear for the small values of correlation distance d. As can be seen in Fig. 3, the results are improved as the number of elements used is increased.

When $d=\infty$, it means the perfect correlation between any two points and each sample function $f(x)_i$, consists of waves of a very large wave length, and therefore, is considered to be a constant f_i over the entire domain of structure. In other words, the elastic modulus E is a contant over the whole domain of the structure. Then the ensemble of $f_1, f_2, f_3, \cdots$ represents a random variable f (Shinozuka 1987). Hence, the stochastic field function $f(x)$ in this case is in effect a random variable with a zero mean and standard deviation σ_{ff}. Since the displacement is linear depending on the elastic modulus the COV of displacement is equal to the standard deviation of stochastic field assumed to be 0.1.

The mean value of displacement for the weighted integral method and the MCS are in good agreement with each other with the maximum difference of 0.45% for 6×6 mesh of simply supported plate. For the case of MCS, a coarse mesh (e.g. 2×2) may lead to an inaccurate results as shown in Fig. 3 and therefore, the use of a too simple mesh should be avoided. It is expected from the fact that the random fields can be well represented when a large number of elements are used in the analysis.

Fig. 4 shows the influence of mesh division on the value of COV when the proposed weighted integral method is used in the analysis. This figure shows the convergence of the COV value as the mesh division is changed, from 2×2 to 8×8 while d value is fixed to be 2.0 as an example. The observation of this figure reveals that only a small change of the COV value is produced as the mesh is refined after 4×4. This indicates that a reasonable result can be obtained with a relatively small number of elements. Since the CPU time increases almost expone-

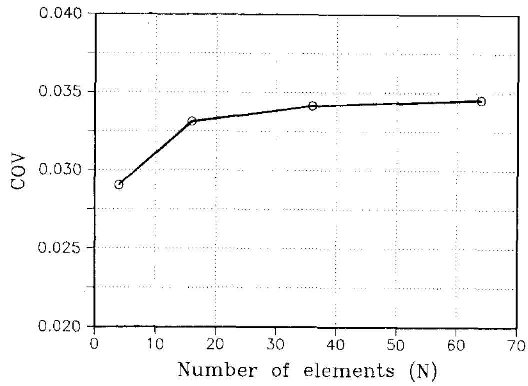


Fig. 4 Influence of element division.

Table 1 Comparison of results ($d=2.0$)

Method	Boundary condition	Simple support			Clamped support		
	Modeling	2×2	4×4	6×6	2×2	4×4	6×6
Weighted Integral	Mean	0.6588	0.6593	0.6591	0.2116	0.2131	0.2129
	Variance	0.0191	0.0218	0.0225	0.0062	0.0066	0.0068
MCS	Mean	0.6617	0.6621	0.6621	0.2116	0.2133	0.2134
	Variance	0.0393	0.0288	0.0260	0.0126	0.0091	0.0081

ntially as the mesh is refined, the above feature is very useful.

Table 1 shows the comparison of the mean and variance values obtained by the two different methods: namely, the weighted integral method and the MCS. As the mesh is refined, the difference between two results decreased. A little larger value of COV is obtained by the MCS which is a general trend which has also been reported previously by Lawrence (1987).

6.2. Results for geometrical randomness

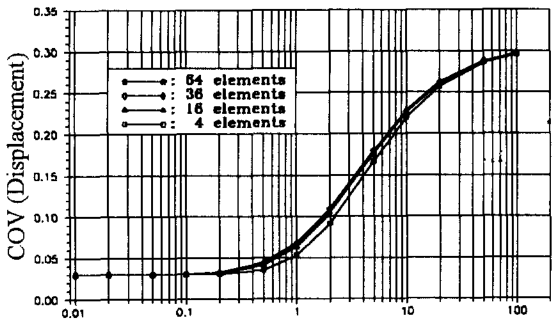
Fig. 5 and Fig. 6 show the results of stochastic finite element analysis for the spatial randomness of the plate thickness for plate in Fig. 1. For this analysis, the newly derived auto-correlation function $\overline{R}_{ff}(\xi)$ which represents the characteristics of the thickness-varying random field is used.

In this case, the converged value of COV at the infinity ($d=\infty$) is approximately 0.3 for both the weighted integral method and MCS which is approximately 3 times as large as that obtained for the spatial randomness of elastic modulus.

