Structural Engineering and Mechanics, Vol. 4, No. 3 (1996) 243-255
DOI: http://dx.doi.org/10.12989/sem.1996.4.3.243

Analysis of natural frequencies of delaminated composite beams based on finite element method

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Abstract. This paper presents a model of a layered, delaminated composite beam. The beam is modelled by beam finite elements, and the delamination is modelled by additional boundary conditions. In the present study, the laminated beam contains only one delaminated region through the thickness direction which extends to the full width of the beam. It is also assumed that the delamination is open. The influence of the delamination length and position upon changes in the bending natural frequencies of the composite laminated cantilever beam is investigated.

Key words: natural frequencies; composite beams; delamination; finite element method.

1. Introduction

High-speed machinery and lightweight structures require high strength-to-weight ratios. For this reason, in recent years, the use of anisotropic reinforced laminated composites, of which the strength-to-weight ratios are very high, has increased substantially in the fields of mechanical and civil engineering.

Delamination is one of the most important failure modes of laminated composite materials. Delaminations introduced during the manufacturing process or produced by impact and other service hazards may substantially reduce the stiffness and buckling load of the structure, thus influencing the vibration and stability characteristics. The effects of delamination on buckling and post-buckling deformation and delamination growth with various geometrical parameters, loading conditions, material properties and boundary conditions have been studied extensively in the past (Chai *et al.* 1981, Bottega and Maewal 1983, Whitcomb 1986, Yin *et al.* 1986, Chen 1991). However, only a few investigations have been conducted to study the effect of delamination on vibration characteristics. Natural vibrations of delaminated beams have been studied by Ramkumar *et al.* (1979) on the basis of the Timoshenko beam theory. The authors, however, did not take into account the effect of coupling of the transverse vibration with the longitudinal wave motion in the upper and lower split layers. Their analytical results predicted significant reduction of the fundamental frequency (from that of the perfect beam) and this prediction was found to disagree with the experimental observation. Wang *et al.* (1982) used the classical beam theory but in the contrast to Ramkumar *et al.* (1979) they considered the coupling effect. With the inclusion of coupling, the calculated fundamental frequency was not appreciably reduced

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by the presence of a relatively short delamination and the results were in close agreement with experimental measurements. The natural frequencies of a composite beam with delamination emanating from a transverse crack have been analysed by Ostachowicz and Krawczuk (1995).

The present work is restricted to the analysis of natural vibrations of a layered composite beam with delamination. The beam is modelled by beam finite elements with three nodes and three degrees of freedom at a node (i.e., the transverse and axial displacements and the independent rotation). In the delaminated region additional boundary conditions are applied. In the present study, the laminated beam contains only one delaminated region through the thickness direction which extends to the full width of the beam. It is also assumed that the delamination is open (i.e., the contact forces between lower and upper parts are neglected in the model). The influence of the delamination length and position upon changes in bending natural frequencies of the composite laminated cantilever beam is investigated.

2. Formulation of a discrete model

A discrete model of a delaminated part of the beam is presented in Fig. 1. The delaminated region is modelled by three beam finite elements which are connected at the tip of the delamination by additional boundary conditions.

The layers are located symmetrically with respect to the x-z plane. Each element has three nodes at $x=-L/2$, $x=0$, $x=L/2$. At each node there are three degrees of freedom which are axial displacement $q_i (i=1, 4, 7)$, transverse displacement $q_i (i=3, 6, 9)$ and the independent rotation $q_i (i=2, 5, 8)$, Additionally it is assumed that the number of degrees of freedom is independent of the number of layers.

2.1. Description of the element number 1

Neglecting warping, the displacements u and v of a point can be expressed as:

$$u(x, y)=u^0(x)-y\cdot\phi(x)$$
$$v(x, y)=v^0(x) \tag{1}$$

where $u^0(x)$ denotes the axial displacement, $\phi(x)$ the independent rotation, and $v^0(x)$ the transverse displacement.

