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On eigenvalue problem of bar structures with stochastic spatial stiffness variations

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Abstract. This paper presents an analysis of stochastic eigenvalue problem of plane bar structures. Particular attention is paid to the effect of spatial variations of the flexural properties of the structure on the first four eigenvalues. The problem of spatial variations of the structure properties and their effect on the first four eigenvalues is analyzed in detail. The stochastic eigenvalue problem was solved independently by stochastic finite element method (stochastic FEM) and Monte Carlo techniques. It was revealed that the spatial variations of the structural parameters along the structure may substantially affect the eigenvalues with quite wide gap between the two extreme cases of zero- and full-correlation. This is particularly evident for the multi-segment structures for which technology may dictate natural bounds of zero- and full-correlation cases.

Keywords: eigenvalue problem; bar structure; spatial stiffness variation; stochastic finite element method; Monte Carlo method; random field; midpoint method

1. Introduction

Solution of the eigenvalue problem reflects fundamental dynamic properties of any structural system and depends on its detailed elastic and inertial data, yet for many types of structural systems, particularly for typical civil engineering structures, the material properties of structural members are random. This randomness is clearly confined for steel structures but it increases for concrete, masonry or geotechnical structures. For this reason the stochastic eigenvalue problem attracted the attention of researchers since early sixties of the past century. One of the very early paper was written by Soong and Bogdanoff (1963) and concerned parameter analysis of a discrete system of stochastic masses and springs. More papers soon appeared and dealt with random cross-sections of beams and frames (Fox and Kappor 1968), axially vibrating bars (Collins and Thompson 1969), systems “beam-column” (Hoshiya and Shah 1971). In the landmark paper from 1972 Shinozuka and Astill carried out detailed analysis of the effect of Young modulus, cross-section, moment of inertia, mass density and axial force on stochastic eigenvalue problem of “beam-column” structural system. Hasselman and Hart (1972) dealt with stochastic eigenvalue problem of discretized beams and

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frames. In the same year Sobczyk (1972) analyzed random eigenvalue problem of a plate. The list of references regarding stochastic eigenvalue problem solved for various types of structures is rather long (vom Scheidt and Purkert 1983, Augusti *et al.* 1984, Hisada and Nakagiri 1985, Vanmarcke *et al.* 1986). The first review papers were written by Ibrahim in 1987 and by Benaroya and Rehak in 1988. Mironowicz and Śniady analyzed random effects of soils (1987), Zhu and Wu (1991) analyzed beams while Ghanem and Spanos (1991b) analyzed general discrete systems. Kleiber and Hien (1992) formulated random eigenvalue problem in terms of second-order perturbation method. Ramu and Ganesan (1991, 1993) analyzed stochastic eigenvalue problem of "beam-column" systems.

Song *et al.* (1995) were the first to consider stochastic eigenvalue problem of trusses with random cross-sections. Lin and Cai (1995) analyzed disordered discrete systems. Qiu *et al.* (1996) dealt with stochastic eigenvalue problem of trusses with random cross-sections and Young modulus using interval analysis. Mehlhose *et al.* (1999) also analyzed the effect of various random factors on eigenvalue problem of beams. Recently the stochastic eigenvalue problem of large structural systems rose into prominence (Székely and Schuëller 2001, Pradlwarter *et al.* 2002) as well as the stochastic eigenvalue problem of beams resting on two-parameter stochastic sub-soil was analyzed in detail (Kaleta and Zembaty 2007). And finally, two recent articles on the stochastic eigenvalue problem of multi-storey frames with random masses and stiffness (Zhu and Chen 2009) and sensitivity of the eigenvalue problem of frames and bridges (Xia *et al.* 2010).

Although from the foregoing brief review, the stochastic eigenvalue problem may seem a field quite exploited by the researchers, but with the increasing practical applications of structural health monitoring and modal analyses, the role of eigenvalue problem again became more pronounced in the recent years (see e.g., Doebling *et al.* 1996, Sohn *et al.* 2002). From this point of view, particularly interesting is the dependence of natural frequencies (eigenvalues) on the random properties of structures as they are the easiest to measures of global dynamic systems parameters.

This paper revists the old stochastic eigenvalue problem of bar structures with particular attention paid to random spatial variations of their flexural stiffness in modeled form of stochastic field. The solution is obtained using stochastic finite element method and, independently, by applying the Monte Carlo approach. For the sake of simplicity, instead of natural frequencies the respective eigenvalues are analyzed. Various practical forms of random properties characteristic for civil engineering structures are considered and respective parameter analyses carried out. In particular the problems of cross-correlation of the stochastic properties among structural members is analyzed in detail. This correlation derives from the type of the applied material as well as technology of production of the structural members.

2. Description of random variability of stiffness of structural members

Consider flexural, shear or axial stiffness of a beam or frame member. Their values depend on the material elastic parameters like Young and shear modulae E, G and on the geometric data of the cross-section. For steel structures the situation is clear. The variations in Young or shear modulus derive from the steel quality and its technology. Eventual variations in the cross-section derive from the technology used at the steel plant or from the corrosion.

For the reinforced concrete structures the situation is more complicated as not only the coefficients of variation of material parameters are greater for concrete, but the respective stiffness

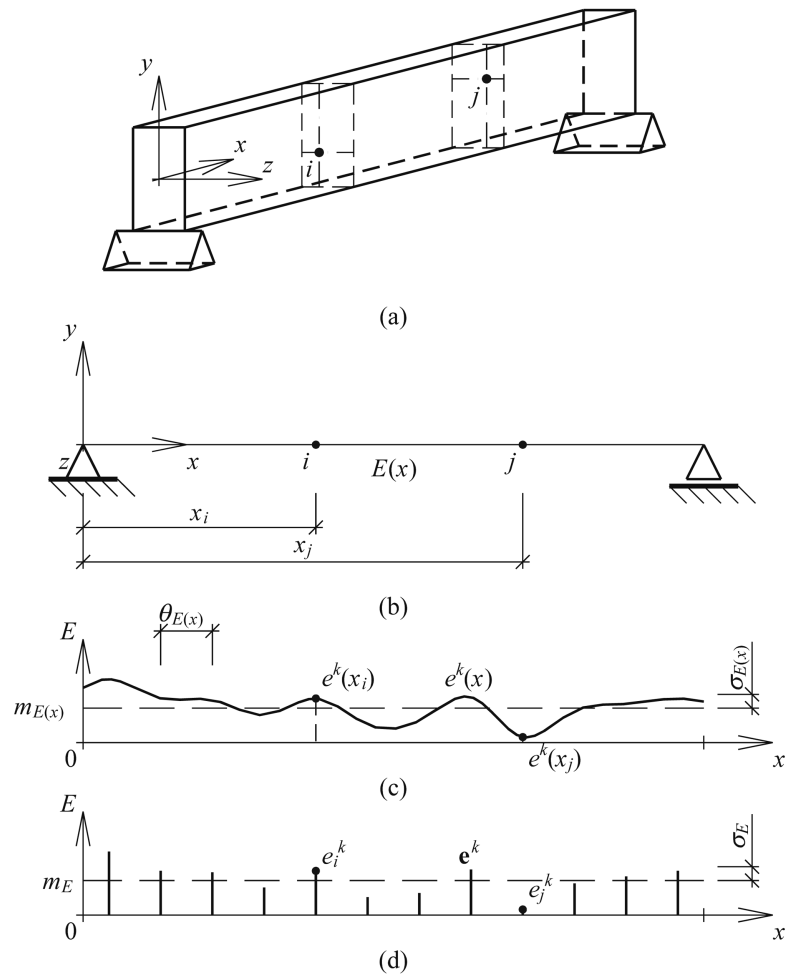

Fig. 1 Simply supported beam (a), (b) with spatial random field $E(x)$, its k-th sample $e^k(x)$ (c), and discretization of the random field (d)

depends also on the distribution of cracks, practically inevitable during the life time of most of the reinforced concrete structures. In addition the strength of concrete (and Young modulus) increases in a wide span of time after erection of the structure. The random material variability is even greater for other civil engineering materials e.g., masonry. This paper concentrates on spatial variations of stiffness among members of structural systems.

