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Static and dynamic behaviour of square plates with inhomogeneity subjected to non-uniform edge loading (compression and tension)

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Abstract. The tension and compression buckling behaviour of a square plate with localized zones of damage and subjected to non-uniform loading is studied using a finite element analysis. The influence of parameters such as position of damage, extent of damage, size of damage and position of load on instability behaviour are discussed. The dynamic behaviour for certain load and damage parameters are also presented. It is observed that the presence of damage has a marked effect on the static buckling load and natural frequency of the plate.

Key words: stability; plates, tension buckling; compression buckling; vibration; in-plane load

1. Introduction

It is known that most engineering materials contain some defects in the form of cracks, voids, inclusions or second phase particles. The local subdomain having the elastic modulii different from those of the rest of the main domain is called an inhomogeneity or damage or a flaw. The inhomogeneity problem has received considerable interest, as evidenced by extensive reviews of Mura (1982, 1988) including the plates with flaws subjected to in-plane loading. Such flaws may exist in beams and plates, which are most commonly used structural elements. These components are subjected to a variety of static and dynamic loads. Even though in the design one assumes the material property to be invariant through out its designed life, there are many situations in practice in which these properties could change because of continuous wear and tear suffered during operations. The growth of microcracks, cracks in welds, localized corrosion or presence of voids are some cases of the problem of inhomogeneity. The presence of such a flaw results in the reduction of local bending stiffness in one and two dimensional components (Cawley and Adams 1979, Yuen 1985). The presence of a flaw results in a redistribution of membrane stress, which in turn affect the static and dynamic behaviour of plates. The changes are further compounded if the plate is subjected to a non-uniform in-plane edge loads such as concentrated or patch loads. These damages usually alter the dynamics of structures, and by identifying these changes it is theoretically possible to detect the damage (Stanley 1992, Rizos and Aspragathos 1990, Joshi and Madhusudhan 1991). The effective and reliable damage assessment methodology will be a valuable tool in the timely determination of damage and deterioration of structural members. Although damage detection in realistic structures is practical, reliable damage location requires much more information.

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Literally hundreds of papers dealing with the buckling and vibration of plates subjected to in-plane initial stresses may be found in the literature. However, in nearly all of them the in-plane stresses are assumed to be caused by uniform in-plane stresses applied at the boundaries of a homogeneous plate. This simplifies the problem in two ways:

(1) solution of the plane elasticity problem to determine the internal stress field is trivial and
(2) the resulting governing differential equation has constant coefficients, yielding the possibility of an exact solution which also satisfies the exact boundary conditions.

But in the problems of either a plate with inhomogeneity and/or a plate subjected to non-uniform in-plane edge loads, the problem is complex in nature. The dynamic and buckling behaviour of a plate with a flaw, subjected to an uniform in-plane edge load is extensively studied by the authors (Prabhakara and Datta 1993). Ayoub and Leissa have studied the vibration and buckling behaviour of plates subjected to a pair of concentrated compressive loads in homogeneous and isotropic plates (Leissa and Ayoub 1988) using a semi-analytical treatment. It is interesting to note that the plates can also buckle when subjected to forces that are only tensile (Leissa and Ayoub 1989).

The present paper deals with static and dynamic behaviour of plates with inhomogeneity subjected to either compressive or tensile non-uniform in-plane loads using a finite element method. The effect of a region of damage is introduced by use of an idealized model having a degradation of flexural rigidity at the zone of damage. The individual effects of the damage and the load parameters on the static stability and dynamic behaviour are studied.

2. Analysis

2.1. Formulation

The expression for the total potential energy of the plate with a zone of damage, subjected to in-plane forces and considering the effects of shear deformation can be expressed as

$$\begin{aligned}
U=&\frac{D}{2}\int_A\left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2-2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right\}\right]dA\\
&+\frac{D}{2}\int_A\frac{6\kappa(1-\nu)}{h^2}[(\phi_x)^2+(\phi_y)^2]dA+\frac{(D_d-D)}{2}\int_{A_d}\left[\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}\right)^2\right.\\
&\left.-2(1-\nu)\left\{\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2}-\left(\frac{\partial^2 w}{\partial x\,\partial y}\right)^2\right\}\right]dA+\frac{(D_d-D)}{2}\int_{A_d}\frac{6\kappa(1-\nu)}{h^2}\\
&[(\phi_x)^2+(\phi_y)^2]dA+\frac{1}{2}\int_A\left[N'_x\left[\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial v}{\partial x}\right)^2+\left(\frac{\partial w}{\partial x}\right)^2\right]+N'_y\left[\left(\frac{\partial u}{\partial y}\right)^2\right.\right.\\
&\left.\left.+\left(\frac{\partial v}{\partial y}\right)^2+\left(\frac{\partial w}{\partial y}\right)^2\right]+2N'_{xy}\left[\frac{\partial u}{\partial y}\frac{\partial u}{\partial z}+\frac{\partial v}{\partial z}\frac{\partial v}{\partial x}+\frac{\partial w}{\partial x}\frac{\partial w}{\partial y}\right]\right]dA
\end{aligned}\tag{1}$$

where A is the area of full plate, A_d is the area of damage, w is the deflection, κ is the shear deformation coefficient, ϕ_x, ϕ_y are average shear strains, N'_x, N'_y and N'_{xy} are in-plane stress resultants.