In the finite-element modelling, the bending displacements $v^0(x)$ are assumed to be cubic polynomials in x, while the axial displacement $u^0(x)$ and the rotation $\phi(x)$ are assumed to be

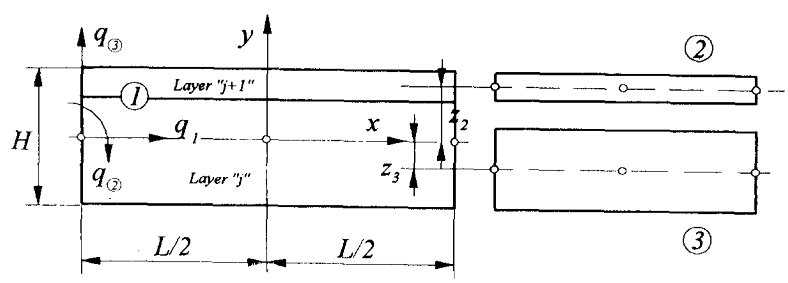

Fig. 1 The delaminated region of a beam modelled by finite elements.

quadratic. Additionally it is assumed that shear strain variation is linear, as proposed by Tessler and Dong (1981). Employing the above conditions, the displacements and rotation in the element may be written in the following forms:

$$\begin{aligned}
&u^0(x)=a_1+a_2x+a_3x^2\\
&\phi(x)=a_4+a_5x+3a_9x^2\\
&v^0(x)=a_6+a_7x+a_8x^2+a_9x^3
\end{aligned} \tag{2}$$

The constants a_1-a_9 can be expressed in terms of the element degrees of freedom by using the nodal conditions in the following forms:

$$\begin{aligned}
&u^0(x=-L/2)=q_1\\
&v^0(x=-L/2)=q_3\\
&\phi(x=-L/2)=q_2\\
&u^0(x=0)=q_4\\
&v^0(x=0)=q_6\\
&\phi^0(x=0)=q_5\\
&u^0(x=L/2)=q_7\\
&v^0(x=L/2)=q_9\\
&\phi^0(x=L/2)=q_8
\end{aligned} \tag{3}$$

Finally we obtain:

$$\begin{aligned}
&a_1=q_4\\
&a_2=\frac{-q_1+q_7}{L}\\
&a_3=\frac{2(q_1-2q_4+q_7)}{L^2}\\
&a_4=q_6\\
&a_5=\frac{-q_3+q_9}{L}\\
&a_6=q_5\\
&a_7=\frac{-6q_2-q_3L+2q_6L+6q_8-q_9L}{6L}\\
&a_8=\frac{2(q_2-2q_5+q_8)}{L^2}\\
&a_9=\frac{2(q_3-2q_6+q_9)}{3L^2}
\end{aligned} \tag{4}$$

Taking into account Eqs. (4) and Eq. (2) we can determine the matrix of the shape function for the single layer of the element.

$$\mathbf{N}=\mathbf{X}\cdot\mathbf{A} \tag{5}$$

where matrix **X** has the form:

$$\mathbf{X}=\begin{bmatrix} 1 & x & x^2 & -y & -xy & 0 & 0 & 0 & -3x^2y \\ 0 & 0 & 0 & 0 & 0 & 1 & x & x^2 & x^3 \end{bmatrix}, \tag{6}$$

whereas the matrix $\mathbf{A}$ can be expressed as:

$$\mathbf{A}=\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ -\frac{1}{L} & 0 & 0 & 0 & 0 & 0 & \frac{1}{L} & 0 & 0 \\ \frac{2}{L^2} & 0 & 0 & -\frac{4}{L^2} & 0 & 0 & \frac{2}{L^2} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{L} & 0 & 0 & 0 & 0 & 0 & \frac{1}{L} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{L} & -\frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{L} & -\frac{1}{6} \\ 0 & \frac{2}{L^2} & 0 & 0 & -\frac{4}{L^2} & 0 & 0 & \frac{2}{L^2} & 0 \\ 0 & 0 & \frac{2}{3L^2} & 0 & 0 & -\frac{4}{3L^2} & 0 & 0 & \frac{2}{3L^2} \end{bmatrix} \tag{7}$$