First consider a beam shown in Fig. 1(a). If we analyze its properties modeled in one dimensional (1-D) the Young modulus E of the beam can be described in two points i and j along the beam (Fig. 1(b)). Their actual values would differ due to the random material properties of the beam. Thus the variations of E along the beam length can be considered as a spatial random field $E(x)$ with a particular k-th sample $e^k(x)$ shown in the Fig. 1(c). For the purpose of practical computations the continous random field is discretized (Fig. 1(d)). It was Shinozuka (1972) who first considered such discretized random spatial distribution for engineering structures and in particular, he analyzed the influence of the size effect on the strength of a concrete beam. He also analyzed spatial distribution of strength in failure analysis with the application of Monte Carlo method. The one dimensional random field $E(x)$ along the bar can be assumed Gaussian and fully described using its mean value

$$m_{E(xi)} = \int_{-\infty}^{+\infty} e_i \cdot f_{E(xi)}(e_i) de_i \tag{1}$$

variance function

$$C_{E(xi)} = \int_{-\infty}^{+\infty} (e_i - m_{E(xi)})^2 \cdot f_{E(xi)}(e_i) de_i \tag{2}$$

and respectively standard deviation

$$\sigma_{E(xi)} = \sqrt{C_{E(xi)}} \tag{3}$$

as well as coefficient of variation

$$v_{E(xi)} = \frac{\sigma_{E(xi)}}{m_{E(xi)}} \tag{4}$$

The spatial variations of the random field can be described in terms of its covariance function

$$C_{E(xi), E(xj)} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (e_i - m_{E(xi)}) \cdot (e_j - m_{E(xj)}) \cdot f_{E(xi), E(xj)}(e_i, e_j) de_j de_i \tag{5}$$

correlation function and correlation coefficient

$$R_{E(xi), E(xj)} = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e_i \cdot e_j \cdot f_{E(xi), E(xj)}(e_i, e_j) de_j de_j \tag{6}$$

$$\rho_{E(xi), E(xj)} = \frac{C_{E(xi), E(xj)}}{\sigma_{E(xi)} \cdot \sigma_{E(xj)}}, \quad -1 \le \rho_{E(xi), E(xj)} \le 1 \tag{7}$$

To simplify further analyses this random field is assumed homogeneous (stationary). Thus, the mean value and standard deviations are constant along the length of the bar (Fig. 1(c))

$$m_{E(xi)} = m_{E(x)} = const. \tag{8}$$

$$\sigma_{E(xi)} = \sigma_{E(x)} = const. \tag{9}$$

while the correlation function depends only on the relative distance between two points

$$\Delta x = x_i - x_j \tag{10}$$

Three types of correlation functions are frequently used to model spatial properties of random fields (e.g., Vanmarcke 1984). These are
triangular function

$$R_{E(xi), E(xj)} = \begin{cases} 1 - \dfrac{|\Delta x|}{a} & \text{for } \Delta x \le a \\ 0 \ \text{for} \ \Delta x > a \end{cases} \tag{11}$$

exponential function type I

$$R_{E(xi),E(xj)} = \exp\left(-\frac{|\Delta x|}{b}\right) \tag{12}$$

and exponential function type II

$$R_{E(xi),E(xj)} = \exp\left(-\left(\frac{|\Delta x|}{c}\right)^2\right) \tag{13}$$

where a, b and c are parameters. The width of the random field is represented by the so called "scale of correlation" given by following formula

$$\theta_{E(x)} = \int_{-\infty}^{+\infty} R_{E(xi),E(xj)} d\Delta x \tag{14}$$

For the three types of correlation functions from Eqs. (11)-(13), the scale of correlation $\theta_{E(x)}$ equals respectively a, $2b$, and $c\sqrt{\pi}$. The exponential function type II (Eq. (13)) can be rewritten as

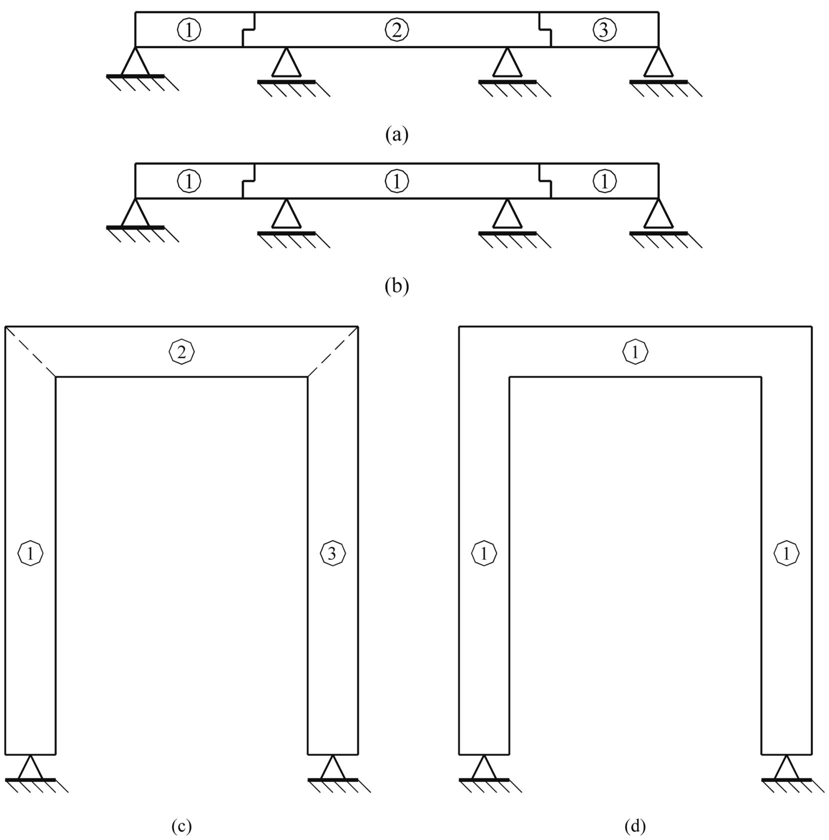

Fig. 2 Examples of the effect of technology on the properties of finally assembled structure: (a) three-span reinforced concrete beam made of the material coming from different suppliers, (b) the same three-span beam made of the material coming from one supplier, (c) steel frame made of the material coming from different suppliers, (d) the same steel frame made of the material coming from one supplier

$$R_{E(xi),E(xj)} = \exp\left(-\pi\left(\frac{|\Delta x|}{\theta_{E(x)}}\right)^2\right) \tag{15}$$

When scale of correlation tends fast to zero the random field becomes uncorrelated on short distances. In the limit when $\theta_{E(x)} \to 0$ the random field becomes white noise (in spatial domain). On the other hand when $\theta_{E(x)} \to \infty$ the random field becomes fully correlated and can be described by a single random variable. In reality partial spatial correlation can be met.

