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New twelve node serendipity quadrilateral plate bending element based on Mindlin-Reissner theory using Integrated Force Method

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Abstract. The Integrated Force Method (IFM) is a novel matrix formulation developed for analyzing the civil, mechanical and aerospace engineering structures. In this method all independent/internal forces are treated as unknown variables which are calculated by simultaneously imposing equations of equilibrium and compatibility conditions. This paper presents a new 12-node serendipity quadrilateral plate bending element MQP12 for the analysis of thin and thick plate problems using IFM. The Mindlin-Reissner plate theory has been employed in the formulation which accounts the effect of shear deformation. The performance of this new element with respect to accuracy and convergence is studied by analyzing many standard benchmark plate bending problems. The results of the new element MQP12 are compared with those of displacement-based 12-node plate bending elements available in the literature. The results are also compared with exact solutions. The new element MQP12 is free from shear locking and performs excellent for both thin and moderately thick plate bending situations.

Keywords: displacement fields; stress-resultant fields; Mindlin-Reissner plate theory; Integrated Force Method.

1. Introduction

The Mindlin-Reissner theory based plate bending elements consider C_0 continuity and avoid C_1 continuity which is rather difficult to adopt for higher order finite elements. Quite a good number of 8-node and lower order quadrilateral plate bending elements are available in the literature. Few of them are (Choi and Park 1999, Choi *et al.* 2002, Kim and Choi 2005, Kanber and Bozkurt 2006,

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Ozgan and Daloglu 2007, Pian 1964, Tong 1970, Lee *et al.* 1982, Pian *et al.* 1982, Chen and Cheung 1987, Dimitris *et al.* 1984, Darılmaz 2005, Darılmaz and Kumbasar 2006, Spilker 1982, Dhananjaya *et al.* 2007, 2009). Though considerable amount of research work has been carried out over the past few decades by research engineers and scientists on the Mindlin-Reissner theory based plate bending elements, a very few higher order (12 node and above) quadrilateral plate bending elements have been developed in displacement-based finite element method and it is almost void in hybrid or mixed finite element method. Higher order (12 node and above) quadrilateral plate bending elements give results with good accuracy even for the coarse mesh size of the order 3×3 and also generally avoids shear locking problem. In this paper IFM has been used to develop the Mindlin-Reissner theory based 12-node quadrilateral plate bending element which considers C_0 continuity and effect of shear deformation.

A new novel matrix formulation of the classical force method of analysis termed "Integrated Force Method (IFM)" has been developed (Patnaik 1973) for analyzing civil, mechanical and aerospace engineering structures. In this method, all independent/internal forces are treated as unknown variables which are computed by simultaneously imposing equations of equilibrium and compatibility conditions. Unlike classical force method of analysis, the IFM is independent of redundants and the basic determinate structure. It requires explicit generation of compatibility conditions for skeletal as well as continuum structures. The advantages of IFM compare to displacement-based finite element method are reported in the reference (Patnaik *et al.* 1991).

The applications of IFM on various areas of structural engineering field are briefly summarized in reference (Dhananjaya *et al.* 2007): Generation of compatibility conditions for elasticity and discrete models have been reported by Patnaik *et al.* (2000). Nagabhushanam and Patnaik (1990) have developed a general purpose program to generate compatibility matrix for the IFM. Automatic generation of sparse and banded compatibility matrix for the Integrated Force Method has been reported by Nagabhushanam and Srinivas (1991). IFM has been successfully implemented for analyzing, plane stress problems (Nagabhushanam and Srinivas 1991), two/three dimensional problems (Kaljevic *et al.* 1996a, b), dynamics (Patnaik and Yadagiri 1976), optimization (Patnaik *et al.* 1986) and non-linear problems (Krishnam Raju and Nagabhushanam 2000). A 4-node rectangular plate bending element based on the Kirchhoff theory has been formulated using the IFM (Patnaik *et al.* 1991). The element considers a transverse displacement and two rotations as degrees of freedom at each node. The performance of this element was compared with those obtained by force method (Przemieniecki 1968, Robinson 1973). Dhananjaya *et al.* (2007), developed a 4-node bilinear plate bending element based on the Mindlin-Reissner theory using IFM. The results of this element were compared with those of similar displacement- based 4-node quadrilateral plate bending elements available literature. Dhananjaya *et al.* (2009), proposed a new 8-node quadrilateral plate bending element for the analysis of thin and moderately thick plates using IFM and compared the results with those obtained from similar displacement based quadrilateral plate bending finite elements.

In this paper, a new 12-node serendipity quadrilateral plate bending element MQP12 has been presented by assuming suitable stress-resultants and displacement fields for analysis of thin and moderately thick plate bending problems using Integrated Force Method. Mindlin-Reissner theory has been employed in the formulation which accounts the effect of shear deformation. The shear correction factor as suggested by Reissner (1945) has been considered in the formulation. Many standard plate bending benchmark problems are analyzed to test the accuracy and convergence of the element presented. The results obtained by this element are compared with those of similar displacement-based 12-node quadrilateral plate bending elements available in the literature. Results are also compared with the exact solutions. Numerical results indicate that proposed element MQP12 is free from spurious/zero energy modes and shear locking problem. The proposed element MQP12 has produced, in general, excellent results in the numerical problems considered.