Employing the shape function matrix for the single layer, we can determine the inertia matrix of the whole element using the following formula:

$$\mathbf{M}_e=\sum_{j=1}^{j=R}\mathbf{M}_{j,e}=\sum_{j=1}^{j=R}\rho_j\int_{V_j}\mathbf{N}^T\mathbf{N}\,dV_j=\sum_{j=1}^{j=R}\rho_j\mathbf{A}^T\int_{V_j}\mathbf{X}^T\mathbf{X}\,dV_j\,\mathbf{A} \tag{8}$$

where j denotes the number of a layer, R the global number of layers in the element, V_j the volume of the j-th layer of material and ρ_j the density of the i-th layer.

The values of the integral in Eq. (8) (for the j-th layer) can be expressed in closed form as:

$$\int_{V_j}\mathbf{X}^T\mathbf{X}\,dV_j=BL\begin{bmatrix} \alpha & 0 & \frac{L^2}{12}\alpha & -\frac{1}{2}\beta & 0 & 0 & 0 & 0 & -\frac{L^2}{8}\beta \\ & \frac{L^2}{12}\alpha & 0 & 0 & -\frac{L^2}{24}\beta & 0 & 0 & 0 & 0 \\ & & \frac{L^4}{80}\alpha & -\frac{L^2}{24}\beta & 0 & 0 & 0 & 0 & -\frac{3L^4}{160}\beta \\ & & & \frac{1}{3}\gamma & 0 & 0 & 0 & 0 & \frac{L^2}{12}\gamma \\ & & & & \frac{L^2}{36}\gamma & 0 & 0 & 0 & 0 \\ & & & & & \alpha & 0 & \frac{L^2}{12}\alpha & 0 \\ & & & & & & \frac{L^2}{12}\alpha & 0 & \frac{L^4}{80}\alpha \\ & & & & & & & \frac{L^4}{80}\alpha & 0 \\ & & & sym. & & & & & \frac{L^6}{448}\alpha+\frac{3L^4}{80}\gamma \end{bmatrix} \tag{9}$$

where $\alpha = H_{j+1} - H_j$, $\beta = H_{j+1}^2 - H_j^2$, $\gamma = H_{j+1}^3 - H_j^3$.

The strains of the single layer of material are given by the following formulas:

$$\varepsilon_x = \frac{\partial u(x,\ y)}{\partial x} = \frac{\partial u^0(x)}{\partial x} - y\frac{\partial \phi(x)}{\partial x}$$

$$\gamma_{xy} = \frac{\partial u\,(x,\ y)}{\partial y} + \frac{\partial v\,(x,\ y)}{\partial x} = \frac{\partial v^0(x)}{\partial x} - \phi \tag{10}$$

Taking into account relations Eq. (2) and Eq. (4), the strains in the single layer can be expressed as a function of nodal degrees of freedom:

$$\begin{bmatrix} \varepsilon_x \\ \gamma_{xy} \end{bmatrix} = \mathbf{B} \cdot \begin{bmatrix} q_1 \\ q_2 \\ \vdots \\ q_9 \end{bmatrix} \tag{11}$$

where matrix $\mathbf{B}$ equals

$$\mathbf{B} = \widetilde{\mathbf{X}} \cdot \mathbf{A} \tag{12}$$

and matrix $\widetilde{\mathbf{X}}$ is given as

$$\widetilde{\mathbf{X}} = \begin{bmatrix} 0 & 1 & 2x & 0 & -y & 0 & 0 & 0 & -6xy \\ 0 & 0 & 0 & -1 & -x & 0 & 1 & 2x & 0 \end{bmatrix} \tag{13}$$

The stiffness matrix of the whole element has the form:

$$\mathbf{K}_e = \sum_{j=1}^{j=R} \mathbf{K}_{j,e} = \sum_{j=1}^{j=R} \int_{V_j} \mathbf{B}^T \mathbf{D}_j \mathbf{B}\, dV_j = \sum_{j=1}^{j=R} \mathbf{A}^T \int_{V_j} \widetilde{\mathbf{X}}^T \mathbf{D}_j \widetilde{\mathbf{X}}\, dV_j\, \mathbf{A} \tag{14}$$

where $\mathbf{D}$ denotes the matrix which describes relations between stresses and strains in the j-th layer of the element (see Appendix A).