The foregoing description of a 1-D random field cannot directly cover more complicated structural bar systems. Consider for example the structures shown in Fig. 2. Both, the beam and frame, can be build of the members deriving from: different suppliers (Figs. 2(a) and 2(c)), or one supplier (Figs. 2(b) and 2(d)). In case of reinforced concrete structures this may mean that the whole beam or frame is made "on site" from different or the same portion of concrete.

The first assumption may result in additional partial correlation among the elements or even zero correlation among the technological elements of the structure deriving from independent production sets. However within single elements the correlation function is described by one of the functions (Eqs. (11)-(13)). The second assumption means that one may even extend the correlation functions (Eqs. (11)-(13)) along neighbouring or farer elements, as if the frame was straighten.

In what follows the continuous random field will be analyzed as a discrete 1-D field (Fig. 1(d)). This approximation is necessary to carry out the numerical analyses. This means that continuous random field $E(x)$ changes into a discrete vector $\mathbf{E}$ consisting of n random variables E_1, E_2, ..., E_n

$$\mathbf{E} = [E_1 \ \dots \ E_n]^T \tag{16}$$

In what follows discrete versions of Eqs. (1)-(7) will be used.

3. Random eigenvalue problem and its solution by stochastic FEM

The equation of motion of a discrete dynamic system with n_d degrees of freedom takes following, familiar form

$$\mathbf{B} \cdot \ddot{\mathbf{q}} + \mathbf{K} \cdot \mathbf{q} = 0 \tag{17}$$

where $\mathbf{B}$ and $\mathbf{K}$ stand for matrices of mass and stiffness respectively. From Eq. (17) one obtaines. Well known algebraic equation for eigenvalues λ_1, λ_2, …, λ_{nd}

$$(\mathbf{K} - \lambda_i \cdot \mathbf{B}) \cdot \mathbf{u}_i = 0 \tag{18}$$

with natural frequencies $\omega_1 = \sqrt{\lambda_1}$, $\omega_2 = \sqrt{\lambda_2}$ ..., $\omega_{nd} = \sqrt{\lambda_{nd}}$, and eigenvectors $\mathbf{u}_1$, $\mathbf{u}_2$, …, $\mathbf{u}_{nd}$.

Assuming that the elements of stiffness or mass matrices are random, the respective natural frequencies are random variables depending in a complicated way on the stochastic structure of matrices $\mathbf{B}$ and $\mathbf{K}$. In this paper it is assumed that the matrix of mass is deterministic and only elements of stiffness matrix $\mathbf{K}$ are random variables. For any structural system consisting of bars these elements depend on respective flexural, shear and axial stiffness. In this paper only flexural and axial stiffness is taken into account.

Consider now, a simply supported beam from Fig. 3(a) (but it can also be any more complicated multi-span beam or frame - see e.g., Fig. 2). When discretized, as shown in Fig. 3(b) the Eq. (17) can be solved for natural frequencies using classic numerical methods. When the Young modulus is

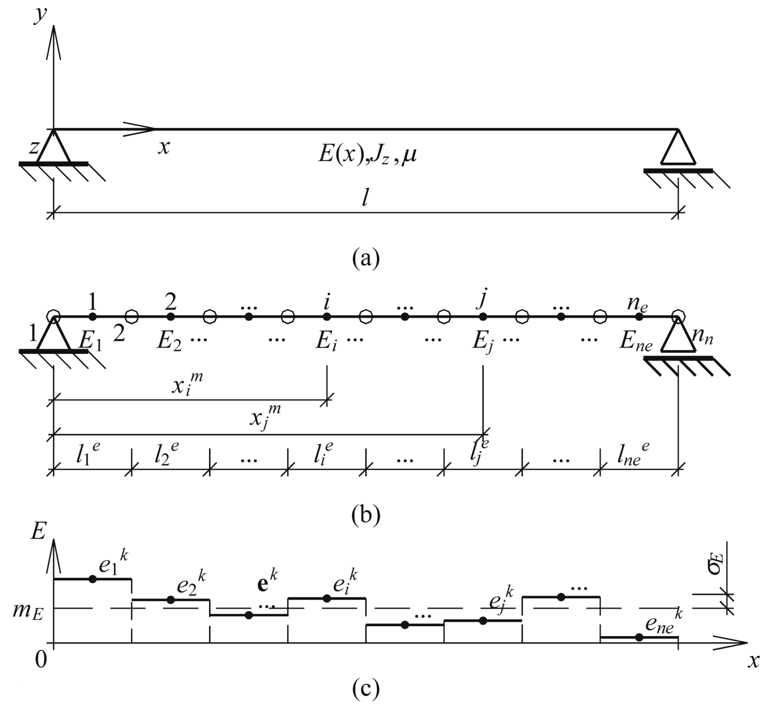

Fig. 3 Detailed discretization of the simply supported beam from Fig. 1(a, b) by applying the midpoint method (c)

a 1-D random field $E(x)$ then its discrete form can be formulated using so called "midpoint method" (see a particular discrete sample $\mathbf{e}^k$ of $\mathbf{E}$ in Fig. 3(c)).

The further analysis can be carried out using the stochastic FEM. For the purpose of stochastic analysis of bar structures simple implementation of a spectral method (e.g., Ghanem and Spanos 1991b) or a perturbation method based on Taylor series (Kleiber and Hien 1992) can be chosen. The frame element matrices of stiffeness and mass of classic implemenetation of FEM are given by following equations in local coordinate system

$$\hat{\mathbf{K}}_i^e = \frac{E_i \cdot J_{zi}}{(l_i^e)^3} \cdot \begin{bmatrix} \frac{A_i \cdot (l_i^e)^2}{J_{zi}} & 0 & 0 & -\frac{A \cdot (l_i^e)^2}{J_{zi}} & 0 & 0 \\ 0 & 12 & 6 \cdot l_i^e & 0 & -12 & 6 \cdot l_i^e \\ 0 & 6 \cdot l_i^e & 4 \cdot (l_i^e)^2 & 0 & -6 \cdot l_i^e & 2 \cdot (l_i^e)^2 \\ -\frac{A \cdot (l_i^e)^2}{J_{zi}} & 0 & 0 & \frac{A \cdot (l_i^e)^2}{J_{zi}} & 0 & 0 \\ 0 & -12 & -6 \cdot l_i^e & 0 & 12 & -6 \cdot l_i^e \\ 0 & 6 \cdot l_i^e & 2 \cdot (l_i^e)^2 & 0 & -6 \cdot l_i^e & 4 \cdot (l_i^e)^2 \end{bmatrix} \tag{19}$$

$$\hat{\mathbf{B}}_i^e = \frac{\mu_i \cdot l_i^e}{420} \cdot \begin{bmatrix} 140 & 0 & 0 & 70 & 0 & 0 \\ 0 & 156 & 22 \cdot l_i^e & 0 & 54 & -13 \cdot l_i^e \\ 0 & 22 \cdot l_i^e & 4 \cdot (l_i^e)^2 & 0 & 13 \cdot l_i^e & -3 \cdot (l_i^e)^2 \\ 70 & 0 & 0 & 140 & 0 & 0 \\ 0 & 54 & 13 \cdot l_i^e & 0 & 156 & -22 \cdot l_i^e \\ 0 & -13 \cdot l_i^e & -3 \cdot (l_i^e)^2 & 0 & -22 \cdot l_i^e & 4 \cdot (l_i^e)^2 \end{bmatrix} \tag{20}$$

where A_i stand for cross-section areas, J_{zi} it is the moment of interia, l_i^e is length of the element, and μ_i is mass per unit length.

