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Investigation on efficiency and applicability of subspace iteration method with accelerated starting vectors for calculating natural modes of structures

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Abstract. For efficient calculation of natural modes of structures, a numerical scheme which accelerates convergence of the subspace iteration method by employing accelerated starting Lanczos vectors was proposed in 2005. This paper is an extension of the study. The previous study simply showed feasibility of the proposed method by analyzing structures with smaller degrees of freedom. While, the present study verifies efficiency of the proposed method more rigorously by comparing closeness of conventional and accelerated starting vectors to genuine eigenvectors. This study also analyzes an example structure with larger degrees of freedom and more complex constraints in order to investigate applicability of the proposed method.

Keywords: natural modes of structures; eigenvector; subspace iteration method; accelerated starting vector.

1. Introduction

Natural modes such as natural frequencies and mode shapes are key parameters in dynamic analysis or seismic design of structures. Natural frequencies and mode shapes of structures can be obtained through eigenvalue analysis. Various numerical methods can be applied for eigenvalue analysis of structures. Among them, the subspace iteration method is widely used. The subspace iteration method is originally proposed by Bathe and Wilson (1972). Thereafter, many improved methods have been developed by many researchers (Bathe and Ramaswamy 1980, Akl *et al*. 1982, Wilson and Itoh 1983, Lam and Bertolini 1994, Kim *et al*. 2005, Sui and Zhong 2006, Zhao *et al*. 2007, Chen *et al*. 2008).

Among them, the subspace iteration method using Lanczos vectors as starting iteration vectors is described in the study by Bathe and Ramaswamy (1980). Kim *et al*. (2005) presented a modified version of the method. The modified method uses accelerated starting Lanczos vectors which are obtained from vector generation algorithm using squared dynamic matrix. The numerical technique

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of squaring matrix was originally proposed by Grosso *et al*. (1993) in quantum problems for solving eigenstate of quantum. The similar technique was proposed in the simultaneous inverse iteration process in the modified subspace iteration method (Lam and Bertolini 1994, Qian and Dhatt 1995, Wang and Zhou 1999). Kim *et al*. (2005) applied the numerical technique to generation of accelerated starting vectors in the subspace iteration method.

This paper is the extension of the study by Kim *et al*. (2005). The previous study introduced an accelerated algorithm and simply showed the feasibility of the proposed method. Example structures were also simple building structures with smaller degrees of freedom such as 1008 and 5040 DOFs. This study investigates the efficiency of the proposed method more rigorously by comparing conventional and proposed starting vectors for their closeness to eigenvectors. Natural modes of an example structure with larger degrees of freedom over 20000 DOFs and more complicated constraints are also analyzed to examine the applicability of the proposed method by comparing number of iterations and calculation time of the conventional and proposed methods.

2. Subspace iteration method with accelerated starting vectors

Natural frequencies and mode shapes of structures are obtained by solving the following eigenvalue problem

$$\mathbf{K}\boldsymbol{\phi}_j = \omega_j^2 \mathbf{M}\boldsymbol{\phi}_j \qquad (j = 1, 2, \ldots, n) \tag{1}$$

where $\mathbf{M}$ and $\mathbf{K}$ are symmetric mass and stiffness matrices of order n, respectively. ω_j is the jth natural frequency and $\boldsymbol{\phi}_j$ is the jth eigenvector which describes the jth mode shape of structures.

The first step in the subspace iteration method for solving (1) is to establish a starting subspace. Starting subspace takes the form of a matrix of which columns are starting iteration vectors. The subspace iteration method with Lanczos starting subspace uses Lanczos vectors as starting iteration vectors because Lanczos vectors are good approximations of exact eigenvectors. Lanczos vectors are generated from the Lanczos algorithm which was first proposed by Lanczos (1950). Lanczos algorithm can be summarized as follows. The first Lanczos vector is established by

$$\mathbf{x}_1 = \frac{\mathbf{x}}{\sqrt{\mathbf{x}^T \mathbf{M}\mathbf{x}}} \tag{2}$$

where $\mathbf{x}$ is an arbitrary vector and usually $[1\ 1\ \ldots\ 1]^T$. Then, the following calculations are repeated for $i = 1, 2, \ldots$ with $\beta_0 = 0$

$$\bar{\mathbf{x}}_i = \mathbf{K}^{-1}\mathbf{M}\mathbf{x}_i \tag{3}$$

$$\alpha_i = \bar{\mathbf{x}}_i^T \mathbf{M}\mathbf{x}_i \tag{4}$$

$$\tilde{\mathbf{x}}_i = \bar{\mathbf{x}}_i - \alpha_i \mathbf{x}_i - \beta_{i-1}\mathbf{x}_{i-1} \tag{5}$$

$$\beta_i = \sqrt{\tilde{\mathbf{x}}_i^T \mathbf{M}\tilde{\mathbf{x}}_i} \tag{6}$$

$$\mathbf{x}_{i+1} = \frac{\tilde{\mathbf{x}}_i}{\beta_i} \tag{7}$$

In the steps, $\mathbf{x}_i$ is the ith Lanczos vector. Eq. (3) represents inverse iteration. In the equation, $\mathbf{K}^{-1}\mathbf{M}$ is called dynamic matrix. $\bar{\mathbf{x}}_i$ is not obtained by direct calculation of the dynamic matrix but it is obtained by forward reduction and back substitution. Eqs. (4) and (5) are orthogonalization processes. Eqs. (6) and (7) are steps for normalization. If the number of required eigenpairs (eigenvalue and the corresponding eigenvector) is p, the number of required Lanczos vectors is generally $2p$ and the starting subspace will be $\mathbf{\Phi}_1 = [\mathbf{x}_1 \ \mathbf{x}_2 \ \dots \ \mathbf{x}_q](q = 2p)$.