The values of the integral in Eq. (14) (for the j-th layer of the material) can be presented in closed form as:

$$\int_{V_j} \widetilde{\mathbf{X}}^T \mathbf{D}_j \widetilde{\mathbf{X}}\, dV_j = BL \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & S_{11}\alpha & 0 & -S_{16}\alpha & -\dfrac{S_{11}}{2}\beta & & S_{16}\alpha & 0 & 0 \\ & & \dfrac{S_{11}L^2}{3}\alpha & 0 & -\dfrac{S_{16}L^2}{6}\alpha & 0 & 0 & \dfrac{S_{16}L^2}{3}\alpha & -\dfrac{S_{11}L^2}{2}\beta \\ & & & S_{66}\alpha & \dfrac{S_{16}}{2}\beta & 0 & -S_{66}\alpha & 0 & 0 \\ & & & & \dfrac{S_{66}L^2\alpha + 4S_{11}\gamma}{12} & 0 & -\dfrac{S_{66}L^2}{6}\alpha & -\dfrac{S_{66}L^2}{6}\alpha & 0 \\ & & & & & 0 & 0 & 0 & 0 \\ & & & & & & S_{66}\alpha & 0 & 0 \\ & & & & & & & \dfrac{S_{66}L^2}{3}\alpha & -\dfrac{S_{16}L^2}{2}\beta \\ & & sym. & & & & & & S_{11}L^2\gamma \end{bmatrix} \tag{15}$$

where $\alpha=H_{j+1}-H_j$, $\beta=H_{j+1}^2-H_j^2$, $\gamma=H^3_{j+1}-H_j^3$.

2.2. Description of elements number 2 and 3

In order to connect element 1 with elements 2 and 3, the following boundary conditions are applied at the tip of the delamination:

$$
\begin{aligned}
&u_1^0(x)=u_2^0(x)+z_2\phi_2(x)\\
&u_1^0(x)=u_3^0(x)+z_3\phi_3(x)\\
&\phi_1(x)=\phi_2(x)=\phi_3(x)\\
&v_1^0(x)=v_2^0(x)=v_3^0(x)
\end{aligned}
\tag{16}
$$

where z_2 and z_3 denote distances between neutral axes of elements 1-2 and 1-3, respectively (see Fig. 1),

Taking into account relations Eq. (16) and Eq. (2), the relationships between constants a_1-a_9 for the above-mentioned elements can be evaluated as:

$$
\begin{aligned}
&a_1^{II}=a_1^{I}-z_2a_4^{I},\ a_1^{III}=a_1^{I}-z_3a_4^{I}\\
&a_2^{II}=a_2^{I}-z_2a_5^{I},\ a_2^{III}=a_2^{I}-z_3a_5^{I}\\
&a_3^{II}=a_3^{I}-3z_2a_9^{I},\ a_3^{III}=a_3^{I}-3z_3a_9^{I}\\
&a_4^{I}=a_4^{II}=a_4^{III}\\
&a_5^{I}=a_5^{II}=a_5^{III}\\
&a_6^{I}=a_6^{II}=a_6^{III}\\
&a_7^{I}=a_7^{II}=a_7^{III}\\
&a_8^{I}=a_8^{II}=a_8^{III}\\
&a_9^{I}=a_9^{II}=a_9^{III}
\end{aligned}
\tag{17}
$$

where the superscrips I, II and III denote the number of the element in the region of delamination.