Both stiffness and mass matrixes (Eqs. (19) and (20)) can be transformed into global by the similar transformation

$$\mathbf{K}_i^e = \mathbf{\Theta}^T \cdot \hat{\mathbf{K}}_i^e \cdot \mathbf{\Theta}, \quad \mathbf{B}_i^e = \mathbf{\Theta}^T \cdot \hat{\mathbf{B}}_i^e \cdot \mathbf{\Theta} \tag{21}$$

where $\mathbf{\Theta}$ stands for the transformation matrix in 2-D.

By assembling n_e elements using classic FEM approach, the global stiffness and mass matrices can be obtained

$$\mathbf{K} = \sum_{i=1}^{ne} \mathbf{K}_i^e, \quad \mathbf{B} = \sum_{i=1}^{ne} \mathbf{B}_i^e \tag{22}$$

The familiar procedure of Eqs. (19)-(22) described in detail in numerous text books and monographs (see e.g., Bathe 1982).

Assuming that the Young modulus of the structural system is a random field discretized in form of the vector $\mathbf{E}$, one can write the equation for stochastic eigenvalue problem in following form

$$(\mathbf{K}(\mathbf{E}) - \lambda \cdot \mathbf{B}) \cdot \mathbf{q} = 0 \tag{23}$$

in which matrix $\mathbf{B}$ is assumed deterministic. Following Kleiber and Hien (1992) and expanding each of the eigenvalues into the Taylor series about a particular value $\bar{\lambda}$ leads to

$$\lambda = \bar{\lambda} + \sum_{i=1}^{ne} \frac{\partial \lambda}{\partial E_i}\bigg|_{\bar{E}i} \cdot (E_i - \bar{E}_i) + \frac{1}{2} \cdot \sum_{i=1}^{ne}\sum_{j=1}^{ne} \frac{\partial^2 \lambda}{\partial E_i \partial E_j}\bigg|_{\bar{E}i, \bar{E}j} \cdot (E_i - \bar{E}_i) \cdot (E_j - \bar{E}_j) + \dots \tag{24}$$

where $\bar{E}_i, \bar{E}_j$ are respective certain values of Young modulus, $\partial\lambda/\partial E_i|_{\bar{E}i}$ is first partial derivative at argument $\bar{E}_i$, and $\partial^2\lambda/\partial E_i \partial E_j|_{\bar{E}i, \bar{E}j}$ is second partial derivative at arguments $\bar{E}_i$ and $\bar{E}_j$.

Applying the operator of expectation to Eq. (24) one obtaines its mean value m_λ

$$m_\lambda = \bar{\lambda} + \sum_{i=1}^{ne} \frac{\partial \lambda}{\partial E_i}\bigg|_{mEi} \cdot \overset{1}{\mu}_{Ei} + \frac{1}{2} \cdot \sum_{i=1}^{ne}\sum_{j=1}^{ne} \frac{\partial^2 \lambda}{\partial E_i \partial E_j}\bigg|_{mEi, mEj} \cdot Cov[E_i, E_j] \tag{25}$$

where m_{Ei}, m_{Ej} are mean value of the Young modulus of elements i and j, $\overset{1}{\mu}_{Ei}$ is central moment of the first-order of element i, and $Cov[E_i, E_j]$ is covariance of Young modulus of elements i and j.

In fact one shall keep in mind that the mean values of the components of all the discretized values of vector of Young modulus are known, so the mean values of eigenvalues can be obtained by straightforward application of standard numerical solutions of eigenvalue problem. To obtain standard deviations of eigenvalues one needs to raise both sides of Eq. (25) to square and take again the operator of mathematical expectation. This yields variance of eigenvalue as follows

$$Var[\lambda] = \sum_{i=1}^{ne}\sum_{j=1}^{ne}\left.\frac{\partial\lambda}{\partial E_i}\right|_{mEi}\cdot\left.\frac{\partial\lambda}{\partial E_j}\right|_{mEj}\cdot Cov[E_i,E_j]+$$

$$\sum_{i=1}^{ne}\left.\frac{\partial\lambda}{\partial E_i}\right|_{mEi}\cdot\mu_{Ei}^{1}\cdot\sum_{i=1}^{ne}\sum_{j=1}^{ne}\left.\frac{\partial^2\lambda}{\partial E_i\partial E_j}\right|_{mEi,mEj}\cdot Cov[E_i,E_j]+$$

$$\frac{1}{4}\cdot\sum_{i=1}^{ne}\sum_{j=1}^{ne}\sum_{k=1}^{ne}\sum_{l=1}^{ne}\left.\frac{\partial^2\lambda}{\partial E_i\partial E_j}\right|_{mEi,mEj}\cdot\left.\frac{\partial^2\lambda}{\partial E_k\partial E_l}\right|_{mEk,mEl}\cdot(Cov[E_i,E_l]\cdot Cov[E_j,E_k]+$$

$$Cov[E_i,E_k]\cdot Cov[E_j,E_l]) \tag{26}$$

We assume that the random field of Young modulus is Gaussian. Therefore central moment of the first-order $\mu_{Ei}^{1} = 0$. Next one simplifies Eqs. (25) and (26) to

$$m_\lambda = \bar{\lambda}+\frac{1}{2}\cdot\sum_{i=1}^{ne}\sum_{j=1}^{ne}\left.\frac{\partial^2\lambda}{\partial E_i\partial E_j}\right|_{mEi,mEj}\cdot Cov[E_i,E_j] \tag{27}$$

$$Var[\lambda] = \sum_{i=1}^{ne}\sum_{j=1}^{ne}\left.\frac{\partial\lambda}{\partial E_i}\right|_{mEi}\cdot\left.\frac{\partial\lambda}{\partial E_j}\right|_{mEj}\cdot Cov[E_i,E_j]+$$

$$\frac{1}{4}\cdot\sum_{i=1}^{ne}\sum_{j=1}^{ne}\sum_{k=1}^{ne}\sum_{l=1}^{ne}\left.\frac{\partial^2\lambda}{\partial E_i\partial E_j}\right|_{mEi,mEj}\cdot\left.\frac{\partial^2\lambda}{\partial E_k\partial E_l}\right|_{mEk,mEl}\cdot(Cov[E_i,E_l]\cdot Cov[E_j,E_k]+$$

$$Cov[E_i,E_k]\cdot Cov[E_j,E_l]) \tag{28}$$

Introducing matrix notation one obtains

$$\mathbf{m}_\lambda = \boldsymbol{\lambda}^0+\frac{1}{2}\cdot\sum_{i=1}^{ne}\sum_{j=1}^{ne}\boldsymbol{\lambda}^{Ei,Ej}\cdot Cov[E_i,E_j] \tag{29}$$

which represents vector of the mean eigenvalues. Calculating not only variances, but also respective covariances of the eigenvalues leads to the matrix

$$\mathbf{Cov}_\lambda = \sum_{i=1}^{ne}\sum_{j=1}^{ne}\boldsymbol{\lambda}^{Ei}\cdot(\boldsymbol{\lambda}^{Ej})^T\cdot Cov[E_i,E_j]+$$

$$\frac{1}{4}\cdot\sum_{i=1}^{ne}\sum_{j=1}^{ne}\sum_{k=1}^{ne}\sum_{l=1}^{ne}\boldsymbol{\lambda}^{Ei,Ej}\cdot(\boldsymbol{\lambda}^{Ek,El})^T\cdot(Cov[E_i,E_l]\cdot Cov[E_j,E_k]+Cov[E_i,E_k]\cdot Cov[E_j,E_l]) \tag{30}$$