The kinetic energy of the free vibration of the plate considering both the effects of transverse inertia and rotary inertia can be written as

$$T=\frac{\rho}{2}\int_A\left[\left(\frac{\partial w}{\partial t}\right)^2+\frac{h^2}{12}\left\{\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial x}\right)\right\}^2+\frac{h^2}{12}\left\{\frac{\partial}{\partial t}\left(\frac{\partial w}{\partial y}\right)\right\}^2\right]dA \tag{2}$$

where ρ is the mass per unit area of the plate.

Upon assuming the polynomial expression for w and substituting in Eqs. (1) and (2), the potential energy and kinetic energy can be witten in matrix form as

$$U=\frac{1}{2}\{q\}^T[K]\{q\} \text{ and } T=\frac{1}{2}\{\dot{q}\}^T[M]\{\dot{q}\},$$

where $[K]=[K_e]-[S]$, in which $\{q\}$ and $\{\dot{q}\}$ are the vectors of generalized co-ordinates, $[K_e]$ is the elastic stiffness matrix taking care of shear deformation parameter, $[S]$ is the stress stiffness matrix, and $[M]$ is the consistent mass matrix of the plate. The explanations and the formulations of $[S]$, $[K_e]$ and $[M]$ matrices are given in the Appendix 1. A list of notation is given in Appendix 2.

The equation of equilibrium in matrix form for the free vibration of a plate subjected to in-plane forces can be written as

$$[M]\{\ddot{q}\}+[[K_e]-[S]]\{q\}=\{0\} \tag{3}$$

The governing equations for the following problems can be obtained by setting appropriate matrix equal to zero in Eq. (3).

(1) Static buckling, $[M]=[0]$

$$[[K_e]-[S]]\{q\}=\{0\}. \tag{4}$$

(2) Free vibration problem without in-plane load, $[S]=[0]$, $[M]\{\ddot{q}\}+[K_e]\{q\}=\{0\}$ and for harmonic motion of the form $\{q\}=\{A\}\sin\omega t$, the above equation reduces to

$$[[K_e]-\omega^2[M]]\{q\}=\{0\}. \tag{5}$$

The Eqs. (4) and (5) represent the eigenvalue problems. The eigenvalues of Eq. (4) give the critical buckling loads (λ_{ij}) and the eigenvalues of Eq. (5) give the squares of the natural frequencies of free vibration (ω_{ij}^2). The corresponding eigenvectors give the mode shapes of buckling and vibration respectively.

The non-dimensional parameters are defined as:

Non-dimensional frequency $\bar{\omega}_{ij}=\frac{\omega_{ij}a^2}{h}\sqrt{\frac{\rho}{D}}$

Non-dimensional compressive buckling load $\bar{\lambda}_{c,ij}=\frac{p_{c,ij}b}{D}$ and

Non-dimensional tensile buckling load $\bar{\lambda}_{t,ij}=\frac{P_{t,ij}b}{D}$

in which $D=\frac{Eh^3}{12(1-\nu^2)}$ is the flexural rigidity, $P_{c,ij}$ and $P_{t,ij}$ are respectively the compressive and tensile buckling loads, i is the number of approximate half waves in the x-direction and j is the number of approximate half waves in the y-direction.

2.2. Method of analysis

The element used in this analysis is isoparametric quadratic which has advantages such as accommodating irregular boundaries and accounting for shear deformation. The formulations of the elastic stiffness matrix, consistent mass matrix and stress stiffness matrix for the plate are based on the references Mukhopadhyay and Mukherjee. (1990), Rock and Hinton (1976), etc.

The summary of element matrices are presented here.

$$\text{Elastic stiffness matrix } [k_e] = \iint [B]^T [D] [B] \, dx \, dy,$$

$$\text{Stress stiffness matrix } [s] = \iint [B_G]^T [\bar{\sigma}] [B_G] \, dx \, dy \text{ and}$$

$$\text{Consistent mass matrix } [m] = \iint [N]^T [P] N] \, dx \, dy,$$

The details of $[B]$, $[D]$, $[B_G]$, $[\bar{\sigma}]$, $[N]$ and $[P]$ are given in Appendix 1.

In the formulation of elastic stiffness matrix, shear deformation is taken into account so that the formulation could well be used in the analysis of Mindlin plates. Reduced 2×2 integration technique is adopted in order to avoid possible shear locking.