The next step is to perform subspace iteration which includes simultaneous inverse iteration, system reduction, eigenvalue analysis for reduced system and computation of updated subspace. The simultaneous inverse iteration is performed as follows for $k = 1, 2, \dots$.

$$\bar{\mathbf{\Phi}}_{k+1} = \mathbf{K}^{-1}\mathbf{M}\mathbf{\Phi}_k \tag{8}$$

Then, the reduced system is calculated by

$$\mathbf{K}_{k+1} = \bar{\mathbf{\Phi}}_{k+1}^T\mathbf{K}\bar{\mathbf{\Phi}}_{k+1} \tag{9}$$

$$\mathbf{M}_{k+1} = \bar{\mathbf{\Phi}}_{k+1}^T\mathbf{M}\bar{\mathbf{\Phi}}_{k+1} \tag{10}$$

The order of reduced system is q because $\mathbf{\Phi}_k$ is a n by q matrix. The following eigenvalue problem for the reduced system is solved.

$$\mathbf{K}_{k+1}\mathbf{Q}_{k+1} = \mathbf{M}_{k+1}\mathbf{Q}_{k+1}\Lambda_{k+1} \tag{11}$$

The generalized Jacobi method can be effectively used for the reduced small system. Using the eigensolution of reduced system, the updated subspace is obtained by

$$\mathbf{\Phi}_{k+1} = \bar{\mathbf{\Phi}}_{k+1}\mathbf{Q}_{k+1} \tag{12}$$

As k increases, Λ_{k+1} converges to a diagonal matrix whose diagonal entries are exact eigenvalues and $\mathbf{\Phi}_{k+1}$ converges to a matrix whose columns are exact eigenvectors. If the number of lower eigenpairs whose errors are within the predetermined tolerance is p, the iteration loop is terminated.

Subspace iteration method with accelerated starting vectors uses modified Lanczos algorithm to generate accelerated Lanczos vectors. The modified algorithm uses (13) instead of (3).

$$\bar{\mathbf{x}}_i = (\mathbf{K}^{-1}\mathbf{M})^2\mathbf{x}_i \tag{13}$$

In the equation, squared dynamic matrix is used while conventional algorithm uses non-squared dynamic matrix as in (3). Squared dynamic matrix means doubled inverse iteration. In (13), $\bar{\mathbf{x}}_i$ is not calculated by direct square of the dynamic matrix. It is obtained by repeated forward reduction and back substitution. Since $\bar{\mathbf{x}}_i$ in (13) is the vector calculated from the repeated inverse iteration, it can separate approximate eigenvalues more rapidly than $\bar{\mathbf{x}}_i$ in (3). Therefore, modified starting subspace with accelerated Lanczos vectors generated from the proposed algorithm are closer to exact eigenvector space than conventional Lanczos starting subspace, resulting in reduction of the number of subspace iterations. Of course, squared dynamic matrix requires additional cost for forward reduction and back substitution. However, the degree of cost reduction due to less iteration overwhelms that of cost increase due to additional forward reduction and back substitution.

Simultaneous inverse iteration, system reduction, eigenvalue analysis for reduced system and computation of updated subspace have the same procedures as the conventional method.

3. Starting vectors' closeness to eigenvectors

In order to verify efficiency of proposed method, comparison of iteration numbers or CPU time for example structures might be valid. However, prior to such direct procedure, investigation on the characteristics of starting vectors due to conventional and proposed algorithms and their comparisons will lead to more basic insight. In this paper, staring vectors of conventional and proposed method and their closeness to eigenvectors are examined and compared. Two standard structures are considered for examples.

The first example is a cantilever beam. Length, section area, second moment of inertia, torsion constant and density are all assumed to be 1. $p = 7$ and $q = 14$. Figs. 1-7 present mode shapes of eigenvectors and starting vectors of the cantilever beam. Those figures show which of the conventional and proposed starting vectors are closer to exact eigenvectors. It is shown that closeness of conventional and proposed starting vectors are similar for the first, second and third mode. However, proposed starting vectors are closer to eigenvectors than conventional starting vectors for 4th~7th modes. Consequently, number of iterations of the proposed method is smaller