2. Formulation of element equilibrium and flexibility matrices

In this section brief formulation on the development of equilibrium and flexibility matrices of plate bending element is described. The Mindlin-Reissner theory has been employed in the formulation. In the Mindlin-Reissner theory, a line that is straight and normal to mid-surface of the un-deformed plate remain straight but not necessarily normal to the mid-surface of the deformed plate. This leads to the following definition of the displacement components u, v, w in the x, y, z Cartesian coordinates system

$$u = -z\theta_x(x,y); \quad v = -z\theta_y(x,y); \quad w = w(x,y) \tag{1}$$

where

x, y are coordinates in the reference mid-surface

z is the coordinate through the thickness of the plate t with $-t/2 \le z \le t/2$

w is the transverse (lateral) displacement

θ_x, θ_y represent the rotations of the normal in x-z and y-z planes respectively

Engineering strains for the Mindlin-Reissner plate theory can be written as

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} = -\begin{Bmatrix} z\dfrac{\partial\theta_x}{\partial x} \\ z\dfrac{\partial\theta_y}{\partial y} \\ z\left(\dfrac{\partial\theta_x}{\partial y} + \dfrac{\partial\theta_y}{\partial x}\right) \\ \theta_y - \dfrac{\partial w}{\partial y} \\ \theta_x - \dfrac{\partial w}{\partial x} \end{Bmatrix} \tag{2}$$

The stress-strain relations for an isotropic two-dimensional plate material is given by

$$\{\sigma\} = [C_{con}]\{\varepsilon\} \tag{3}$$

where $\{\sigma\} = [\sigma_x \;\; \sigma_y \;\; \tau_{xy} \;\; \tau_{yz} \;\; \tau_{xz}]^T$ = vector of stress components

$\{\varepsilon\} = [\varepsilon_x \;\; \varepsilon_y \;\; \gamma_{xy} \;\; \gamma_{yz} \;\; \gamma_{zx}]^T$ = vector of strain components

$$[C_{con}] = \text{constitutive matrix} = \frac{E}{(1-\nu^2)}\begin{bmatrix} 1 & \nu & 0 & 0 & 0 \\ \nu & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{(1-\nu)}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{(1-\nu)}{2} & 0 \\ 0 & 0 & 0 & 0 & \frac{(1-\nu)}{2} \end{bmatrix}$$

E = Young's modulus
ν = Poisson's ratio

The stress-resultants for plates can be written as

$$\begin{aligned} M_x &= \int_{-t/2}^{t/2} z\sigma_x dz \\ M_y &= \int_{-t/2}^{t/2} z\sigma_y dz \\ M_{xy} &= \int_{-t/2}^{t/2} z\tau_{xy} dz \\ Q_y &= \int_{-t/2}^{t/2} \tau_{yz} dz \\ Q_x &= \int_{-t/2}^{t/2} \tau_{xz} dz \end{aligned} \tag{4}$$

Eqs. (2), (3) and (4) yield the moment-curvature relations as

$$\{M\} = [C_1]\{k\} \tag{5}$$

Where $\{M\}$ = vector of stress-resultants
$= [M_x \;\; M_y \;\; M_{xy} \;\; Q_y \;\; Q_x]^T$
$[C_1]$ = matrix relating stress-resultants to curvatures
$\{k\}$ = vector of curvatures

$$= \left[\frac{\partial\theta_x}{\partial x} \;\; \frac{\partial\theta_y}{\partial y} \;\; \frac{\partial\theta_x}{\partial y}+\frac{\partial\theta_y}{\partial x} \;\; \theta_y-\frac{\partial w}{\partial y} \;\; \theta_x-\frac{\partial w}{\partial x}\right]^T$$

From the Eq. (5), the curvature-moment relations can be written as

$$\{k\} = [C_1]^{-1}\{M\} = [H]\{M\} \tag{6}$$

where $[H] = [C_1]^{-1}$
= matrix relating curvatures to stress-resultants

The matrix $[H]$ for the Mindlin-Reissner plate with Reissner's shear correction factor (Reissner 1945) of 5/6 can be written as

$$[H] = \frac{1}{D_1}\begin{bmatrix} 1 & -\nu & 0 & 0 & 0 \\ -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & \frac{t^2(1+\nu)}{5} & 0 \\ 0 & 0 & 0 & 0 & \frac{t^2(1+\nu)}{5} \end{bmatrix} \tag{7}$$

where $D_1 = Et^3/12$

The strain energy U_p of the elastic plate in bending and shear is written as

$$U_p = \iint \frac{1}{2}\left[M_x\frac{\partial\theta_x}{\partial x} + M_y\frac{\partial\theta_y}{\partial y} + M_{xy}\left(\frac{\partial\theta_x}{\partial y} + \frac{\partial\theta_y}{\partial x}\right) + Q_y\left(\theta_y - \frac{\partial w}{\partial y}\right) + Q_x\left(\theta_x - \frac{\partial w}{\partial x}\right)\right]dxdy \tag{8}$$

The vectors $\{M\}$ and $\{k\}$ for a discrete plate bending element can be expressed in matrix notations in terms of assumed stress-resultants and displacement fields respectively as

$$\{M\} = [\psi]\{F_e\} \tag{9}$$

$$\{k\} = [D_{op}][\phi_1]\{\alpha\} = [D_{op}][\phi]\{X_e\} \tag{10}$$

where

$[\psi]$ = matrix of polynomial terms for stress-resultant fields
$\{F_e\}$ = vector of force components of the discrete element
$[\phi_1]$ = matrix of polynomial terms for displacement fields
$[\phi] = [\phi_1][A]^{-1}$
$[A]$ = matrix formed by substituting the coordinates of the element nodes into the polynomial of displacement fields
$\{\alpha\}$ = coefficients of the displacement field polynomial
$\{X_e\}$ = vector of displacements of the discrete element

$$[D_{op}] = \text{differential operator matrix} = \begin{bmatrix} 0 & \frac{\partial}{\partial x} & 0 \\ 0 & 0 & \frac{\partial}{\partial y} \\ 0 & \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \\ -\frac{\partial}{\partial y} & 0 & 1 \\ -\frac{\partial}{\partial x} & 1 & 0 \end{bmatrix}$$