It should be noted that the stress stiffness matrix $[s]$ is essentially a function of the plane stress distribution in the element due to a given combination of in-plane loads on the edges of the plate. Since the stress field is non-uniform, mainly due to the presence of damaged region, plane stress analysis is carried out using finite element techniques to determine the stresses at 2×2 Gauss points of each element. The stresses obtained at Gauss points are used to determine $[\bar{\sigma}]$ and the stress stiffness matrix $[s]$ is evaluated using numerical integration.

Both transverse and rotary inertia are considered in the formulation of consistent mass matrix. This matrix has been evaluated using a 3×3 Gaussian quadrature rule.

These element matrices are assembled using skyline technique. Subspace iteration technique is adopted throughout to extract eigenvalues.

2.3. Geometry and load

The model used to represent the behaviour of interest is shown in Fig. 1(a). The plate is rectangular $a \times b$ and a flaw is bounded by $a_1 \leq x \leq a_2$ and $b_1 \leq y \leq b_2$. The centroid (c, d) of the damaged area represents the position of the damage. The plate is subjected to in-plane loading as shown in Fig. 1(b)-(c). The following non-dimensional parameters are used to study the effects of the damage:

$\psi = e/b$,	position of load;
$\zeta = c/a$, $\eta = d/b$	position of damage;
$\xi = (D - D_d)/D$,	extent of damage and
$\varphi = (a_2 - a_1)(b_2 - b_1)/ab$,	a measure of the size of the damage.

where, D_d is the flexural rigidity of the damaged zone of plate, D is the flexural rigidity of the undamaged plate and e is the distance from one corner of the plate to the line of action of a concentrated load or the band width of a partially distributed uniform load.

2.4. Fortran coding, computation and verification

A general purpose program has been written to analyse plates of rectangular shape with internal

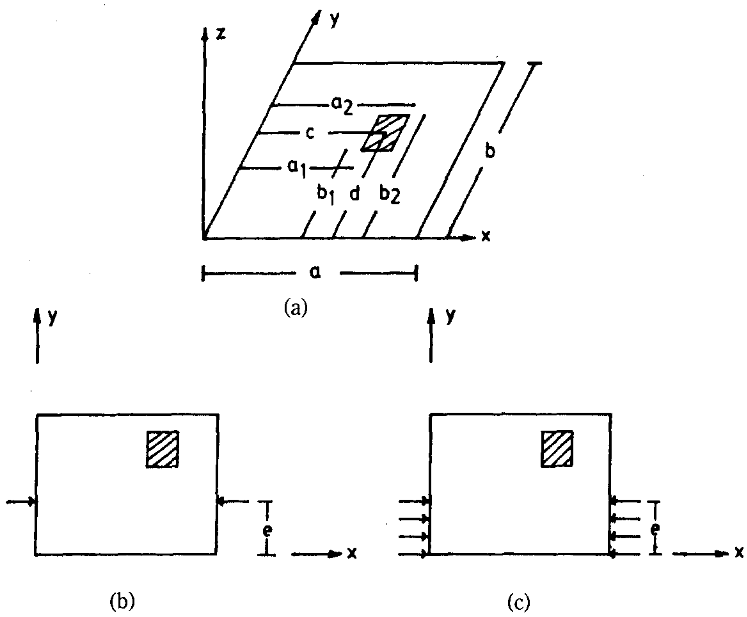


Fig. 1 Description of the plate problem with damage;
(a) Geometry (b) Buckling with point load (c) Buckling with partial U.D.L.

Table 1 Comparison of non-dimensional buckling loads, for a plate without flaw subjected to concentrated load

a/b	ψ	mode	present		leissa and ayoub(1988, 1989)	
			$\bar{\lambda}_{c,11}$	$\bar{\lambda}_{t,11}$	$\bar{\lambda}_{c,11}$	$\bar{\lambda}_{t,11}$
1.0	0.1	SX-SY	48.665	–	48.620	–
	0.25	SX-SY	36.405	1163	37.000	1221
	0.5	SX-SY	25.720	609	25.814	614
0.5	0.25	SX-SY	37.608	1295	38.282	1158
	0.5	SX-SY	29.852	1527	30.061	1667

inhomogeneity. The program is capable of solving the plane stress, vibration, static stability and dynamic stability problems for plates subjected to a variety of in-plane edge loads. The present version of the program utilises eight-noded isoparametric quadratic element with three degrees of freedom per node. This element formulation is based on Mindlin's plate theory which is the most general one and can accommodate both the thin and thick plate with appropriate shear deformation parameters in the energy expressions. An extensive series of verification analyses of the program have been undertaken. As a check of validity of the program, the results are first obtained for the case of the undamaged plate subjected to a pair of concentrated loads and are compared in Table 1 with the available results in the literature for static stability (Leissa and Ayoub 1988, 1989). The result shows that they compare well with those of Leissa and Ayoub (1988, 1989). However, for the case of $a/b=0.5$ and $\psi=0.25$ with *SX-SY* mode, the difference is about 9%. This can be improved by taking finer mesh size in the finite element analysis. Table 2 shows the convergence of natural frequencies for an undamaged square plate which