The shape function matrices for the elements number 2 and 3 will have the following forms:

$$\mathbf{N}_2=\mathbf{X}\cdot\mathbf{A}_2 \tag{18}$$

$$\mathbf{N}_3=\mathbf{X}\cdot\mathbf{A}_3 \tag{19}$$

where

$$\mathbf{A}_2 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & z_2 & 0 & 0 & 0 \\ -\frac{1}{L} & 0 & \frac{z_2}{L} & 0 & 0 & 0 & \frac{1}{L} & 0 & -\frac{z_2}{L} \\ \frac{2}{L^2} & 0 & -\frac{2z_2}{L^2} & -\frac{4}{L^2} & 0 & \frac{4z_2}{L^2} & \frac{2}{L^2} & 0 & -\frac{2z_2}{L^2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{L} & 0 & 0 & 0 & 0 & 0 & \frac{1}{L} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{L} & -\frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{L} & -\frac{1}{6} \\ 0 & \frac{2}{L^2} & 0 & 0 & -\frac{4}{L^2} & 0 & 0 & \frac{2}{L^2} & 0 \\ 0 & 0 & \frac{2}{3L^2} & 0 & 0 & -\frac{4}{3L^2} & 0 & 0 & \frac{2}{3L^2} \end{bmatrix} \quad (20)$$

$$\mathbf{A}_3 = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & z_3 & 0 & 0 & 0 \\ -\frac{1}{L} & 0 & \frac{z_3}{L} & 0 & 0 & 0 & \frac{1}{L} & 0 & -\frac{z_3}{L} \\ \frac{2}{L^2} & 0 & -\frac{2z_3}{L^2} & -\frac{4}{L^2} & 0 & \frac{4z_3}{L^2} & \frac{2}{L^2} & 0 & -\frac{2z_3}{L^2} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{L} & 0 & 0 & 0 & 0 & 0 & \frac{1}{L} \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & -\frac{1}{L} & -\frac{1}{6} & 0 & 0 & \frac{1}{3} & 0 & \frac{1}{L} & -\frac{1}{6} \\ 0 & \frac{2}{L^2} & 0 & 0 & -\frac{4}{L^2} & 0 & 0 & \frac{2}{L^2} & 0 \\ 0 & 0 & \frac{2}{3L^2} & 0 & 0 & -\frac{4}{3L^2} & 0 & 0 & \frac{2}{3L^2} \end{bmatrix} \quad (21)$$

Taking into account matrices $\mathbf{A}_2$ and $\mathbf{A}_3$ we can determine (using relation 8) the inertia matrix of elements 2 and 3.

In a similar way matrices $\mathbf{B}_2$ and $\mathbf{B}_3$ of elements 2 and 3 can be evaluated, and finally the stiffness matrices (using relation 14) of these elements can be calculated.

3. Numerical calculation

The formulation of the elements and the method of modelling of the delaminated region

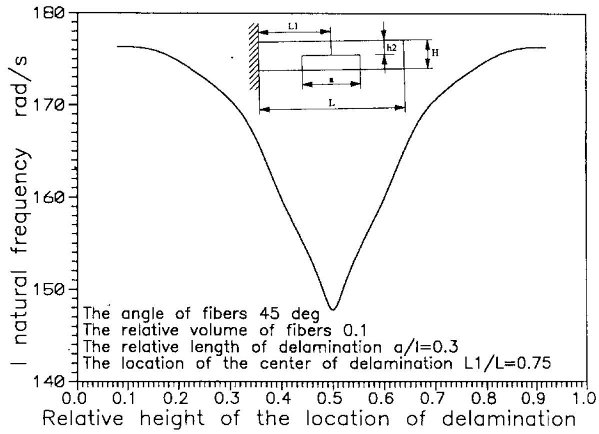

Fig. 2 The effect of the location of delamination along the beam height on the first bending natural frequency.