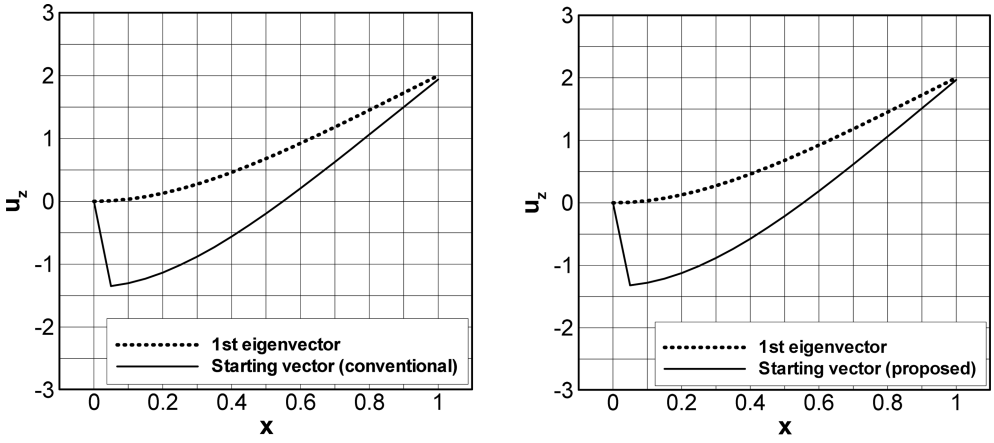


Fig. 1 Mode shapes of 1st eigenvector and starting iteration vectors of cantilever

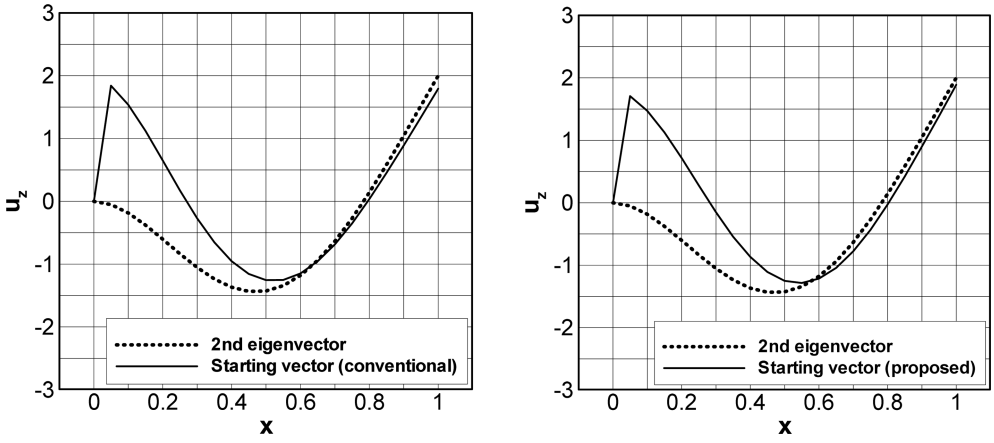


Fig. 2 Mode shapes of 2nd eigenvector and starting iteration vectors of cantilever

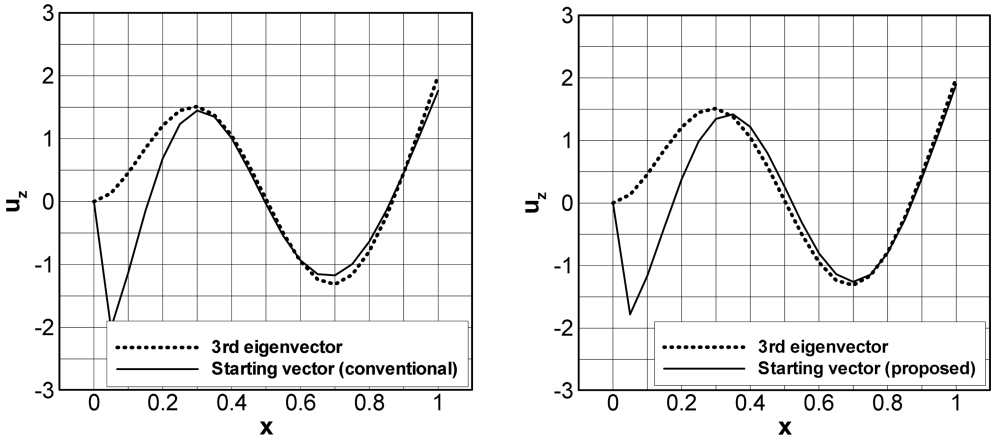


Fig. 3 Mode shapes of 3rd eigenvector and starting iteration vectors of cantilever

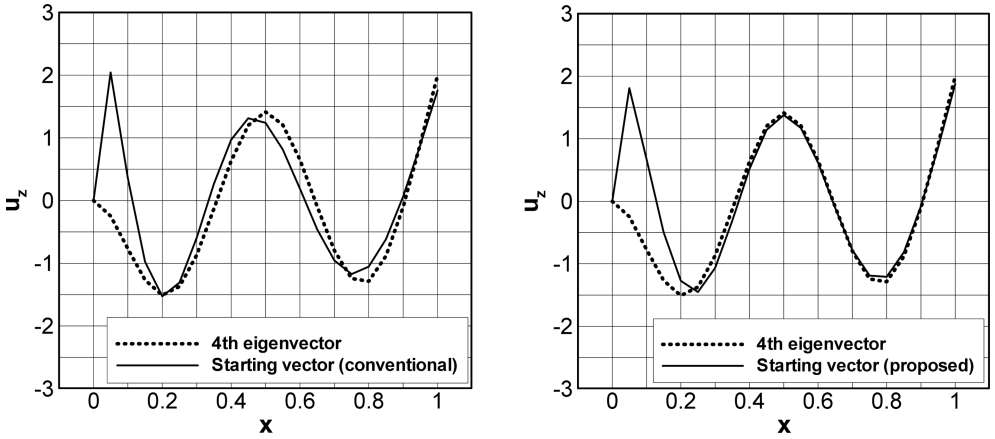


Fig. 4 Mode shapes of 4th eigenvector and starting iteration vectors of cantilever