Table 2 Convergence of non-dimensional natural frequencies, $\bar{\omega}_{ij}$, for a plate without in-plane load and without damage

a/b	mesh division for half plate	mode SX-SY	AX-SY
1.0	4×8	19.732	49.208
	5×10	19.729	49.148
	6×12	19.728	49.135

(Classical results for a/b=1.0 and are $\bar{\omega}_{11}$=19.739 and $\bar{\omega}_{12}$=49.348

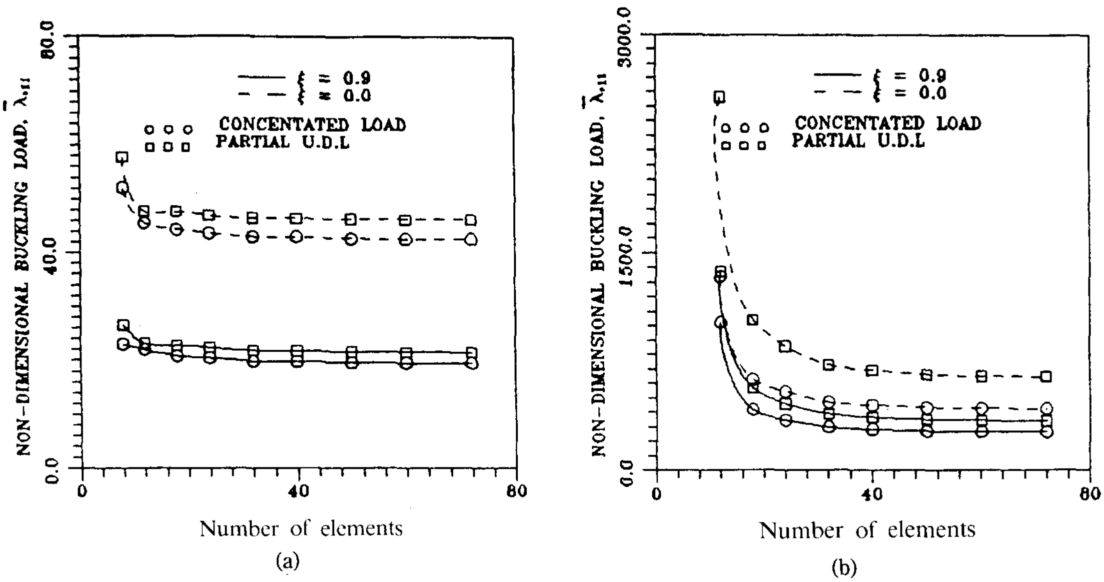


Fig. 2 Plots of buckling load vs. no. of elements (a) Compressive load (b) Tensile load (a/b=1.0; B.C: S-S-S-S; ζ=0.5; η=0.1)

agree extremely well with that of results given in reference Leissa (1969). In the results shown, *SX, SY* denote symmetric mode shapes along x and y directions and *AX, AY* represent the corresponding antisymmetric modes. The problem under investigation is the static and dynamic behaviour of plates due to the non-uniform stress field caused by both the presence of inhomogeneity and the application of non-uniform in-plane loads. The absence of any literature for this class of problem has necessitated the study of convergence of the results which are shown in Fig. 2(a)-(b). On the basis of this study, a 5×10 mesh (50 elements) is adapted in the symmetrical half plate.

2.5. Problem considered

A simply supported square plate with internal inhomogeneity (flaw or damage) subjected to four different cases of loading is considered for analyses. The loading cases are,

Case 1 - A pair of compressive concentrated loads
Case 2 - A pair of compressive partial loads
Case 3 - A pair of tensile concentrated loads

Case 4 - A pair of tensile partial loads

The compressive loading Cases 1 and 2 are shown in Fig. 1(b)-(c). The corresponding tensile loading Cases 3 and 4 can be obtained by reversing the direction of the load. A damage of the size of 4% of the plate area located on the symmetrical y-axis of the plate has been taken in the analyses. The effects of various damage and load paramters such as the extent of damage (ξ varying from 0.0 to 0.9), the position of damage (ζ=0.5 and η changing from 0.1 to 0.9) and position of load (ψ varying from 0.0 to 1.0) on the static stability and dynamic behaviour are studied.

3. Results and discussions

The results obtained for the static and dynamic behaviour of a simply supported square thin plate having an internal inhomogeneity for different cases of loading and damage parameters are presented as follows. The effects of damage have been compared with the corresponding undamaged plate for all the cases, as depicted in the Figs. 3 to 6. In the plots, the continuous curves refer to the damaged plate and dotted lines refer to the undamaged plate.