Vector $\boldsymbol{\lambda}^0$ is called the "zeroth-order" vector of eigenvalues. It can be obtain from equation

$$(\mathbf{K}^0-\lambda^0\cdot\mathbf{B}^0)\cdot\mathbf{q}^0 = 0 \tag{31}$$

where $\mathbf{K}^0$ and $\mathbf{B}^0$ are classic FEM stiffenes and mass matrixes given by Eq. (22).

Vectors λ_m^{Ei} and $\lambda_m^{Ei,Ej}$ are called the "first-" and "second-order" vectors of eigenvalues and their elements are given by following equations

$$\lambda_m^{Ei} = (\mathbf{u}_m^0)^T \cdot \mathbf{K}^{Ei} \cdot \mathbf{u}_m^0, \quad m = 1, 2, \ldots, n_d \tag{32}$$

$$\lambda_m^{Ei,Ej} = (\mathbf{u}_m^0)^T \cdot (\mathbf{K}^{Ej} - \lambda_m^{Ej} \cdot \mathbf{B}^0) \cdot \mathbf{u}_m^{Ei} + (\mathbf{u}_m^0)^T \cdot (\mathbf{K}^{Ei} - \lambda_m^{Ei} \cdot \mathbf{B}^0) \cdot \mathbf{u}_m^{Ej} \tag{33}$$

Vectors $\mathbf{u}^{Ei}$ and $\mathbf{u}^{Ei,Ej}$ are called "first-" and "second-order" eigenvectors. Their elements can be determine by equations

$$\mathbf{u}_m^{Ei} = \mathbf{U}^0 \cdot \mathbf{a}_m^{Ei} \tag{34}$$

$$\mathbf{u}_m^{Ei,Ej} = \mathbf{U}^0 \cdot \mathbf{a}_m^{Ei,Ej} \tag{35}$$

where $\mathbf{U}^0$ is matrix of the eigenvectors, and $\mathbf{a}_m^{Ei}, \mathbf{a}_m^{Ei,Ej}$ are vectors of first- and second-order coefficients. Detailed description of the algorithm to obtain the above two vectors can be found in the book by Kleiber and Hien (1992).

4. Notes on Monte Carlo method as applied to solve stochastic eigenvalue problem

The Monte Carlo method is very well established in stochastic structural mechanics (e.g., Collins and Thomson 1969, Shinozuka and Astill 1972, Baecher and Ingra 1981, Liu *et al.* 1986, Spanos and Ghanem 1989, Ghanem and Spanos 1991a, Székely and Schuëller 2001 and Kaleta and Zembaty 2007). The Monte Carlo method is universal and versatile. With the aid of this method various technical problems can be analyzed. The Monte Carlo method introduce no limitation in the "size" of the analyzed stochastic variations (value of coefficient of variation) in contrast to the applied stochastic finite element method which is limited to small variations. Its disadvantage however may be the long computation time due to repeated eigenproblem solutions for each sample of the data. In this paper it is applied as an alternative verification of the examples computed by stochastic FEM. The respective samples of Young modulus random field are generated by applying Gaussian probability distribution and the Cholesky method of generation of the covariances (Yamazaki *et al.* 1988, Zielinski 1970). Following Marek *et al.* (2003) one can presume, that the accuracy of the Monte Carlo method reach 1% when at least 500 samples are generated. In this paper the number of correlated samples of the Young modulus E equals 1000. Formally the Gaussian random field Young modulus should be defined as a truncated probability distribution. However for the applied number of samples it was never the case in the numerical simulations to follow to reject any negative E value.

5. Example one: simply supported beam

Consider a simply supported beam (Fig. 3(a)) with the span $l = 3$ m, rectangular cross-section $b \times h = 0.25$ m × 0.5 m, and moment of inertia $J_{zi} = 0.002604$ m^4. The beam has been divided onto 8 finite elements. It is also assumed that it is made of reinforced concrete with mass per unit

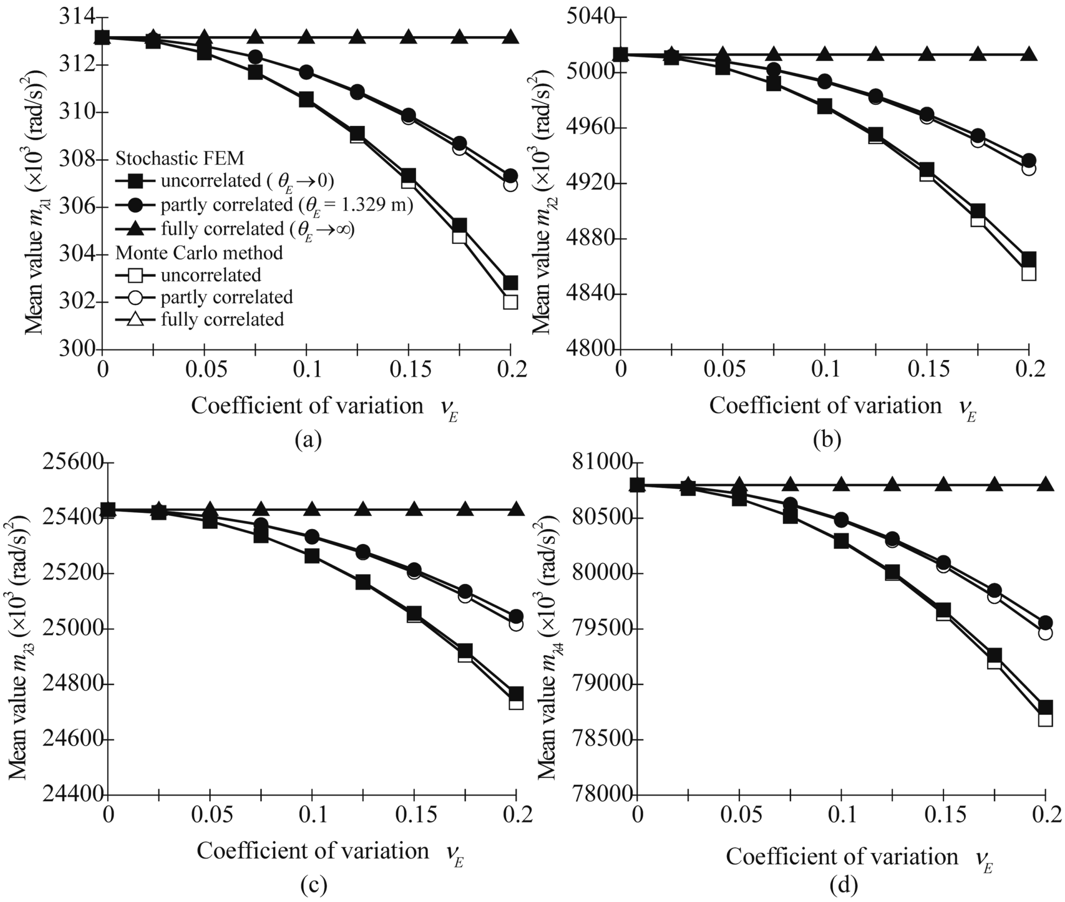

Fig. 4 Dependence of the mean values of the first four (a), (b), (c), (d) eigenvalues $m_{\lambda1}$, ..., $m_{\lambda4}$ of simply supported beam on the coefficient of variation v_E of the spatial random field