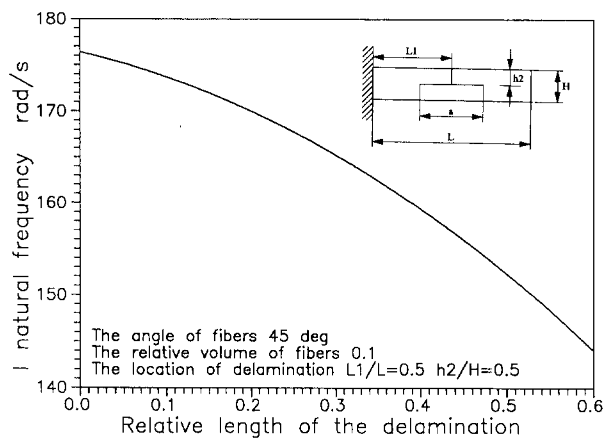

Fig. 3 The influence of the length of delamination on the first bending natural frequency.

of the beam have been evaluated by performing several example calculations.

Numerical calculations have been made for the cantilever beam of the following dimensions: length 600 mm, height 25 mm and width 50 mm. The beam was made of graphite-polyamide composite. It was assumed that all layers of the beam have the same mechanical properties, i.e., the volume fraction of fibers and the angle of fibers in each layer are identical. The mechanical properties of the material are given in Appendix B.

The first example illustrates the influence of the delamination position along the beam height upon the changes of the first bending natural frequencies. The length of delamination was equal to 180 mm ($a/L=0.3$) and the centre of delamination was located 510 mm from the free end of the beam ($L1/L=0.85$). The angle of fibers (measured from x-axis of the beam) was 45 deg., whereas the volume fraction of fibers was equal to 10% the volume of the beam. The beam was discretized by 13 finite elements (6 elements in 2 layers around the region of delamination and 7 elements outside the delaminated region). The results of numerical calculations are given in Fig. 2. It is clearly shown that the largest drop in the natural frequency is observed when

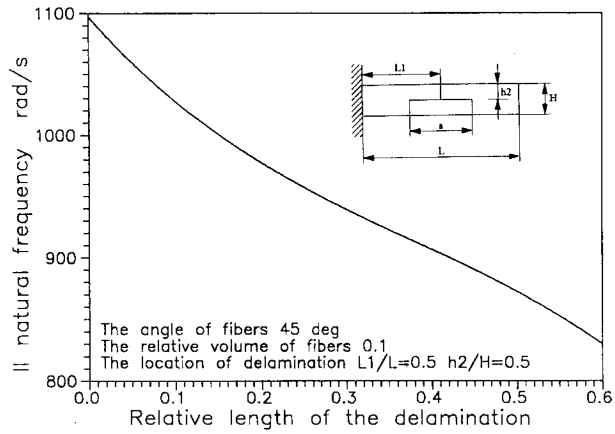

Fig. 4 The influence of the length of delamination on the second bending natural frequency.

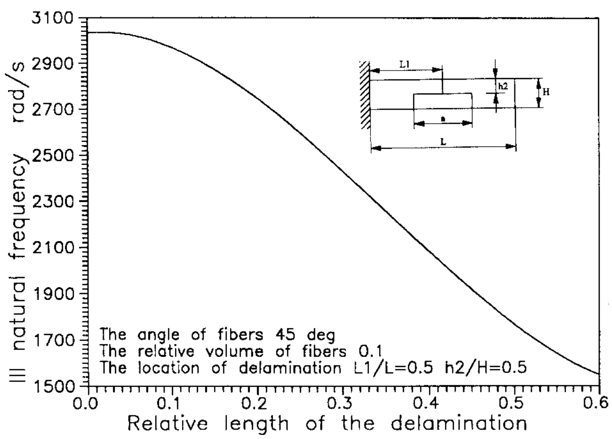

Fig. 5 The influence of the length of delamination on the third bending natural frequency.

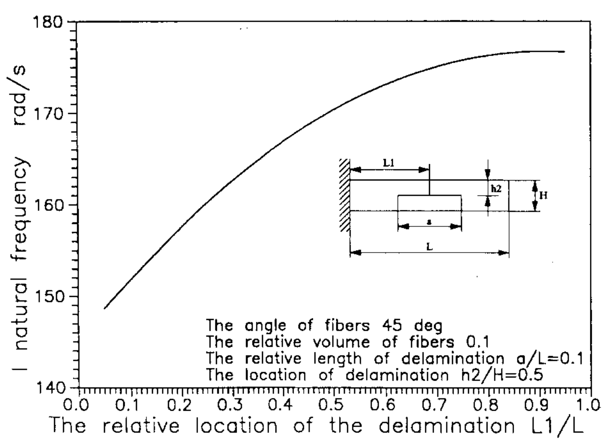

Fig. 6 The effect of the location of delamination on the first bending natural frequency.