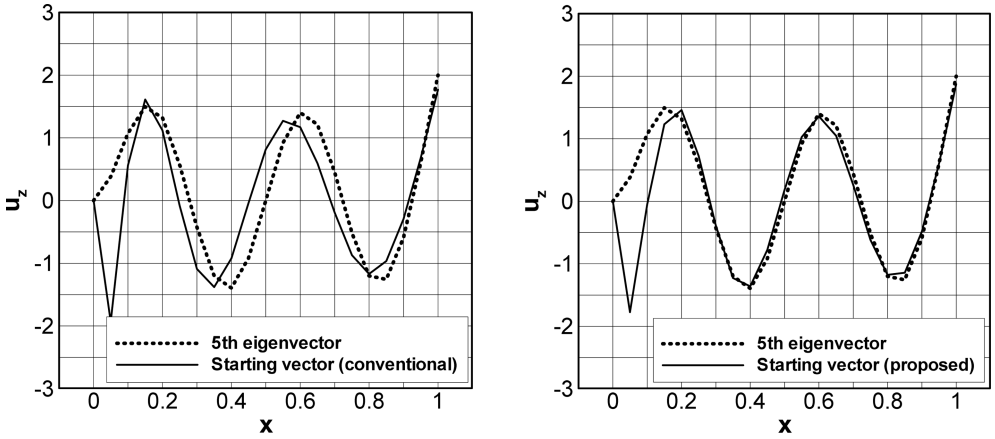


Fig. 5 Mode shapes of 5th eigenvector and starting iteration vectors of cantilever

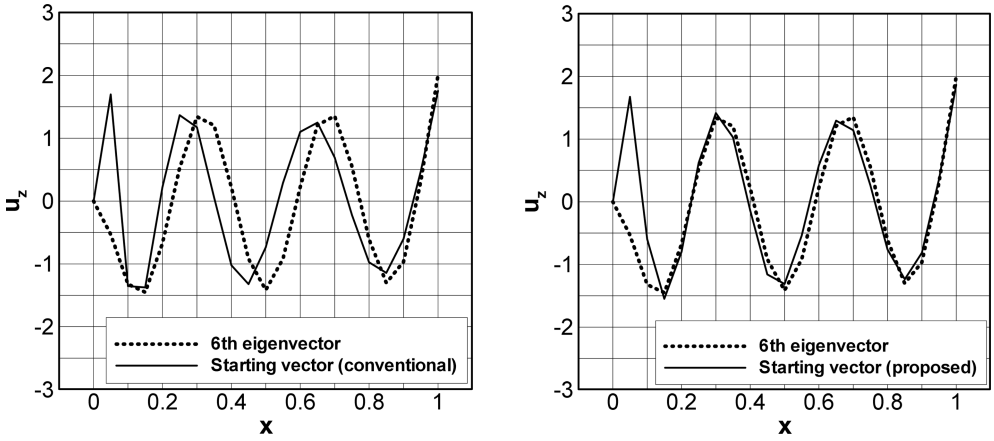


Fig. 6 Mode shapes of 6th eigenvector and starting iteration vectors of cantilever

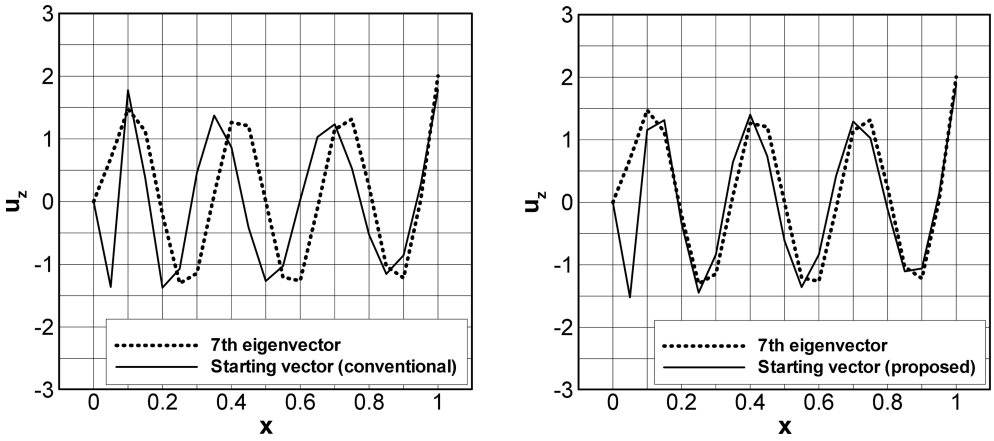


Fig. 7 Mode shapes of 7th eigenvector and starting iteration vectors of cantilever