Substituting Eqs. (9) and (10) into the Eq. (8), the strain energy for the discrete element can be expressed as

$$U_p = \frac{1}{2}\{X_e\}^T[B_e]\{F_e\} \tag{11}$$

where $[B_e]$ represents the element equilibrium matrix and is given by

$$[B_e] = \iint[\phi]^T[D_{op}]^T[\psi]dxdy \tag{12}$$

The complementary strain energy for the elastic plate in bending and shear is expressed as

$$U_c = \iint\frac{1}{2}\frac{1}{D_1}\left[M_x^2 + M_y^2 - 2\nu M_x M_y + 2(1+\nu)M_{xy}^2 + Q_y^2\frac{t^2(1+\nu)}{5} + Q_x^2\frac{t^2(1+\nu)}{5}\right]dxdy$$

Using the Eq. (7), the complementary strain energy for the discrete element is written as

$$U_c = \frac{1}{2}\{F_e\}^T[G_e]\{F_e\} \tag{13}$$

where $[G_e]$ represents the element flexibility matrix and is given by

$$[G_e] = \iint[\psi]^T[H][\psi]dxdy \tag{14}$$

The Eqs. (12) and (14) are used to obtain element equilibrium matrix $[B_e]$ and element flexibility matrix $[G_e]$ respectively. These element matrices $[B_e]$ and $[G_e]$ of all elements are assembled to obtain the global equilibrium matrix $[B]$ and global flexibility matrix $[G]$ of the structure and they are used to setup the IFM governing equation to analyze the plate problems by IFM.

Displacement and stress resultant fields for MQP12

The typical 12-node quadrilateral plate bending element is shown in the Fig. 1. Three degrees of freedom namely a transverse displacement w and two rotations θ_x, θ_y are considered at each node of this element.

The assumed polynomials for displacement fields in Integrated Force Method should satisfy the

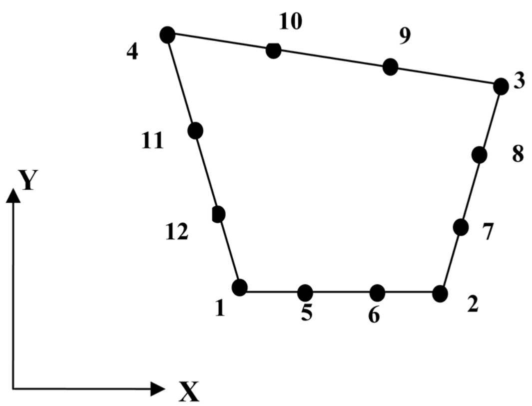


Fig. 1 A typical 12-node quadrilateral plate bending element

convergence requirements. Assumed displacement fields for w, θ_x and θ_y in terms of generalized displacement parameters $\alpha_1\ \alpha_2\ \dots\ \alpha_{36}$ are given in the Eq. (15) for this proposed 12-node quadrilateral element.

$$
\begin{aligned}
w &= \alpha_1+\alpha_2x+\alpha_3y+\alpha_4x^2+\alpha_5xy+\alpha_6y^2+\alpha_7x^3+\alpha_8x^2y+\alpha_9xy^2+\alpha_{10}y^3+\alpha_{11}x^3y+\alpha_{12}xy^3\\
\theta_x &= \alpha_{13}+\alpha_{14}x+\alpha_{15}y+\alpha_{16}x^2+\alpha_{17}xy+\alpha_{18}y^2+\alpha_{19}x^3+\alpha_{20}x^2y+\alpha_{21}xy^2+\alpha_{22}y^3+\alpha_{23}x^3y+\alpha_{24}xy^3\\
\theta_y &= \alpha_{25}+\alpha_{26}x+\alpha_{27}y+\alpha_{28}x^2+\alpha_{29}xy+\alpha_{30}y^2+\alpha_{31}x^3+\alpha_{32}x^2y+\alpha_{33}xy^2+\alpha_{34}y^3+\alpha_{35}x^3y+\alpha_{36}xy^3
\end{aligned}
\tag{15}
$$

Shape functions (N_i) in terms of natural coordinate system (ξ and η) for 12 node element using displacement fields as given in the Eq. (15) can be written as:

For the corner nodes (i =1, 2, 3, 4) : $N_i = 1/32\ (1+\xi\,\xi_i)(1+\eta\,\eta_i)[9(\xi^2+\eta^2)-10]$

where $\xi_i = -1, 1, 1, -1$ for $i = 1, 2, 3, 4$

$\eta_i = -1, -1, 1, 1$ for $i = 1, 2, 3, 4$

For the nodes i = 7, 8, 11, 12 : $N_i = 9/32\ (1+\xi\,\xi_i)(1+\eta\,\eta_i)\ (1-\eta^2)$, with $\xi_i = \pm 1$, $\eta_i = \pm 1/3$

For the nodes I = 5, 6, 9, 10 : $N_i = 9/32\ (1+\eta\,\eta_i)\ (1+\xi\,\xi_i)\ (1-\xi^2)$, with $\eta_i = \pm 1$, $\xi_i = \pm 1/3$

In the Integrated Force Method, the assumed stress-resultant fields must satisfy the equilibrium equations. Eq. (16) shows the assumed stress-resultant fields for this proposed 12-node quadrilateral plate bending element. in terms of polynomials with independent generalized force parameters $F_1\ F_2\ \dots\ F_{33}$. The stress-resultant fields for the shear forces Q_y and Q_x are obtained by considering plate equilibrium equations of the element.