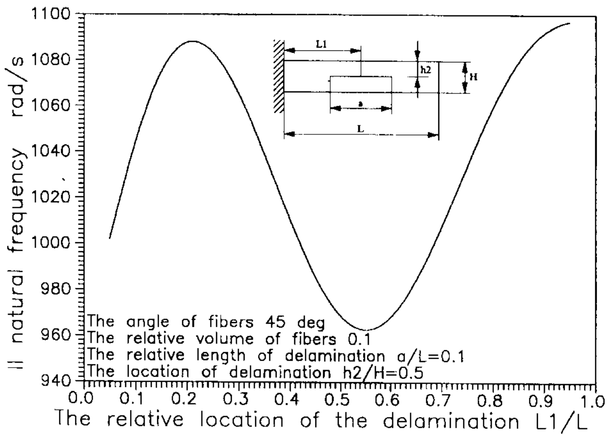

Fig. 7 The effect of the location of delamination on the second bending natural frequency.

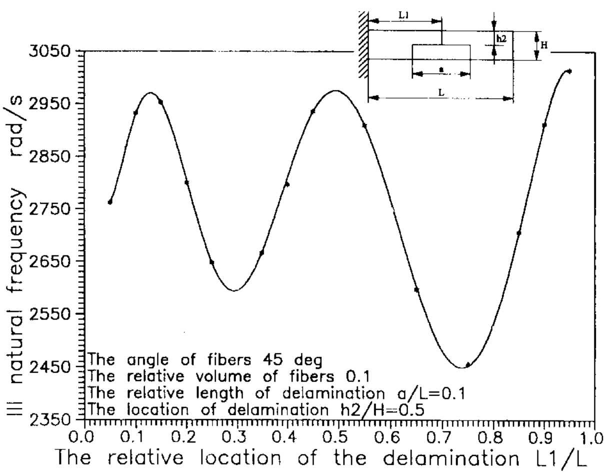

Fig. 8 The effect of the location of delamination on the third bending natural frequency.

the delamination is located along the neutral axis of the beam. When the delamination is located near the upper or the lower surface of the beam the changes in natural frequencies can be neglected.

In the next example the influence of the length of the delamination upon the drop in bending natural frequencies of the analysed beam was observed. It was assumed that the delamination is located along the neutral axis of the beam and the distance between the centre of the delamination and the free end of the beam is equal to 300 mm ($L1/L=0.5$). The other parameters were the same as in the first example. The results of numerical calculations are presented in Figs. 3-5. It is noted that when the length of the delamination increases the values of natural frequencies are greatly reduced. The intensity of these changes also depends on the number of natural frequencies (i.e., the mode shape and the location of delamination along the beam).

The third example shows the influence of the location of delamination along the beam on the drop in bending natural frequencies. As in the first and second examples the beam was made of polyamide-graphite composite material. The delamination was located along the neutral axis of the beam. The length of delamination was equal to 60 mm ($a/L=0.1$). Figs. 6-8 illustrate the changes of analysed frequencies for different locations of the delamination. It is clearly shown that the changes in natural frequencies strongly depend on the location of delamination. For the analysed beam the largest drop in natural frequency is observed when the centre of the delamination is located at the node of mode shape associated with this frequency.

4. Conclusions

A finite-element-based model was developed to study the bending natural frequencies of a cantilever composite beam with delamination. The method of modelling the delamination in the beam is versatile and allows analysis of the influence of multiple delaminations on natural frequencies of beams with various boundary conditions. Using the elaborated model the effects of location and size of delamination on bending natural frequencies of composite were studied.