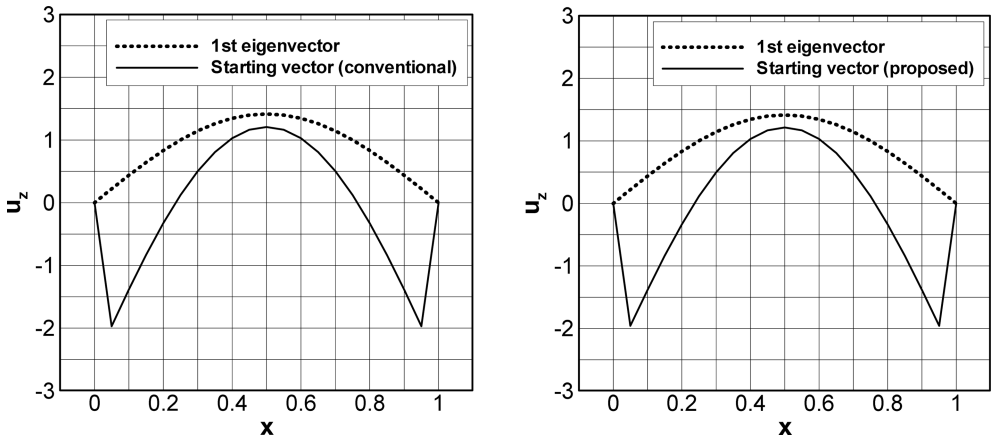


Fig. 8 Mode shapes of 1st eigenvector and starting iteration vectors of simple beam

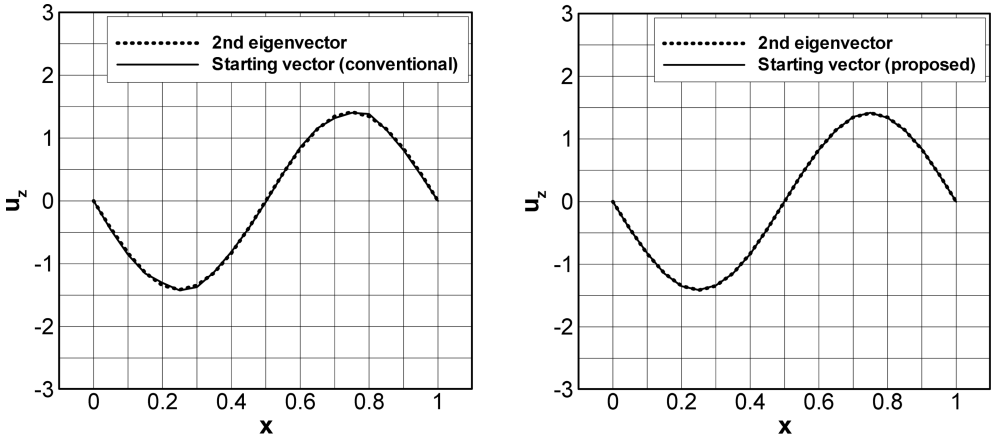


Fig. 9 Mode shapes of 2nd eigenvector and starting iteration vectors of simple beam

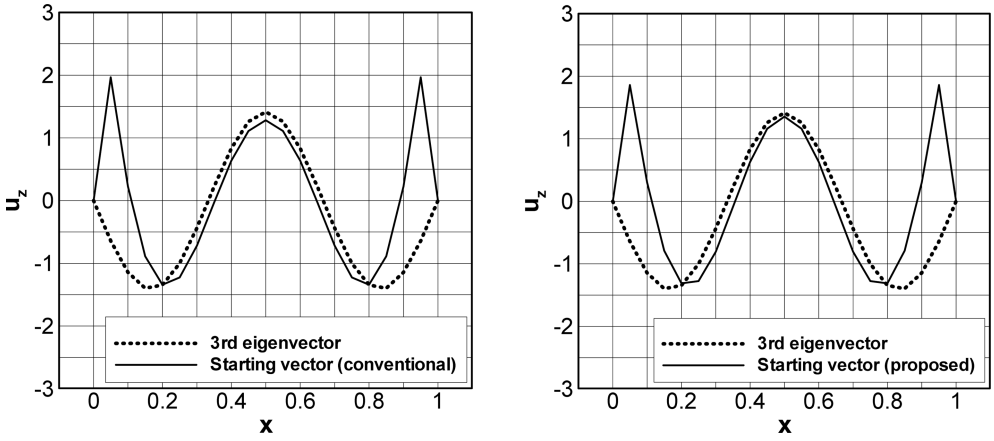


Fig. 10 Mode shapes of 3rd eigenvector and starting iteration vectors of simple beam

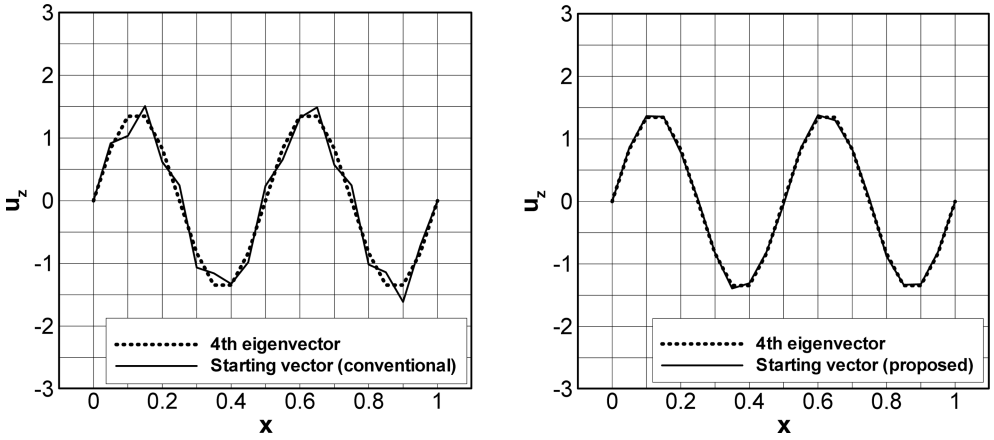


Fig. 11 Mode shapes of 4th eigenvector and starting iteration vectors of simple beam