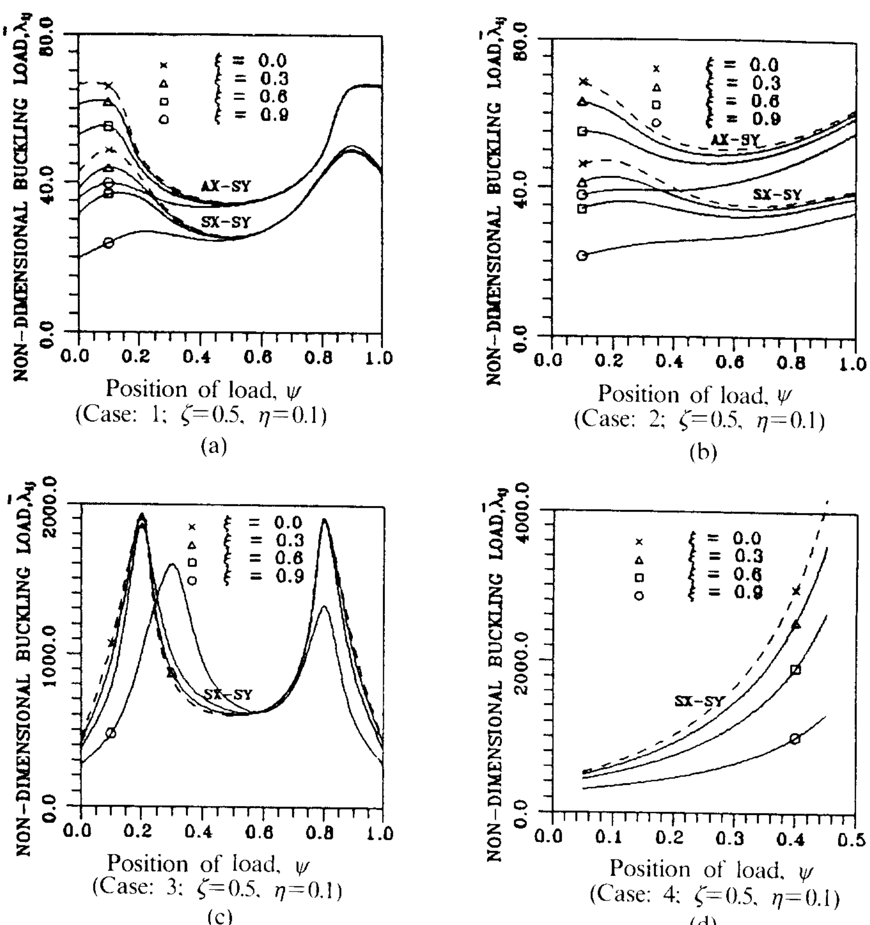


Fig. 3 Plots of non-dimensional buckling load vs. position of load for load Cases: 1-4.

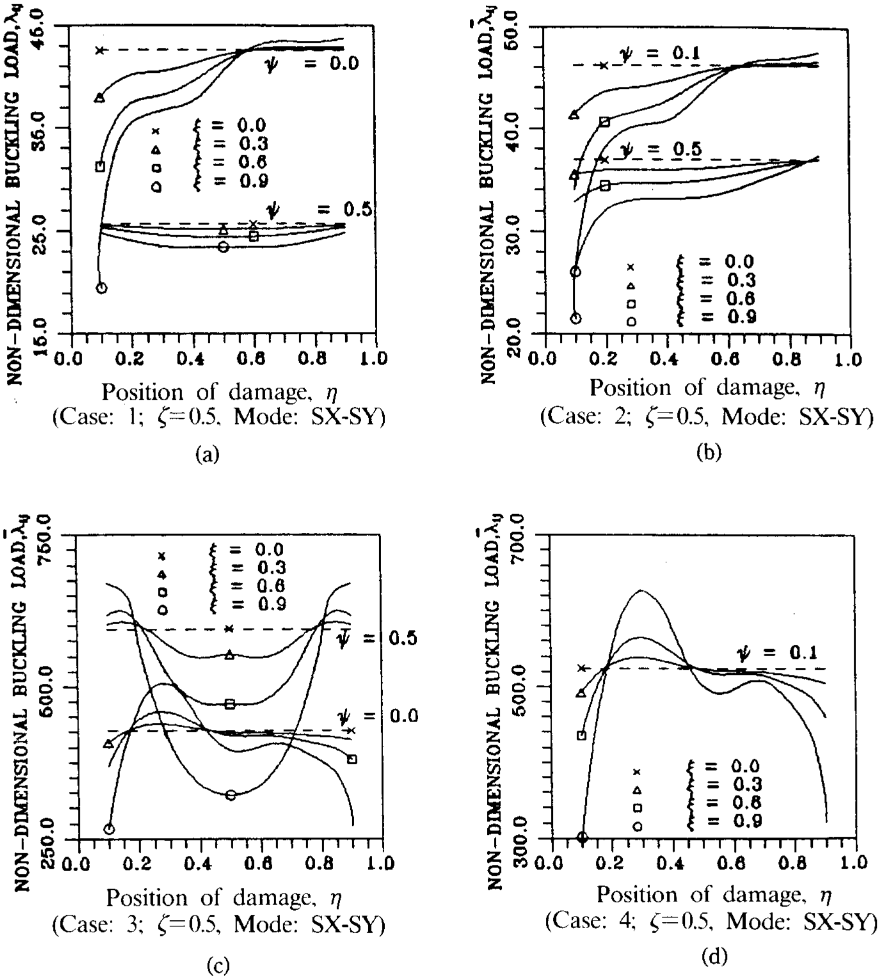


Fig. 4 Plots of non-dimensional buckling load vs. position of damage for load Cases: 1-4

3.1 Static buckling behaviour

3.1.1 Position of load

The buckling behaviour of the plate for a particular position of the damage ($\zeta=0.5$ & $\eta=0.1$) is examined for different values of ψ.