$$
\begin{aligned}
M_x &= F_1+F_2x+F_3y+F_4x^2+F_5xy+F_6y^2+F_7x^3+F_8x^2y+F_9xy^2+F_{10}y^3+F_{11}x^3y+F_{12}xy^3\\
M_Y &= F_{13}+F_{14}x+F_{15}y+F_{16}x^2+F_{17}xy+F_{18}y^2+F_{19}x^3+F_{20}x^2y+F_{21}xy^2+F_{22}y^3+F_{23}x^3y+F_{24}xy^3\\
M_{xy} &= F_{25}+F_{26}x+F_{27}y+F_{28}x^2+F_{29}xy+F_{30}y^2+F_{31}x^2y+F_{32}xy^2+F_{34}y^3+F_{33}x^2y^2\\
Q_y &= (F_{15}+F_{26})+(F_{17}+2F_{28})x+(2F_{18}+F_{29})y+F_{20}x^2+2(F_{21}+F_{31})xy+(3F_{22}+F_{32})y^2\\
&\quad +F_{23}x^3+(3F_{24}+2F_{33})xy^2\\
Q_x &= (F_2+F_{27})+(2F_4+F_{29})x+(F_5+2F_{30})y+(3F_7+F_{31})x^2+2(F_8+F_{32})xy+F_9y^2\\
&\quad +(3F_{11}+2F_{33})x^2y+F_{12}y^3
\end{aligned}
\tag{16}
$$

The equilibrium and flexibility matrices of the element MQP12 are obtained by substituting Eqs. (15) and (16) into the Eqs. (12) and (14).

3. Numerical results and discussions

A square plate with simply supported/clamped boundary conditions, cantilever strip plate subjected to a point load/uniform load over the entire plate, the Morley's plate problem and the Razzaque's plate problem are considered. These problems are analyzed for deflections and moments considering various meshes using the proposed 12-node quadrileral plate bending element MQP12 via IFM. The performance of the proposed element MQP12 is examined with respect to accuracy and convergence comparing results with the exact solutions (Timoshenko1959, Liu *et al.* 2000). The results of MQP12 are also compared with those of a few existing 12-node displacement-based quadrilateral plate bending elements available in the reference (Spilker 1980) and in the commercial software (NISA, Version 9.3). The details of the example problems considered are given below.

1. A square thin plate (t/L = 0.01) with simply supported/clamped boundary conditions subjected to uniform load/central point load. The parameters of the problem are : size of the plate = 10 × 10, t = 0.1, $E = 10^7$, ν = 0.3, q = 10 and P = 100 (Spilker 1980)
2. A square moderately thick plate (t/L = 0.1) with simply supported boundary conditions subjected to uniform load. The parameters of the problem are : size of the plate = 10 × 10, t = 1, $E = 10^7$, ν = 0.3 and q = 10 (Spilker 1980)
3. A long cantilever beam (strip plate Fig. 2) subjected to point load at the tip or uniform load over the entire plate. The parameters of the problem are: L = 1000, B = 30, t = 5, $E = 2 \times 10^5$, ν = 0.0, P = 25 and q = 0.01. Here the Poisson's ratio is considered as zero to compare the results with the beam solution.
4. The Morley's plate problem (Fig. 3). Parameters of the problems are: L = 100, B = 100, t = 1, E = 1092000.0, ν = 0.3 and q = 1, inclination of the plate $\theta = 30^\circ$, w = 0 on all boundaries (Morley 1963)
5. The Razzaque's plate problem (Fig. 4). Parameters of the problems are: L = 100, B = 100, t = 1, E = 1092000.0, ν = 0.3 and q = 1, inclination of the plate $\theta = 60^\circ$ (Razzaque 1973)

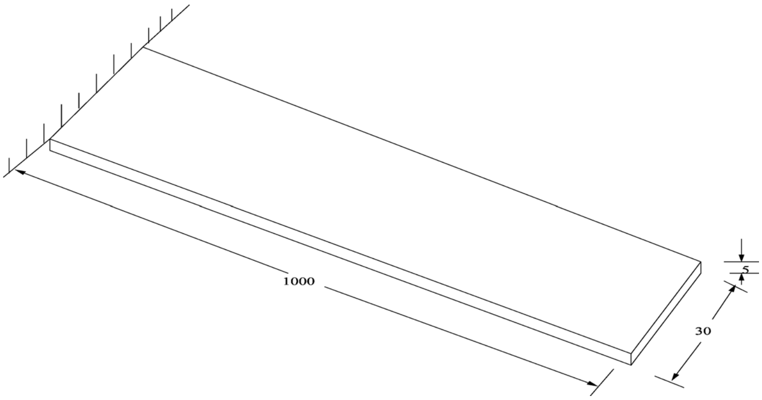


Fig. 2 Cantilever plate (strip plate): L = 1000, B = 30, t = 5, $E = 2 \times 10^5$, ν = 0.0, P = 25, q = 0.01

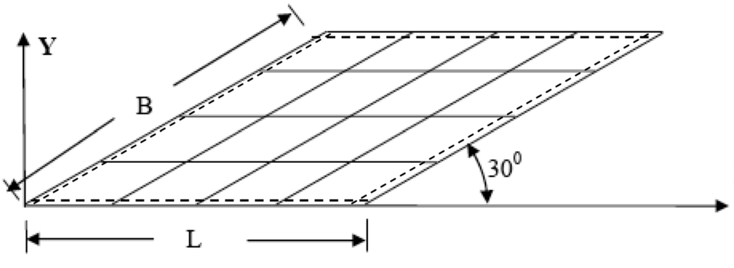


Fig. 3 Morley's plate: L = 100, B = 100, t = 1, E = 1092000.0, ν = 0.3 and q = 1, inclination of the plate $\theta = 30^\circ$, w = 0 on all sides of the plate