Based on the numerical results, the following conclusions are drawn:

1) The delamination in the cantilever composite beam causes, as expected, reduction in bending natural frequencies.
2) The changes in natural frequencies are a function of the location (along the length of the beam and also along the height of it) and length of the delamination.
3) When the centre of delamination is located at a point where the bending moment has the maximum value (for analysed mode shape) the reduction in the bending natural frequency associated with this mode is largest.
4) The largest drop in the bending natural frequencies is observed when the delamination is located along the neutral axis of the beam.
5) When the length of the delamination increases the drop in natural frequencies also increases.

Acknowledgements

This work has been made possible through the financial support and sponsorship of the U.S. Government through its European Research Office of the U.S. Army; Contract Number N68171-94-C9108.

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Appendix A

In the case of the element analysed, the stress-strain relations matrix has the form (Vinson and Sierakowski 1991):

$$\mathbf{D}_j = \begin{bmatrix} \overline{S}_{11} & \overline{S}_{16} \\ \overline{S}_{16} & \overline{S}_{66} \end{bmatrix}$$

where the elements of the matrix $\mathbf{D}_j$ are expressed by relations:

$$\begin{aligned} \overline{S}_{11} &= S_{11} m^4 + (S_{12} + 2S_{66}) m^2 n^2 + S_{22} n^4 \\ \overline{S}_{16} &= (S_{11} - S_{12} - 2S_{66}) m^3 n + (S_{12} - S_{22} + 2S_{66}) n^3 m \\ \overline{S}_{66} &= (S_{11} - 2S_{12} + S_{22} - 2S_{66}) m^2 n^2 + S_{66} (m^4 + n^4) \end{aligned}$$

where $m = \cos(\alpha)$ and $n = \sin(\alpha)$ (α denotes the angle between the fiber direction and the axis of the beam perpendicular to the delamination).

The terms S_{ij} corresponding to the material principal axes are determined by the following formulas:

$$S_{11} = \frac{E_{11}}{1 - \nu_{12}^2 \dfrac{E_{22}}{E_{11}}}$$

$$S_{22} = \frac{E_{22}}{1 - \nu_{12}^2 \dfrac{E_{22}}{E_{11}}}$$

$$S_{12} = \frac{\nu_{12} E_{22}}{1 - \nu_{12}^2 \dfrac{E_{22}}{E_{11}}}$$

$$S_{66} = G_{12}$$

where E_{11}, E_{22}, ν_{12} and G_{12} are given in Appendix B.

Appendix B

The properties of the graphite-fiber reinforced polyamide composite analysed in the paper are assumed as follows (Vinson and Sierakowski 1991);

	matrix (polyamide)	fiber (graphite)
elastic modulus	$E_m = 2.756$ GPa	$E_f = 275.6$ GPa
Poisson's ratio	$\nu_m = 0.33$	$\nu_f = 0.2$
rigidity modulus	$G_m = 1.036$ GPa	$G_f = 114.8$ GPa
mass density	$\rho_m = 1600$ kg/m^3	$\rho_f = 1900$ kg/m^3

The material is assumed to be orthotropic with respect to its axes of symmetry which lie along and perpendicular to the direction of the fiber. The gross mechanical properties of the composite are calculated using the following formulas:

$$\rho = \rho_f \upsilon + \rho_m (1 - \upsilon)$$
$$E_{11} = E_f \upsilon + E_m (1 - \upsilon)$$
$$E_{22} = E_m \left[\frac{E_f + E_m + (E_f - E_m)\upsilon}{E_f + E_m - (E_f - E_m)\upsilon} \right]$$
$$\nu_{12} = \nu_f \upsilon + \nu_m (1 - \upsilon)$$
$$G_{12} = G_m \left[\frac{G_f + G_m + (G_f - G_m)\upsilon}{G_f + G_m - (G_f - G_m)\upsilon} \right]$$

where υ denotes the volume fraction of the fiber. The principal axes 1 and 2 are in the plane of the composite specimen aligned along and perpendicular to the fiber direction.