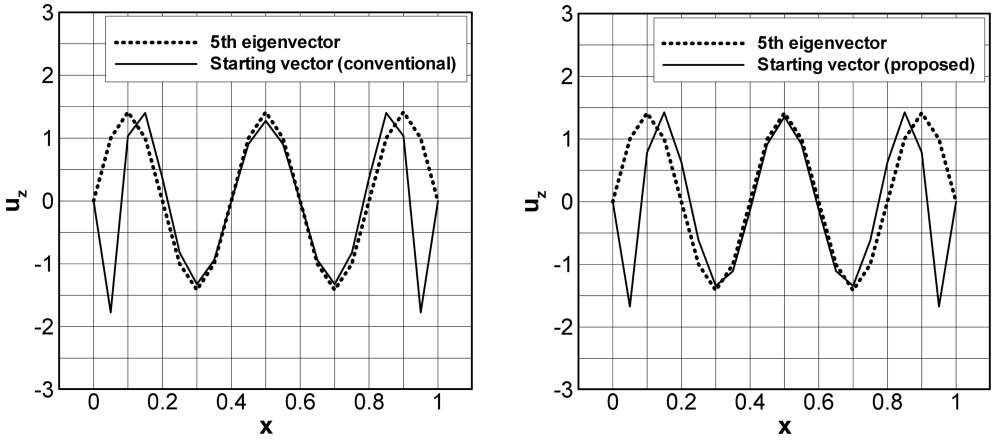


Fig. 12 Mode shapes of 5th eigenvector and starting iteration vectors of simple beam

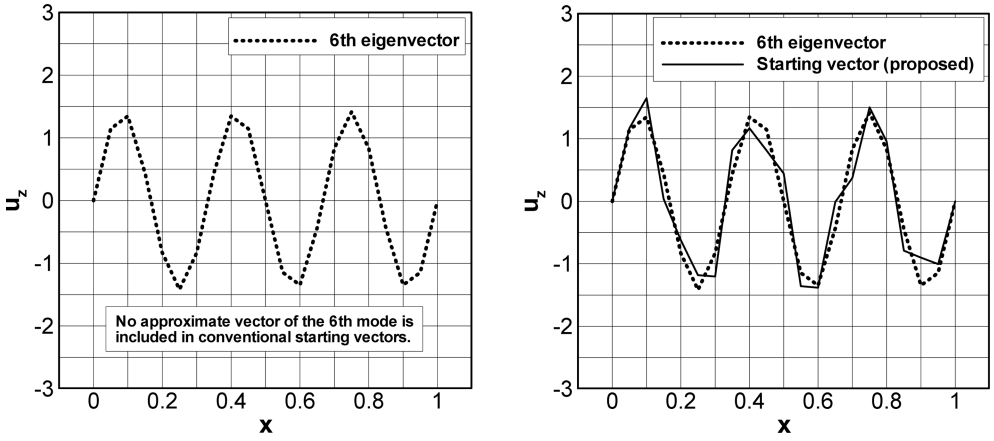


Fig. 13 Mode shapes of 6th eigenvector and starting iteration vectors of simple beam

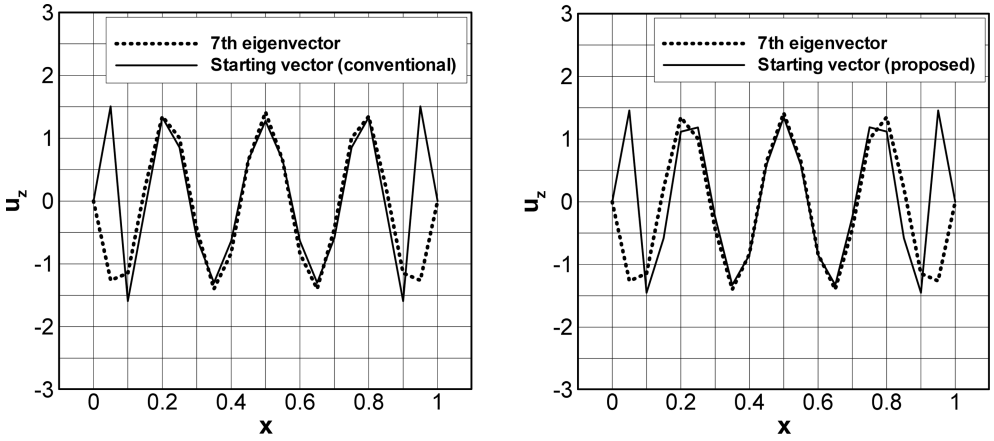


Fig. 14 Mode shapes of 7th eigenvector and starting iteration vectors of simple beam

than that of conventional method. Numbers of iterations of the conventional and proposed methods for getting seven modes are 3 and 1, respectively.

The second example is a simply supported beam of which length, section area, second moment of inertia, torsion constant and density are 1. p is 7 and q is 14. Figs. 8~14 are resulting mode shapes. Closeness of conventional and proposed starting vectors are similar in the case of the first, second, third, fifth and seventh modes. However, proposed starting vector is closer to eigenvector than conventional starting vector in the fourth mode. And, Fig. 13 shows that no approximation for the sixth mode is included in the fourteen conventional starting vectors. Number of iterations to get seven modes is 7 in the conventional method. While, number of iterations of the proposed method is only 2.