For a homogeneous plate with in-plane compression, the buckling load is minimum for the value of $\psi=0.5$ or $\psi\simeq0.7$ for the loading Case 1 or 2 respectively, as can be seen in Fig. 3(a)-(b). In the presence of damage, the buckling load reduces singnificantly for smaller values of ψ. For concentrated loading (Case 1), the change in buckling load is very small for values of ψ beyond 0.5. However, for partial loading (Case 2), there is appreciable difference in buckling load for all values of ψ. For tensile loading (Cases 3 & 4), for an undamaged plate, the buckling load is lowest for $\psi=0$. For concentrated loading (Case 3), the buckling load increases with ψ and attains a maximum value for a particular value of ψ. As ψ increases further, the buckling

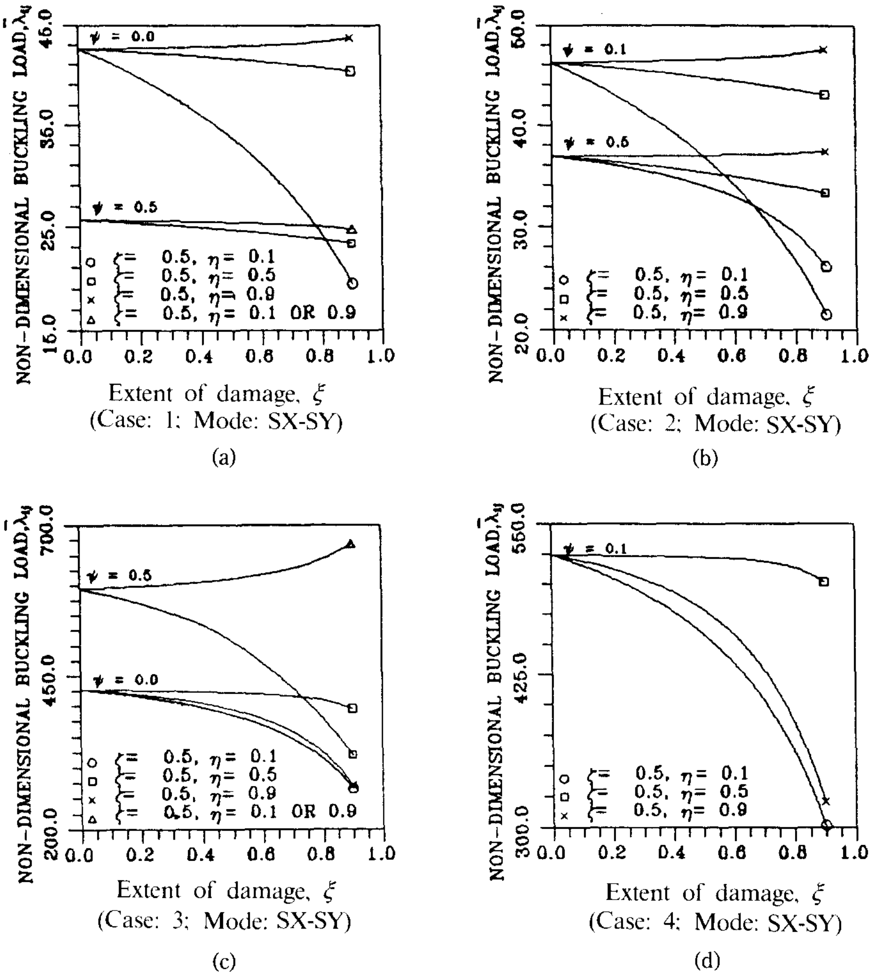


Fig. 5 Plots of non-dimensional buckling load vs. extent of damage for load Cases: 1-4.