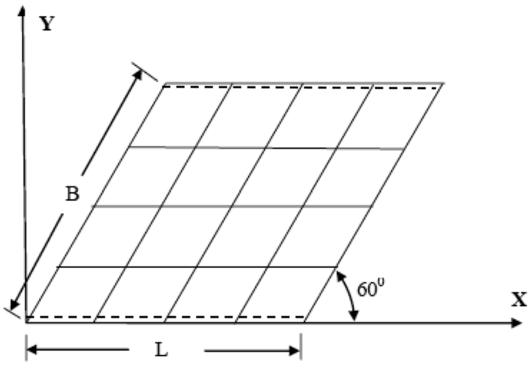
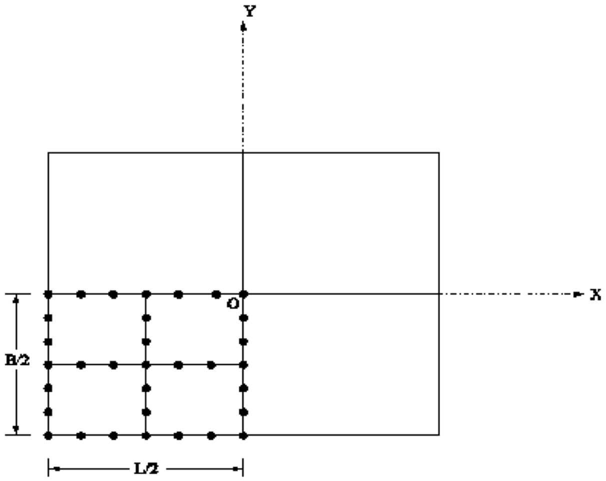


Fig. 4 Razzaque's plate: L = 100, B = 100, t = 1, E = 1092000.0, ν = 0.3 and q = 1, inclination of the plate θ= 60° (Two opposite edges simply supported while other two free)



b. A typical (2×2) mesh in one quadrant of the plate

Fig. 5 A typical (2×2) mesh in one one quadrant of the plate

Due to symmetry of the plate, loading and boundary conditions in the example problems 1 and 2, one quadrant of the plate is analyzed for central deflections and moments. The typical mesh (2×2) in one quadrant of the plate is as shown in the Fig. 5.

Displacements and moments are computed using the proposed element MQP12 via IFM for the above example problems. Computed results are compared with those obtained by displacement-based 12-node quadrilateral plate bending elements CH1 and CSDR available in the reference (Spilker 1980) for the example problems 1 and 2. For example problems 3, the computed results using MQP12 are compared with the results obtained from 12-node quadrilateral plate bending element available in the commercial software (NISA, version 9.3).

Exact displacements and moments for the plates with various boundary conditions and loadings are obtained from the reference (Timoshenko 1959) for the example problems 1. The formulae given in the reference (Liu *et al.* 2000) are used to obtain exact displacements and moments for the example problems 2. The beam solution is used to obtain exact displacements and moments for example problems 3. Central displacements and moments obtained for example problems 1 and 2 are normalized respectively with respect to exact solutions of thin and moderately thick plate bending theories.

The normalized central deflections and moments for various mesh sizes of simply supported square thin plate (t/L = 0.01) with uniform load are summarized in Tables 1 and 2 respectively and corresponding convergence trends are shown in Figs. 6 and 7 respectively. Table 1 shows that the central deflections estimated by the proposed element MQP12 is extremely close to the exact solution (percentage of error = 0.09) for the grid size (3×3). It also shows that the central deflections of MQP12 and CH1 are almost close to each other and are superior to the element CSDR. Table 2

Table 1 Normalized central deflection for a simply supported square thin plate with uniform load

W_c (t/L = 0.01, Example Problem 1)			
Elements	CH1	CSDR	MQP12
1×1	0.935	0.750	0.935
2×2	0.997	0.975	0.997
3×3	0.999	0.985	0.999

Table 2 Normalized central moment for a simply supported square thin plate with uniform load

M_c (t/L = 0.01, Example Problem 1)			
Elements	CH1	CSDR	MQP12
1×1	0.580	1.620	0.740
2×2	0.985	1.025	0.990
3×3	0.999	1.010	0.999

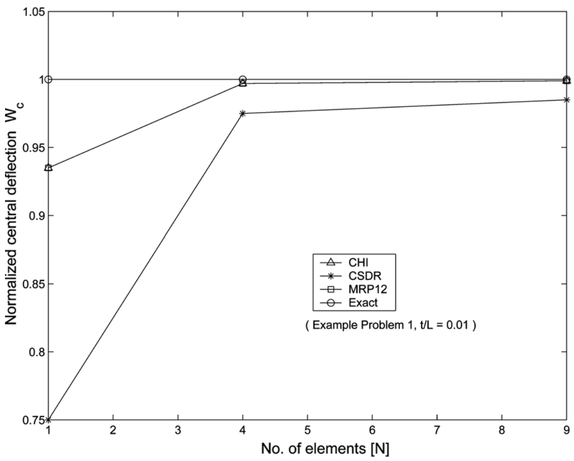


Fig. 6 Normalized central deflection for a simply supported square thin plate with uniform load

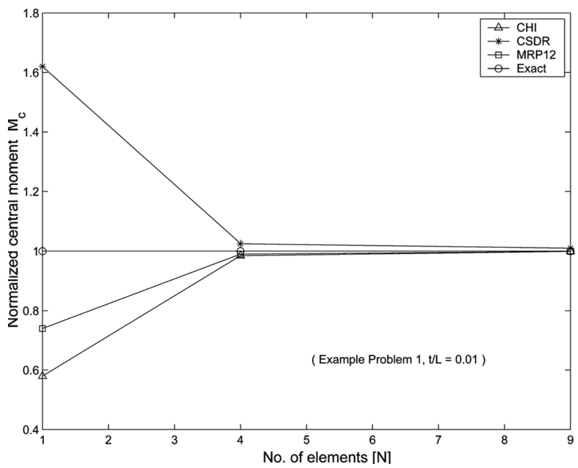


Fig. 7 Normalized central moment for a simply supported square thin plate with uniform load

shows that central moments estimated by MQP12 along with those obtained with elements CH1 and CSDR. The results of MQP12 and CH1 are almost same each other for the grid size (3×3) and are superior to those of element CSDR. The percentage of error of the element MQP12 for the grid size (3×3) is 0.1 with respect to the exact solution. Central deflections for the case of clamped thin plate ($t/L = 0.01$) with uniform load are shown in the Table 3. Fig. 8 shows the corresponding converging trends. It is observed that estimations of central deflections by the element MQP12 are better than those with elements CH1 and CSDR. The MQP12 has estimated the central deflection equal to the exact value for the grid size (3×3).