With Eq. (30), when $d=0$, the function becomes independent of the variable of separation vector ξ, and the auto-correlation function has a value of approximately 0.0, thus the new auto-correlation function becomes

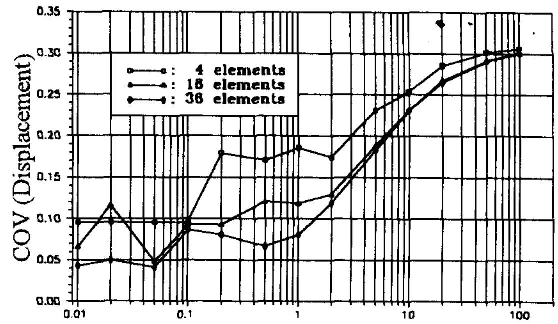
$$\overline{R}_{ff}(\xi)=9\sigma_{ff}^{4} \tag{32}$$

For the case $d=\infty$, the value of $R_{ff}(\xi)$ becomes σ_{ff}^{2}, thus the new auto-correlation function has



Correlation distance d

Fig. 5 COV *vs.* d (Weighted integral method).



Correlation distance d

Fig. 6 COV *vs.* d (MCS).

Table 2 Comparison with the results of Lawrence

Boundary conditions		Simple support		Clamped support	
Method		Proposed	MCS	Proposed	MCS
This study $d=\infty$	Mean	0.01286	0.01348	0.006275	0.006527
	Standard deviation	0.00395	0.00424	0.001929	0.002038
Lawrence $d=\infty$	Mean	0.01339	0.01349	0.006467	0.006491
	Standard deviation	0.00389	0.00413	0.001877	0.001968
Mean ratio (This study/Lawrence (1987))		0.9607	1.0000	0.9703	1.0055
Standard deviation ratio (This study/Lawrence (1987))		1.0154	1.0259	1.0277	1.0356

the value of $9\sigma_{ff}^2+45\sigma_{ff}^4+15\sigma_{ff}^6$, and these values are used in calculating the covariance by Eq. (29).

For the comparison of the results with that of Lawrence (1987), the variability of the deflection of a square plate with the spatial randomness under the deterministic central point load is studied. The same material and geometrical properties are used as used by Lawrence.

Elastic modulus $E=1.0\times10^7$ Poisson's ration $\nu=0.3$ Length of plate $L=100.0$
Central point load $P=100.0$ Mean Thickness $t=1.0$ Standard deviation of thickness 0.10

Table 2 shows the comparison of the results obtained by the weighted integral method using the 6×6 mesh with that obtained by Lawrence for both the simple and the clamped boundary conditions. The results obtained by the MCS is also presented for each case. The correlation distance d is taken to be infinity, which means the thickness at various points are perfectly correlated.

In case of the proposed scheme, the values of the coefficient of variation obtained are 0.3075 for both the simple and the clamped boundary conditions. For the mean value, the result of Lawrence is approximately 4% larger than that of the present study. However, the value of standard deviation appears to be slightly larger for this study than that of Lawrence. Thus, it can be stated that the standard deviation, i.e., the target value of the stochastic analysis, is evaluated more appropriately in the present study. When the Heterosis element is used all the results reveal a less value than given in Table 2, i.e., the nonconforming plate bending element gives the results in the safe side.

For the MCS, only 720 samples are used in this study, which is a rather small number compared with 10,000 which Lawrence used. Even though a small number of samples are used, the results appear to be reasonably good. The small sample number implies the less CPU time, thus endows the computational efficiency.

7. Conclusions

In the present study, the spatial randomness of the elastic modulus E and the plate thickness t are taken into consideration in the stochastic finite element analysis by the weighted integral method. The decomposition of the strain-displacement matrix in the stochastic formulations enabled us for the first time to extend the use of the weighted integral method for quadrilateral finite elements.

The new auto-correlation function in Eq. (30) can be effectively used for the analysis of the response variability of the problems with the randomness of plate thickness. The new function is derived in terms of the original auto-correlation function in Eq. (21) and the coefficient of variation of the stochastic field.

The numerical examples show the robustness of the method, and reveal a fairly good agreement with the results of the MCS for both the material and geometrical randomness and with those of Lawrence for geometrical randomness as well. For the randomness of the thickness, the value of COV is approximately three times as large as that obtained for the randomness of elastic modulus. Thus, the randomness of the plate thickness can be considered to have more influence on the response variability of plate structure than the elastic modulus.

Utilizing the nonconforming plate bending element, a more reasonable results can be obtained as compared with the case of using the Heterosis element. In the weighted integral method, the covariance of response is calculated based on the mean value, thus the use of more improved element is generally recommended.

More research on the method to deal with the randomness in loading and the multiple randomness of the structural parameters in the field of weighted integral method in needed in the future.

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