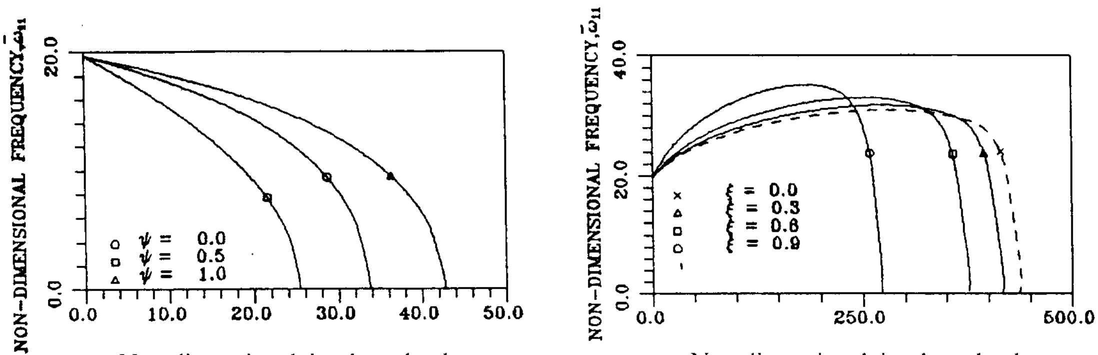
load decreases and attains a minimum value for $\psi=0.5$, as can be seen in Fig. 3(c). For the damaged plate, the change in buckling load is not very significant for tensile loading except for a severely damaged plate.

For partial tensile loading (Case 4), as ψ increases, there is a rapid increase in the buckling load for an undamaged plate. In a damaged plate, as ψ increases, there is a significant change in the buckling load as can be seen in Fig. 3(d).

3.1.2 Position of damage

Fig. 4 shows the variation of buckling load with position of damage for different cases of loading.

For the loading Case 1 with $\psi=0$ and $\zeta=0.5$, the lowest critical load occurs for the value of $\eta=0.1$. It increases rapidly towards the critical load of an undamaged plate as η increases irrespective of extent of damage as can be seen in Fig. 4(a). However, there is a particular



Non-dimensional in-plane load, p
(Case: 1; $\zeta=0.9$, $\zeta=0.5$, $\eta=0.1$)

(a)

Non-dimensional in-plane load, p
(Case: 3; $\xi=0.0$, $\xi=0.5$, $\eta=0.1$)

(b)

Fig. 6 Plots of non-dimensional frequency vs. non-dimensional in-plane load, p for load Cases 1 & 3.

value of η, called an inflection point, beyond which the trend reverses and the critical load is on the higher side to that of an undamaged plate. Similar kind of behaviour can also be observed for a partially loaded plate (Case 2) as can be seen in Fig. 4(b). However, here the inflection point moves further with increase of ψ.

For tensile loading (Cases 3 and 4), the buckling load behaviour shows that there exists two points of inflection as can be seen in the Fig. 4(c)-(d).

3.1.3. Extent of damage

The variation of buckling load with extent of damage for different cases of loading and for other load and damage parameters is shown in Fig. 5.

For in-plane compressive loading cases, the buckling load decreases drastically with the increase in the extent of damage at higher value of ξ and lower value of η as can be seen in the Fig. 5(a)-(b). For tensile loading case, the effect of extent of damage on buckling load is similar in behaviour as shown in Fig. 5(c)-(d).

3.2. Dynamic behaviour

Fig. 6 shows the variation of non-dimensional frequency with in-plane concentrated compressive and tensile loading for different load and damage parameters. As the applied compressive load increases, the fundamental frequency goes on decreasing and becomes zero at the corresponding buckling load as can be seen in Fig. 6(a).

In case of tensile loading, the fundamental frequency rises with applied tension. With the further increase of load, the frequency starts falling down reflecting the onset of compression effect and finally becomes zero at the value of tensile buckling load as depicted in Fig. 6(b).

4. Conclusions

The static and dynamic analyses of thin square plates with an inhomogeneity show that the presence of a flaw has significant effects on the buckling load and vibration behaviour. The following conclusions can be drawn from the analytical results.

(1) The buckling load of a damaged plate reduces significantly compared to the undamaged plate for the position of the concentrated compressive load near the edges, with the flaw being at the middle of the same edge. Similar behaviour can be observed for partial compressive load of smaller band width. For tensile partial loading with higher band width, buckling load of the damaged plate deviates significantly from that of an undamaged plate.

(2) For non-symmetrical compressive load, there is a particular position of damage, called an inflection point at which the buckling load of the damaged plate does not deviate from that of an undamaged plate. Two such inflection points can be observed for plates subjected to non-symmetrical tensile load.

(3) The effect of extent of damage is predominant for the damage situated at the centre of an edge parallel to the direction of loading.

(4) The vibration behaviour of a damaged plate due to in-plane tensile loading shows that for certain combination of load and damage parameters, the plate with lesser flexural rigidity, yields higher frequency compared to an undamaged plate.