4. Numerical example

A numerical example is considered to show applicability of the proposed subspace iteration method. A very large floating structure (VLFS) is considered as an example structure. VLFS is a very innovative structure and many researchers are discussing its design and analysis methods (Pham *et al*. 2008, Riggs *et al*. 2008). This paper analyzes natural modes of a VLFS with hinge-linked breakwater which was designed as a floating airport by Korea Ocean Research & Development Institute (2008). And, the number of iterations and computing time are examined with increasing the number of required modes. Fig. 15 shows geometry of the example structure. Finite plate elements are used to establish a numerical discrete model. Number of nodes is 8466 and total degrees of freedom are 25398. Fig. 16 shows the mesh system. Main particulars of example

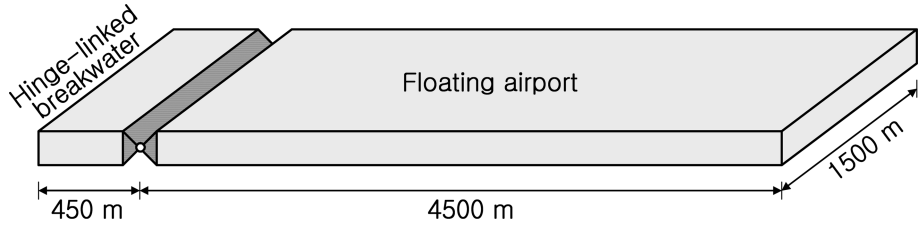


Fig. 15 Geometry of example structure

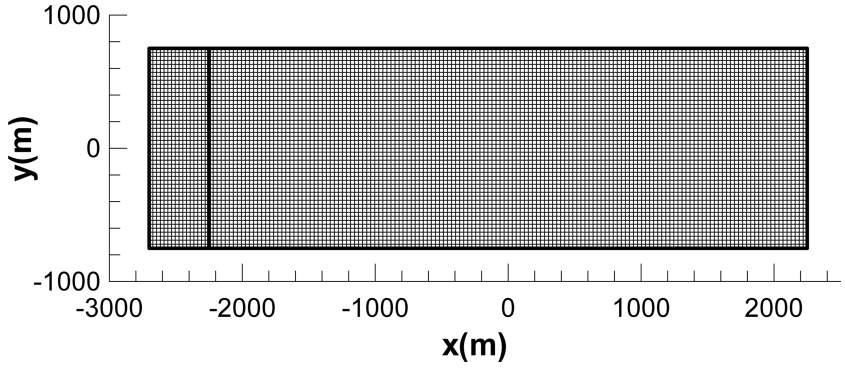
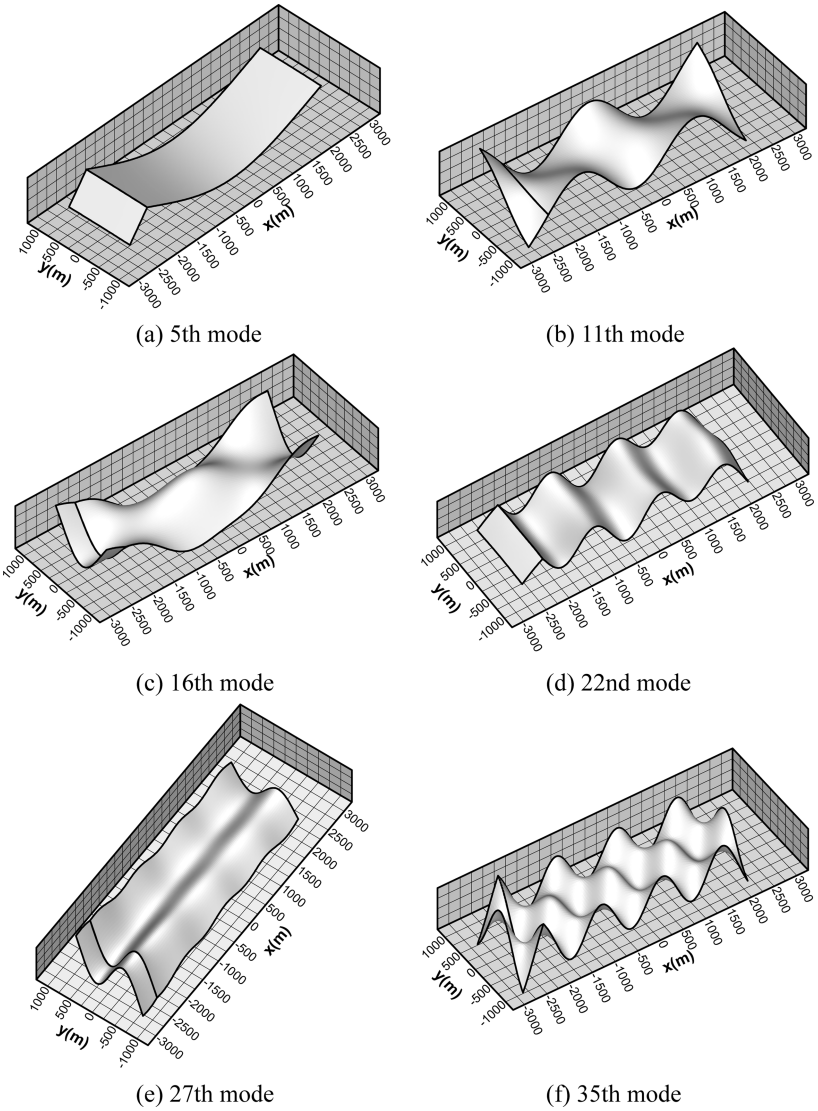