length $\mu = 300$ kg/m. The random variations of stiffness along the beam are defined by 1-D random field of Young modulus E which is defined by its mean value $m_E = 30$ GPa and coefficient of variation v_E changing from 0 to 20%. In what follows three separate cases of random field described by exponential correlation function type II (Eq. (13)) are analyzed: uncorrelated random field ($\theta_E \to 0$), partly correlated random field ($\theta_E = 1{,}329$ m after Yamazaki *et al.* 1988), and fully correlated random field ($\theta_E \to \infty$). In typical practical situations the elastic properties are partly correlated or are close to be totally correlated. Thus the first situation is analyzed here only as a reference state.

In Fig. 4 the plots of mean values of the first four eigenvalues $m_{\lambda1}$, ..., $m_{\lambda4}$ as functions of the coefficient of variation v_E are shown. The calculations using stochastic FEM are verified by the application of the Monte Carlo approach. It can be seen that, as could be expected, the assumption of total correlation of the random field results in constant eigenvalues. It means that the mean eigenvalues do not depend on the loss of the quality of the concrete. In cases of partial or zero correlations the mean eigenvalues drop down with increasing v_E. The results of stochastic FEM and Monte Carlo approach almost coincide for low values of v_E and start to slowly diverge with

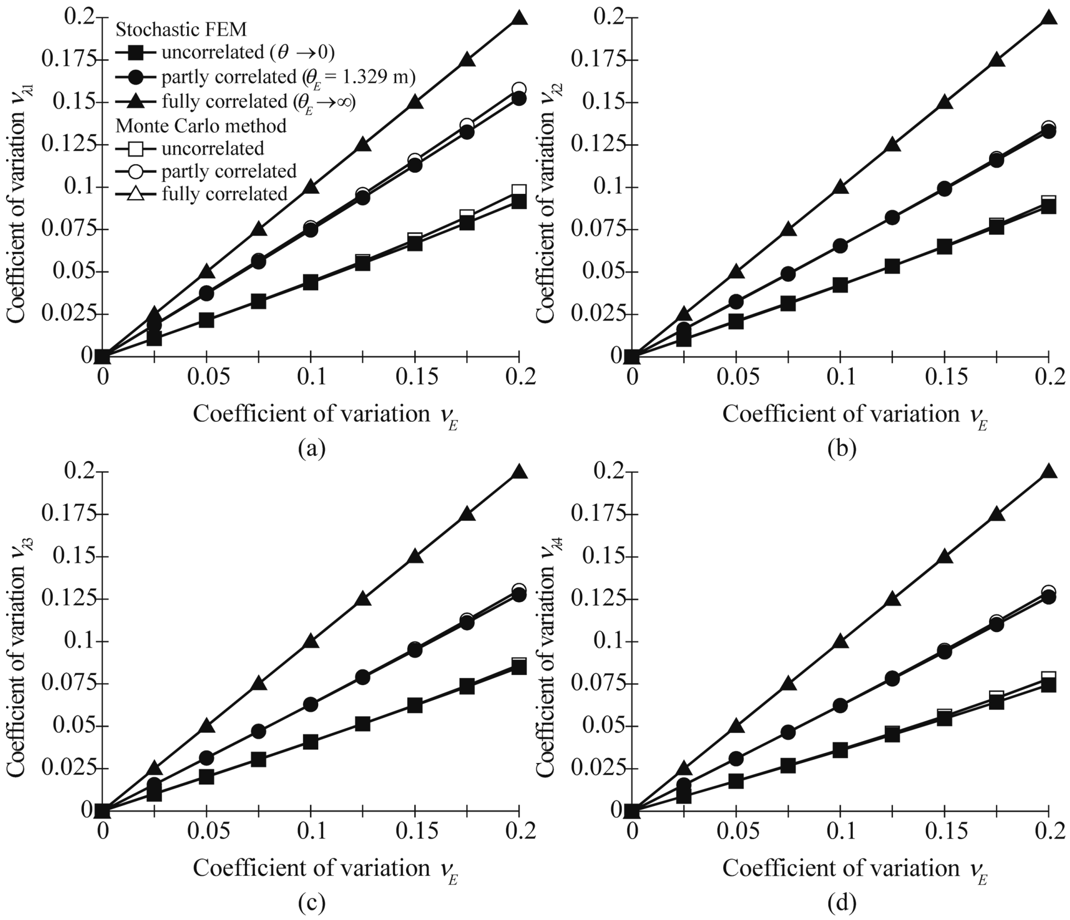

Fig. 5 Dependence of the coefficients of variation of the first four (a), (b), (c), (d) eigenvalues $\nu_{\lambda 1}$, $\nu_{\lambda 2}$, $\nu_{\lambda 3}$ and $\nu_{\lambda 4}$ of simply supported beam on the coefficient of variation ν_E of the spatial random field

increasing ν_E. It is so because the Monte Carlo method holds for high variations of E (high values of standard deviation σ_E) while the stochastic FEM is only second order method and looses its accuracy when ν_E reach about 10%.

In Fig. 5 the plots of the coefficients of variation of the eigenvalues $\nu_{\lambda 1}$, $\nu_{\lambda 2}$, $\nu_{\lambda 3}$ and $\nu_{\lambda 4}$ as functions of the coefficient of variation ν_E are shown again for three cases of spatial correlation. Almost linear dependence between the Young modulus coefficients of variations and respective coefficients of variation of the eigenvalues can be observed. Plots from Fig. 5 clearly reflect the role of coefficients of variations of eigenvalues as measures of the quality of concrete. With decreasing quality of concrete greater variations of eigenvalues are observed. It is interesting to note that the assumption of fully correlation makes this dependence stronger. Thus, when assuming only partial correlation the coefficient of variation ν_λ depends less on the loss of the concrete quality. It is interesting to note that while in structural health monitoring the decay of natural frequencies (eigenvalues) directly reflect the loss of structural stiffness in stochastic eigenvalue problem the loss of material quality is reflected by the increase of the coefficient of variation of the eigenvalues.