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Appendix 1

1.1. Elastic stiffness matrix $[k_e]$

The elastic stiffness matrix of the plate element (Mukhopadhyay and Mukherjee 1990) is given by

$$[k]=\iint [B]^T[D][B]\,dx\,dy \text{ in which,}$$

$$[B_i]=\begin{bmatrix} 0 & 0 & \frac{\partial N_i}{\partial x} \\ 0 & -\frac{\partial N_i}{\partial y} & 0 \\ 0 & -\frac{\partial N_i}{\partial x} & \frac{\partial N_i}{\partial y} \\ \frac{\partial N_i}{\partial x} & 0 & N_i \\ \frac{\partial N_i}{\partial y} & -N_i & 0 \end{bmatrix};\ i=1,\ 2,\ \cdots,\ 8$$

$$\text{and } [D]=\left[\begin{array}{c|c} \frac{Eh^3}{12(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \frac{1-\nu}{2} \end{bmatrix} & 0 \\ \hline 0 & \frac{Eh}{2(1+\nu)}\begin{bmatrix} \kappa & 0 \\ 0 & \kappa \end{bmatrix} \end{array}\right]$$

Assembling the matrix $[k_e]$ yields the elastic matrix $[K]$ for the entire plate.

1.2. Consistent mass matrix $[m]$

The consistent mass matrix of the plate element, (Rock and Hinton 1976) which takes care of both transverse and rotary inertia is given by,

$$[m]=\iint [N]^T[P]\,[N]\,dx\,dy, \text{ in which}$$

$[N]$ is the element shape function matrix and

$$[P]=\rho\begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{h^2}{12} & 0 \\ 0 & 0 & \frac{h^2}{12} \end{bmatrix}$$

Assembling the matrix $[m]$ yields the mass matrix $[M]$ for the entire plate.

1.3. Geometric or stress stiffness matrix $[s]$

Let N'_{x1}, N'_{y1} and N'_{xy1} be the stress resultants on a small element of area $dx\,dy$ due to a normal in-plane edge load N acting parallel to x-axis. The stress resultants can be expressed in matrix form as

$$[\sigma]_x=\begin{bmatrix} N'_{x1} & N'_{xy1} \\ N'_{xy1} & N'_{y1} \end{bmatrix}$$

Further, geometric stiffness matrix Cook, R. D. (1989) is expressed as

$$[s_x]=\iint [B_G]^T[\bar{\sigma}]_x[B_G]\,dy\,dx$$

in which,

$$[\bar{\sigma}]_x=\begin{bmatrix} [\sigma]_x & & \\ & \frac{h^2}{12}[\sigma]_x & \\ & & \frac{h^2}{12}[\sigma]_x \end{bmatrix}$$

and

$$[B_G]=\begin{bmatrix} N_{i,x} & 0 & 0 \\ N_{i,y} & 0 & 0 \\ 0 & 0 & N_{i,x} \\ 0 & 0 & N_{i,y} \\ 0 & N_{i,x} & 0 \\ 0 & N_{i,y} & 0 \end{bmatrix};\ i=1,\ 2,\ \cdots,\ 8$$

Here, N_i is the shape function, $N_{i,x}$ and $N_{i,y}$ are their derivatives with respect to x and y, i refers to the node number of the element.

Similarly, the geometric stiffness matrices $[s_y]$ due to a similar load N acting parallel to y-axis and $[s_{xy}]$ due to an in-plane edge shear load N can be expressed as

$$[s_y]=\iint [B_G]^T[\bar{\sigma}]_y[B_G]\,dy\,dx \text{ and}$$

$$[s_{xy}]=\iint [B_G]^T[\bar{\sigma}]_{xy}[B_G]\,dy\,dx \text{ respectively.}$$

If the actual edge in-plane loads are expressed in terms of N, such that $N_x=\bar{a}N$, $N_y=\bar{b}N$ and $N_{xy}=\bar{c}N$ then the total geometric stiffness matrix of the element is expressed as

$$[s]=\bar{a}[s_x]+\bar{b}[s_y]+\bar{c}[s_{xy}]$$

Assembling the matrix $[s]$ yields the stress stiffness matrix $[S]$ for the entire plate.

Appendix 2

Notations

$a,\ b$	dimensions of plate in x and y directions
$a_1,\ a_2,\ b_1,\ b_2$	boundaries of damage along x and y directions
$c,\ d$	co-ordinates of centre of damage
e	position of concentrated load or band width of a partially distributed uniform load
h	thickness of plate
$x,\ y,\ z$	cartesian co-ordinate system
$u,\ v,\ w$	displacements along $x,\ y,\ z$ directions
$N'_x,\ N'_y,\ N'_{xy}$	in-plane stress resultants at any point in the element
$[K_e]$	elastic stiffness matrix of the plate
$[M]$	consistent mass matrix of the plate
$[S]$	stress stiffiness matrix of the plate
$[k_e]$	elastic stiffness matrix of the plate element
$[m]$	consistent mass matrix of the plate element

$[s]$	stress stiffness matrix of the plate element
$[\bar{\sigma}]$	resultant stress matrix
ν	Poisson's ratio
$[\sigma]$	two dimensional stress tensor
$\bar{\omega}_{ij}$	non-dimensional natural frequency
$\bar{\lambda}_{ij}$	non-dimensional critical load
ψ	e/b, position of concentrated load or band width of partial *u.d.l*
ζ, η	position of damage
ξ	extent of damage
φ	size of damage