Table 4 presents central deflections for simply supported square thin plate (t/L = 0.01) subjected to the central point load and the corresponding converging trends are shown in the Fig. 9. It shows that elements MQP12 and CH1 have estimated almost identical deflections for all grid sizes and

Table 3 Normalized central deflection for a clamped square thin plate with uniform load

W_c (t/L = 0.01, Example Problem 1)			
Elements	CH1	CSDR	MQP12
1×1	0.960	0.300	0.975
2×2	0.970	0.650	0.980
3×3	0.980	0.920	1.000

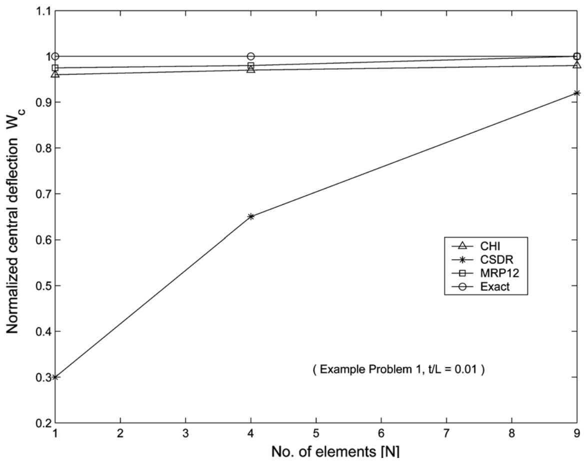


Fig. 8 Normalized central deflection for a clamped square thin plate with uniform load

Table 4 Normalized central deflection for a simply supported square thin plate with point load

W_c (t/L = 0.01, Example Problem 1)			
Elements	CH1	CSDR	MQP12
1×1	0.925	0.820	0.925
2×2	0.990	0.940	0.990
3×3	0.996	0.982	0.996

these deflections are superior to those of the element CSDR. The percentage of deflection error of the element MQP12 for the grid size (3×3) is 0.0047 with respect to the exact solution.

Normalized central deflections of the clamped square thin plate ($t/L = 0.01$) subjected to the central point load are summarized in the Table 5. Fig. 10 presents the corresponding converging

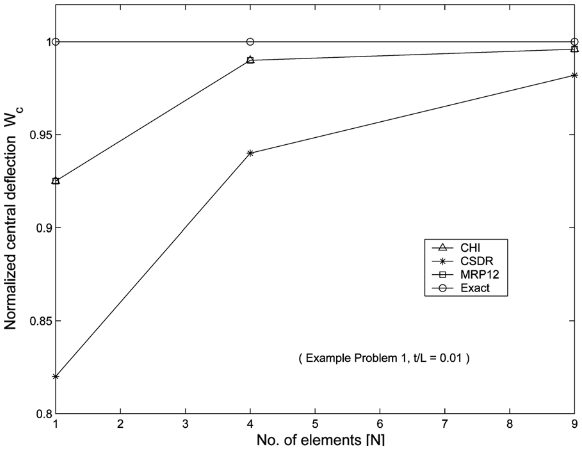


Fig. 9 Normalized central deflection for a simply supported square thin plate with point load

Table 5 Normalized central deflection for a clamped square thin plate with central point load

W_c (t/L = 0.01, Example Problem 1)			
Elements	CH1	CSDR	MQP12
1×1	0.960	0.200	0.970
2×2	0.970	0.670	0.980
3×3	0.990	0.920	0.999

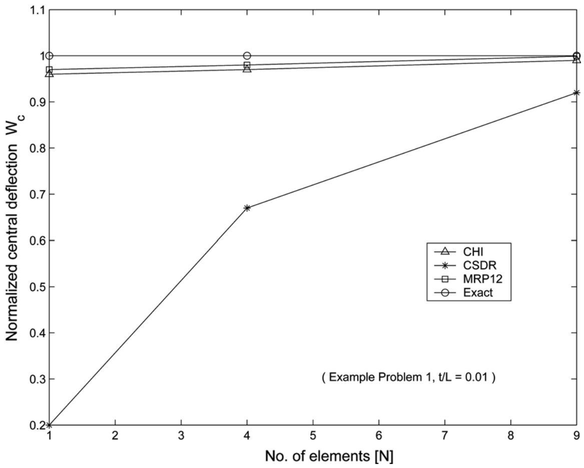


Fig. 10 Normalized central deflection for a clamped square thin plate with central point load

trends. It shows that the predicted central deflection by the element MQP12 are superior to those obtained by elements CH1 and CSDR. The percentage of deflection error of the MQP12 for the grid size (3×3) is 0.0023 with respect to the exact solution.

Concerning the studies on moderately thick plates, Figs. 11 and 12 present the converging trends of central deflections and moments respectively for a simply supported square moderately thick plate ($t/L = 0.1$) with uniform load and their numeric values are summarized in Tables 6 and 7. Central deflections estimated by the elements MQP12 and CH1 are almost identical and are superior to those of the element CSDR. The central deflection estimated by MQP12 for the grid size (3×3) is reached to the exact solution. The predicted normalized central moments of the element

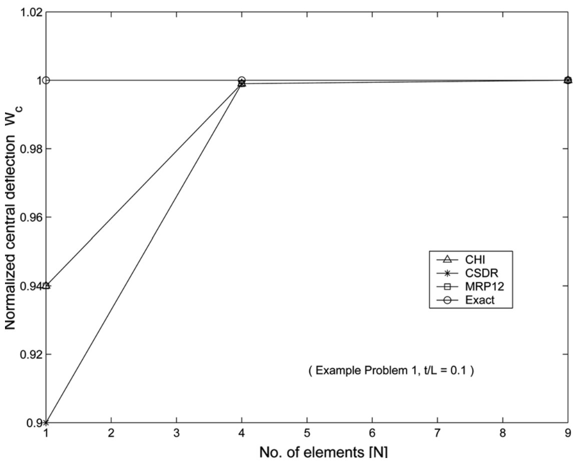


Fig. 11 Normalized central deflection for a simply supported moderately thick square thin plate with uniform load