Fig. 16 Finite element mesh of example structure

Table 1 Main particulars of example structure

Basic dimensions	Breadth (B)	1500 m
	Length (L)	450 m + 4500 m
	Draft (d)	2 m
Material properties	Young's modulus (E)	2.1×10^{11} Pa
	Bending stiffness per unit length (I/B)	0.924 m^3
	Poisson ratio	0.3
Finite element model	Number of nodes	8466
	Number of elements	8200
	Total degrees of freedom	25398
	Half-band width of system matrix	158



(a) 5th mode (b) 11th mode

(c) 16th mode (d) 22nd mode

(e) 27th mode (f) 35th mode

Fig. 17 Mode shapes of example structure

structure are summarized in Table 1. Hinge-linked nodes are imposed to special constraint conditions. If two nodes are linked with hinge and equation numbers for the vertical displacement component of the nodes are i and j, the stiffness matrix is modified as (14) by the penalty method.

$$\begin{aligned}
\mathbf{K}(i,i)_{new} &= \mathbf{K}(i,i) + \lambda \\
\mathbf{K}(i,j)_{new} &= \mathbf{K}(i,j) - \lambda \\
\mathbf{K}(j,i)_{new} &= \mathbf{K}(j,i) - \lambda \\
\mathbf{K}(j,j)_{new} &= \mathbf{K}(j,j) + \lambda
\end{aligned} \tag{14}$$

Where λ is a penalty number such as $10^{10} \times max(diag(\mathbf{K}))$.

Results are shown in Fig. 17, Table 2 and Fig. 18. Fig. 17 shows some selected mode shape of example structure. Table 2 summarizes the number of iterations for obtaining required modes and

Table 2 Number of iterations for calculating natural modes of example structure

Number of required modes	Number of iterations	
	Conventional	Proposed
5	56	53
6	41	37
7	32	27
8	28	22
9	21	17
10	18	14
11	16	12
12	14	10
13	14	9
14	11	8
15	9	8
16	9	8
17	8	4
18	8	2
19	7	1
20	7	1
21	7	1
22	5	1
23	3	1
24	5	1
25	4	2
26	9	2
27	6	2
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33	2	2
34	2	2
35	2	2

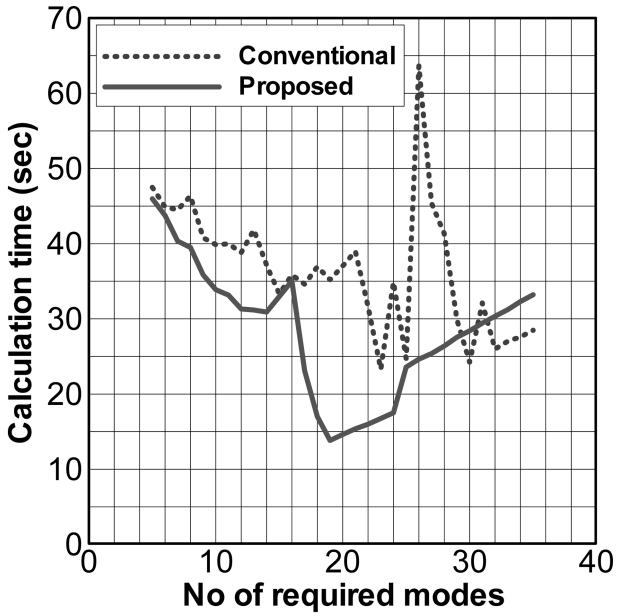

Fig. 18 Computing time for calculating natural modes of example structure

Fig. 18 represents computing time spent on calculating required modes. In general, larger number of modes requires more computing time. However, sometimes, many starting vectors have the chances of including good guesses of eigenvectors when the number of required modes is large. In that case, the number of iterations is small resulting in less computing time. Such trend is also observed in this example.

As shown in Table 2 and Fig. 18, proposed method has smaller number of iterations and less computing time than the conventional method when the number of required modes is generally small. When the number of required modes is large, conventional starting vectors are already good approximations of eigenvectors because the size of starting subspace is sufficiently large. In such cases, accelerated starting vectors have no benefits. However, in practical analysis and design, a few lower dominant modes are important. In that sense, the proposed method has a practical applicability.

5. Conclusions

In this study, the efficiency and applicability of the subspace iteration method with accelerated Lanczos starting vectors are investigated. By examining starting vectors' closeness to eigenvectors, the efficiency of the proposed method is validated. By analyzing a structure with larger DOFs and more complex constraints, the applicability of the proposed method is verified. From numerical analyses, conclusions are summarized as follows.

Since proposed starting vectors are generated from the modified algorithm which uses squared dynamic matrix, starting vectors of the proposed method are closer to exact eigenvectors than those of the conventional method. Therefore, the proposed method has more rapid convergence.

Numerical results of an example structure shows that the proposed method has smaller number of iterations and less computing time than the conventional method when the lower dominant modes are calculated.

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