6. Example two: three-span hinged beam and portal frame

Consider a three-span hinged beam (Fig. 6(a)) with the spans $l_1 = 2$ m, $l_2 = 3$ m and $l_3 = 2$ m and a portal frame with height $l_1 = 4$ m and width $l_2 = 3$ m (Fig. 7(a)). Both the beam and frame are made of concrete and consist of members with rectangular cross-sections $b \times h = 0.25$ m $\times$ 0.5 m, moments of inertia $J_{zi} = 0.002604$ m^4 and masses per unit length $\mu = 300$ kg/m. The beam has been

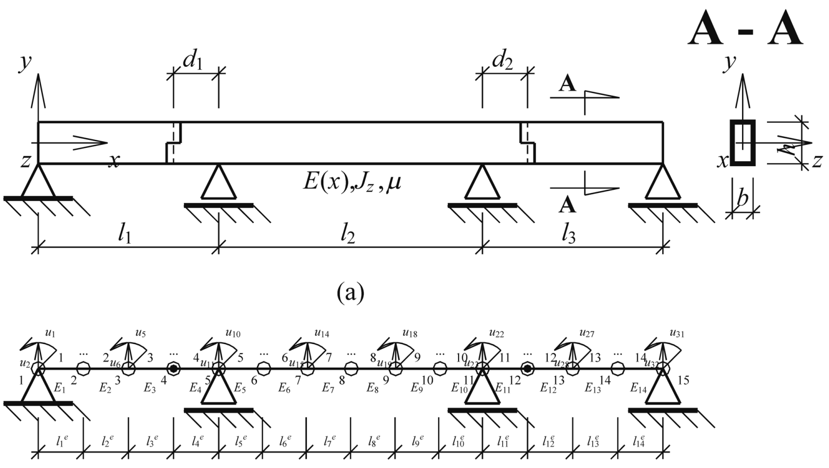

Fig. 6 (a) Three-span hinged beam and (b) its finite element discretization

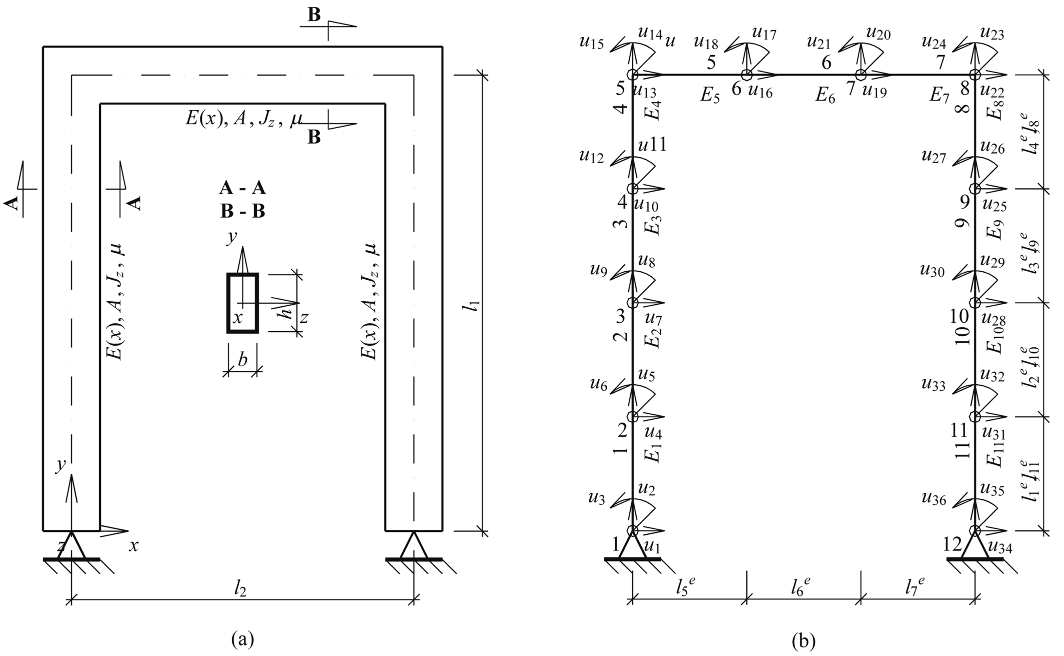

Fig. 7 (a) Portal frame and (b) its finite element discretization divided onto 14 finite elements (Fig. 6(b)) and frame onto 12 finite elements (Fig. 7(b)). As previously the Young modulus E is defined in terms of its mean value $m_E = 30$ GPa and the coefficient of variation ν_E changing from 0 to 20%. Both the beam and the frame can either be made of the material coming from the same supplier (see Figs. 2(b) and 2(d)) or from different ones (see Figs. 2(a) and 2(c)). In the latter case the beam consists of two outer segments of length l_1-$d_1 = 1{,}5$ m and the central segment with length $l_2 + d_1 + d_2 = 4$ m (Fig. 6(a)) while the frame consists of just three separate members (two 4 m high columns and 3 m long spandrel beam). The assumption of the same material among respective segments or members does not introduce anything new compare to the previous example. On the other hand the assumption that the materials of respective segments of the beam or members of the frame derive from different suppliers leads to the introduction of additional cross-correlation cases. This situation is reflected here by three 1-D random fields of Young modulus E defined separately in the three analyzed segments (members) of the beam and frame. The foregoing explanations results here in four separate cases of the random field: uncorrelated random field ($\theta_E \to 0$), partly correlated random field ($\theta_E = 1{,}772$ m for the beam