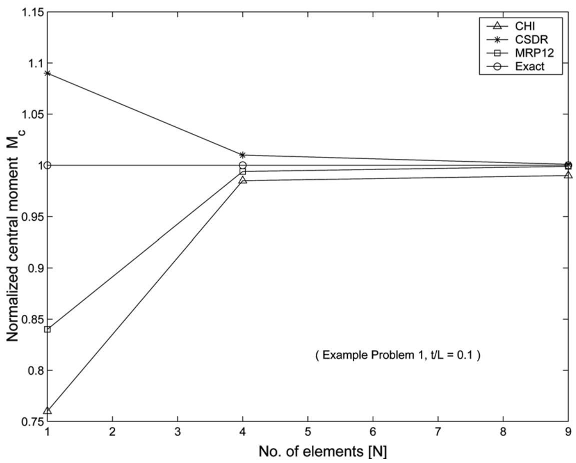


Fig. 12 Normalized central moment for a simply supported moderately thick square plate with uniform load

Table 6 Normalized central deflection for a simply supported moderately thick square thin plate with uniform load

W_c (t/L = 0.1, Example Problem 2)			
Elements	CH1	CSDR	MQP12
1×1	0.940	0.900	0.940
2×2	0.999	0.999	0.999
3×3	1.000	1.000	1.000

Table 7 Normalized central moment for a simply supported moderately thick square plate with uniform load

W_c (t/L = 0.1, Example Problem 2)			
Elements	CH1	CSDR	MQP12
1×1	0.760	1.090	0.840
2×2	0.985	1.010	0.994
3×3	0.990	1.001	0.999

MQP12 are better than those of elements CH1 and CSDR for all grid sizes. The percentage error in estimated central moment of the proposed element MQP12 for the grid size (3×3) is 0.104.

The variations of the moments along the central line of the simply supported thin ($t/L = 0.01$) and moderately thick ($t/L = 0.1$) plates with uniform load (the example problems 1 and 2) are estimated by the proposed element MQP12 for the mesh size (3×3). These values are summarized in the Table 8 along with exact values. It can be seen in the Table 8 that the estimated values are much closer to the exact values at all the points. These moment values of both thin ($t/L = 0.01$) and moderately thick ($t/L = 0.1$) plates are plotted in the Fig. 13.

The cantilever beam (strip plate, example problem 3) subjected point load at tip/uniform load over the entire plate is analyzed for moments and deflections considering various mesh sizes (1×1, 2×1,

Table 8 Variation of moment M_x along central line of the simply supported square thin/thick plate with uniform load for grid size (3×3)

M (t/L = 0.01, 0.1 Example Problem 1 and 2)			
Length of the plate	Moment in thick plate	Moment in thin plate	Exact Values
0.0 ($L = 0$)	0.142	0.148	0.0
0.5556	12.75	12.75	12.69
1.1111	22.72	22.72	22.73
1.6667	30.63	30.63	30.55
2.2222	36.45	36.45	36.51
2.7778	40.96	40.97	40.95
3.3333	44.07	44.07	44.14
3.8889	46.48	46.28	46.27
4.4444	47.51	47.51	47.49
5.0(L/2)	47.79	47.78	47.89

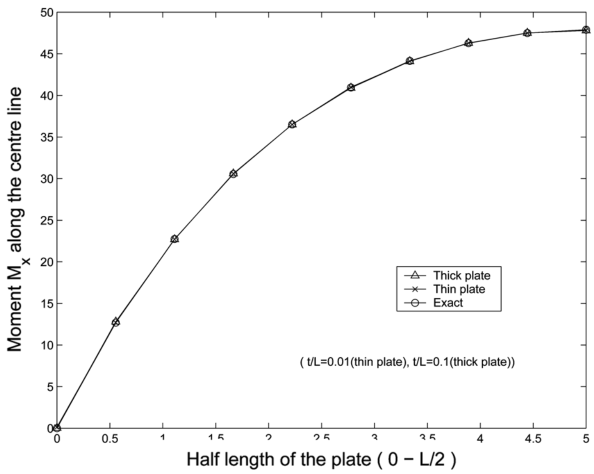


Fig. 13 Variation of moment M_x along central line of the simply supported square thin/moderately thick plate with uniform load for grid size (3×3)

4×1, 8×1, 16×1 and 32×1) using the proposed element MQP12 via Integrated Force Method. To compare results with the beam solution, the Poisson's ratio is considered as zero in this example problem. Deflections at the free edge and moments at the clamped edge are estimated by the proposed element MQP12. These values of deflections and moments are compared with those computed using 12-node quadrilateral plate bending element (NISA12) available in the commercial software NISA (displacement-based FEM package). It is interesting to note that both the elements MQP12 and NISA12 have produced identical results in all the cases and for all grid sizes, and they are equal to the exact beam solutions. For tip load exact values of tip deflection and moment at the clamped edge are 133.33 and 833.33 respectively. Similarly for uniform load over the entire plate, exact values of tip deflection and moment at the clamped edge are 600.00 and 5000.00 respectively.

The Morley's plate (Example problem 4) and the Razzaque's plate (Example problem 5) are analyzed for central deflections considering various mesh sizes using the proposed element MQP12 via the Integrated Force Method. Central deflections of the Morley's plate and the Razzaque's plate are plotted in the Figs. 14 and 15 respectively along with the exact values. These Figs show that the estimated central deflections using the proposed element MQP12 are fast converging to the exact solutions.