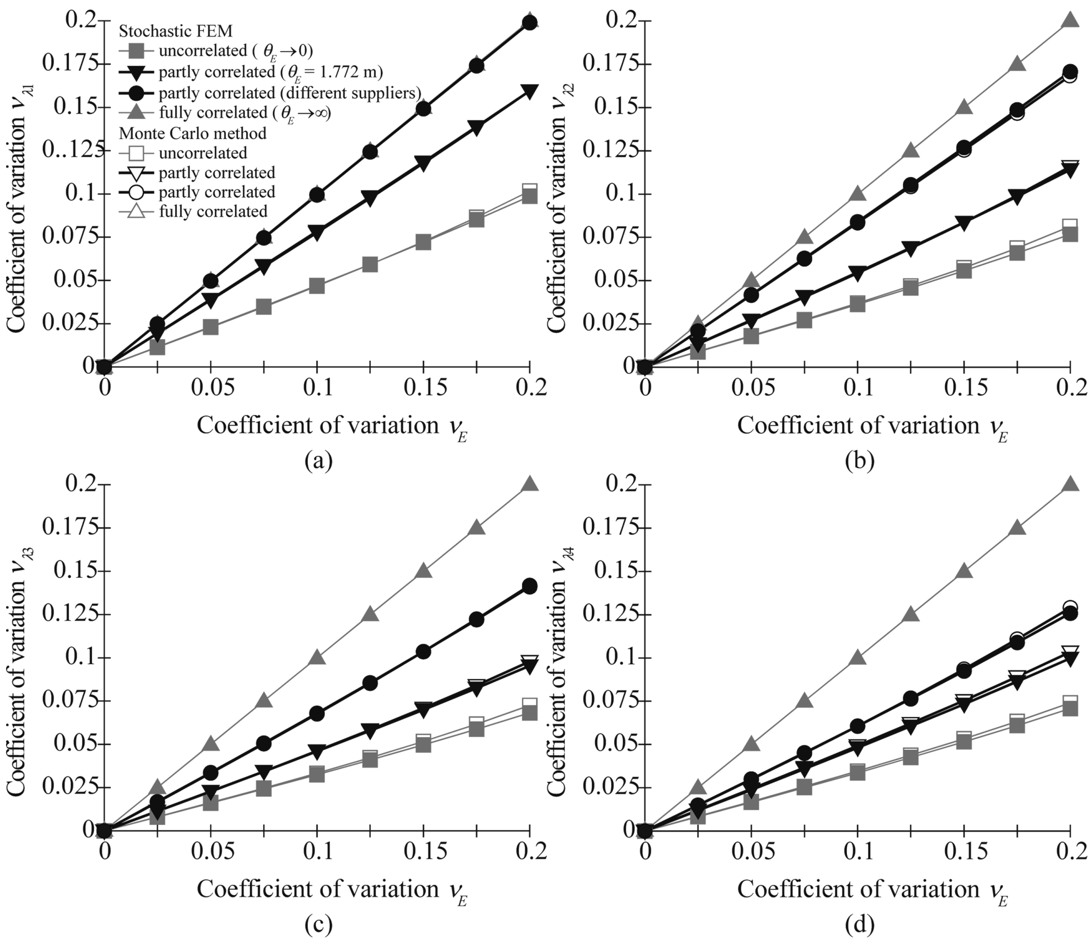

Fig. 8 Dependence of the coefficients of variation of the first four (a), (b), (c), (d) eigenvalues $\nu_{\lambda 1}$, $\nu_{\lambda 2}$, $\nu_{\lambda 3}$ and $\nu_{\lambda 4}$ of three-span hinged beam on the coefficient of variation ν_E of the spatial random field

and $\theta_E = 3{,}545$ m for the frame after Yamazaki *et al.* 1988), partly correlated random field (each segment (member) is fully correlated, but they are uncorrelated among each other), and fully correlated random field ($\theta_E \to \infty$). In this example the same exponential correlation function type II (Eq. (13)) is applied as in the previous example.

In Figs. 8 and 9 the plots of the coefficients of variation of the eigenvalues $v_{\lambda 1}$, $v_{\lambda 2}$, $v_{\lambda 3}$ and $v_{\lambda 4}$ as functions of the coefficient of variation v_E are shown for the beam and frame for the above four cases of spatial correlation. As for the simple beam again almost linear dependence between the Young modulus coefficients of variations and respective coefficients of variation of the eigenvalues can be observed. It is interesting to note that in the third case of correlation (zero correlation among different beam segments and full correlation within each segment) the higher the eigenvalue the lower the coefficient of variation of the eigenvalue Effectively with increasing eigenvalue the final solution is closer and closer to the uncorrelated results. This effect is less pronounced for the three-member frame (see Fig. 9).

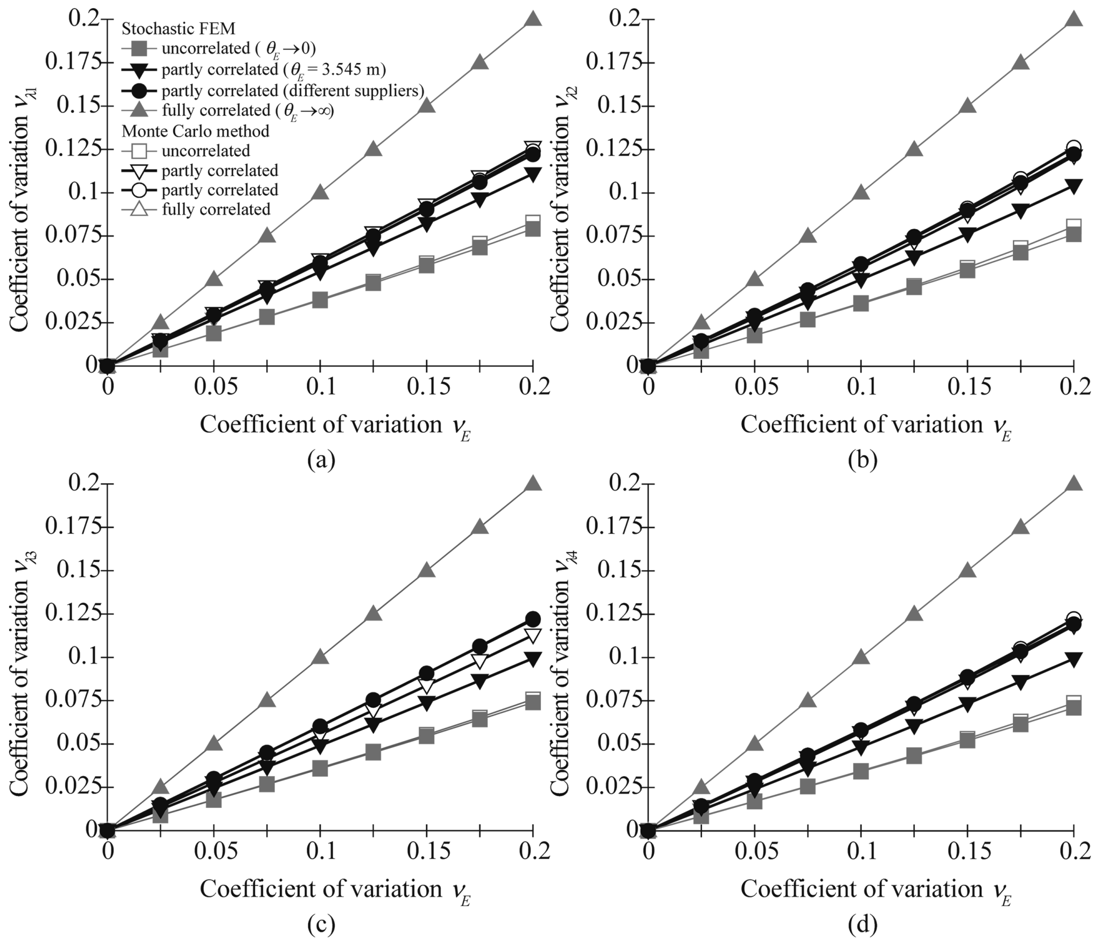

Fig. 9 Dependence of the coefficients of variation of the first four (a), (b), (c), (d) eigenvalues $v_{\lambda 1}$, $v_{\lambda 2}$, $v_{\lambda 3}$ and $v_{\lambda 4}$ of the frame on the coefficient of variation v_E of the spatial random field

8. Conclusions

The problem of the effect of spatial structural properties distribution on the structural response was a subject of only limited research in recent years. The most important ones of Shinozuka and Astill (1972), Pradlwarter *et al.* (2002) are concentrated on random 1-D and 2-D spatial fields rather than on spatial, structural configuration. Thus, in this paper, the stochastic eigenvalue problem of beams and frames was revisited with particular attention paid to the effect of spatial random variability of the stiffness of their members on the eigenvalues. Both stochastic FEM, as implemented in the monograph by Kleiber and Hien (1992) and Monte Carlo technique, were applied with the first one giving meaningful results only for about 10% of the eigenvalues because of the obvious limitations of any second-order approximation. The spatial variations of beam properties along its length were analyzed as well as correlations among different technological segments.

The results of typical exponential correlation cases were bounded by the zero- and full-correlation ones. The dependence of the first four eigenvalues on the spatial parameter variation is, however, not that straightforward when it comes to consider different technological segments of the structure. The difference between zero- and full-correlation cases appeared to be substantial. In most of the analyzed examples it reached about 50%. This means that for the structure consisting of various technologically different segments the effect of the cross-correlation of their properties will strongly influence the solution of the eigenvalue problem.

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