The simply supported square plate with various thickness-span ratio (very thin: $t/L = 0.00001$, 0.0001, thin $t/L = 0.001$, 0.01 and moderately thick $t/L = 0.1$) subjected to uniform load is analyzed using the proposed element MQP12 for the grid size 3×3 in one quadrant of the plate to estimate the central deflections and moments. The parameters of the problem considered are: $L = 50$, $B = 50$, $t = 5$, 0.5, 0.05, 0.005, 0.0005, $E = 200000$, $\nu = 0.3$, $q = 1$. The exact central displacements and moments are calculated from the Kirchhoff theory (Timoshenko and Krieger 1959) and Mindlin theory (Liu *et al.* 2000) solutions for thin and moderately thick plate bending problems respectively. The results are shown in the Figs. 16 and 17. These Figures indicate that the proposed element MQP12 performs quite well for both thin and moderately thick plate bending problems.

In all the above example problems, the proposed new quadrilateral plate bending element MOP12 has consistently, in general, produced excellent results for both thin and moderately thick plate bending problems.

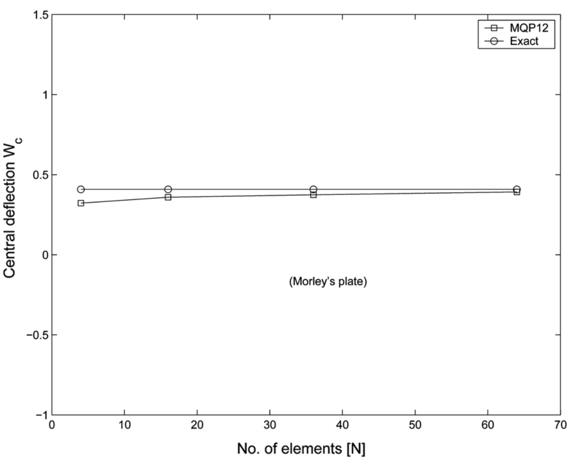


Fig. 14 Central deflection for Morley's plate with uniform load (Example problem 4)

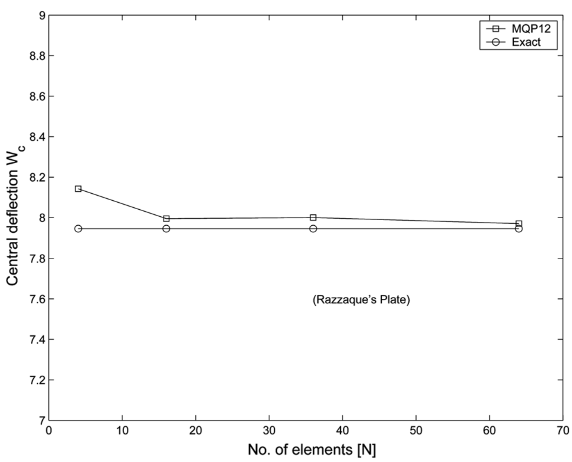


Fig. 15 Central deflection for Razzaque's plate with uniform load (Example Problem 5)

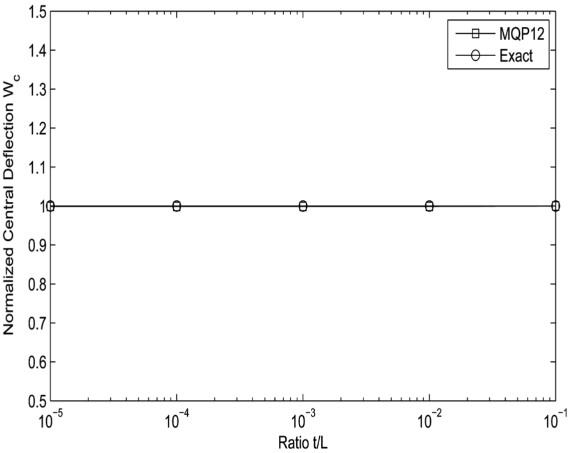


Fig. 16 Normalized central deflections for various thickness-span ratios (t/L = 0.00001, 0.0001, 0.001, 0.01 and 0.1)

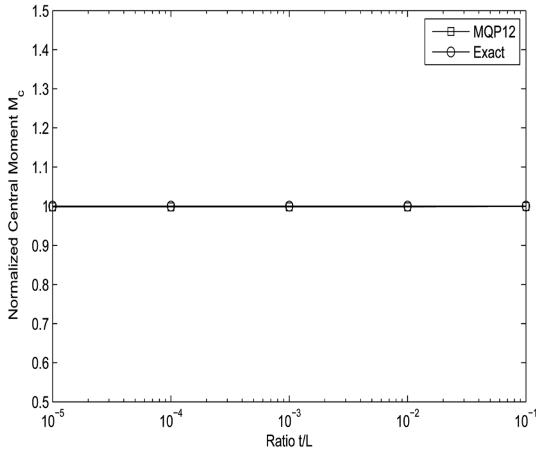


Fig. 17 Normalized central moments for various thickness-span ratios (t/L) (t/L = 0.00001, 0.0001, 0.001, 0.01 and 0.1)

4. Conclusions

New 12-node serendipity quadrilateral plate bending element (MQP12) based on the Mindlin-Reissner theory is presented for the analysis of thin and moderately thick plate bending problems using Integrated Force Method. Three degrees of freedom namely a transverse displacement w and two rotations θ_x, θ_y are considered at each node of element. The proposed element MQP12 is free from zero/spurious energy modes. Further the proposed element MQP12 is free from shear locking. The studies in all the example problems considered here, show that the proposed element MQP12 performs equally well for both thin and moderately thick plate bending situations and produced excellent results. Therefore the proposed lement MQP12 can be used to anaylze both thin and moderately thick plate bending problems. Also this proposed new quadrilateral plate bending element (MQP12) becomes an alternative element to analyze thin and thick plate bending problems compare to displacement based 12-node plate bending elements available in the literature.

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