Structural Engineering and Mechanics, Vol. 36, No. 5 (2010) 529-544
DOI: http://dx.doi.org/10.12989/sem.2010.36.5.529

Large strain analysis of two-dimensional frames by the normal flow algorithm

Tabatabaei, R.*[1] and Saffari, H.[2a]

[1]*Islamic Azad University, Kerman Branch, Civil Engineering Department, PO Box 76175-6114, Kerman, Iran*
[2]*Shahid Bahonar University, Civil Engineering Department, Kerman, Iran*

(Received August 25, 2009, Accepted July 16, 2010)

Abstract. Nonlinear equations of structures are generally solved numerically by the iterative solution of linear equations. However, this iterative procedure diverges when the tangent stiffness is ill-conditioned which occurs near limit points. In other words, a major challenge with simple iterative methods is failure caused by a singular or near singular Jacobian matrix. In this paper, using the Newton-Raphson algorithm based on Davidenko's equations, the iterations can traverse the limit point without difficulty. It is argued that the propose algorithm may be both more computationally efficient and more robust compared to the other algorithm when tracing path through severe nonlinearities such as those associated with structural collapse. Two frames are analyzed using the proposed algorithm and the results are compared with the previous methods. The ability of the proposed method, particularly for tracing the limit points, is demonstrated by those numerical examples.

Keywords: equilibrium path; limit point; snap-through; snap-back; Davidenko's flow.

1. Introduction

Nonlinear analysis is characterized by the non-proportional nature of load-deformation behaviour. It means the structural response against an incremental loading is affected by the instantaneous loading level and the deformed geometry of the structure. In fact, the stiffness matrix of the structure is a function of element force as well as the deflection of the structure and, therefore, for medium to huge size problems, the instantaneous stiffness equation can only be solved numerically by an incremental and iterative procedure allowing for the geometrical change of the structure. Based on the Newton-Raphson scheme, the applied load is first divided into many small increments, and the displacement increment within each increment is computed, using the tangent stiffness matrix. The numerical problem that may be encountered when tracing the nonlinear load-displacement curve is the ill-conditioning of the tangent stiffness matrix near the critical point. The critical points in nonlinear analysis may be classified into limit points with snap-through and limit points with snap-back (Ramm 1981, Pecknold *et al.* 1985). For limit points, the path-tracing scheme to successively compute the regular equilibrium points on the equilibrium path and the pinpointing

*Corresponding author, Assistant Professor, E-mail: tabatabaei@iauk.ac.ir
[a]Associate Professor

scheme to precisely locate the singular equilibrium points are sufficient for the computational stability analysis. For bifurcation points, however, a specific procedure for path-switching is also necessary to detect the branching paths to be traced in the post-buckling region. Fujii and Ramm have described these fundamental strategies as path-tracing, pin-pointing and path-switching in computational bifurcation theory (Fujii and Ramm 1997). There are analytical solutions for simple geometrical nonlinear trusses (Jagannathan *et al*. 1975). The researchers have considered a total Lagrangian formulation with elastic material properties on the geometric nonlinear analysis of space truss using Green-Lagrange representation of the axial strain (Kassimali 1983). The static analysis of elastic-plastic frames including inelastic material and large deformation geometric nonlinearities has been considered by the references (Kassimali and Abbassnia 1991, Levy and Spillers 2003). But in the structures with many degrees of freedom, generally there is no explicit solution for the nonlinear system of equations. Therefore, several numerical solutions have been presented for such equations by the researchers (Toklu 2004). The nonlinear analysis problem, formulated as an application of minimum potential energy principle, is obviously an optimization problem. For optimization of the total potential, there are various techniques that have been used such as simulated annealing algorithm (Baltoz and Dhatt 1979).

Some methods are weak for tracing through these points and are not able to present the real behavior of the structure. Among the methods for solving the nonlinear system of equations, the incremental iterative can be chosen, but it is time consuming and expensive in the case of nonlinear analysis of huge structures, and it diverges during tracing the limit points (Wempner 1971). These disadvantages exist because in the structures with complex behavior the load-displacement curve is a combination of softening and hardening states with limit points. Therefore, the analysis of such structures is not possible by simple incremental iterative methods. The simple incremental iterative methods are conducted in the form of load increments or displacements increments. As the load level is constant in the method of load increments, tracing the limit points is not possible. Also, if there are serious changes in the load-displacement path, the number of iterations for convergence will be increased and make this method time consuming and expensive. Similarly, in the method of incremental displacement tracing the limit points is difficult. For resolving such disadvantages, advanced analysis methods have been developed. In these methods, an auxiliary equation is needed for solving the equilibrium equation. There are several methods for setting up such auxiliary equation like methods by the references (Wempner 1971, Riks 1979, Mallardo and Alessandri 2004, Al-Rasby 1991, Crisfield 1981, 1983).

The advanced incremental iterative methods have been developed based upon the arc-length approach. In this approach, proportional to the load factor obtained in any iteration, the load level will converge to the equilibrium path. This process will be continued until the convergence is achieved with acceptable accuracy. In these methods, contrary to the simple incremental iterative methods, it is possible to pass the limit points. The impossibility of tracing the limit points in the case of highly nonlinear behavior state is one of the disadvantages of the advanced incremental iterative methods (Bashir-Ahmad and Xiao-zu 2004).

In this paper, for tracing the equilibrium path of the two-dimensional frames, the Newton-Raphson iterative algorithm is used along the normal path to the Davidenko's flows with modified convergence rate. A major challenge with Newton's iterative methods is failure caused by a singular or near singular tangent stiffness matrix. To circumvent this problem, in this algorithm, the Newton-Raphson iterative along the normal path to the Davidenko's flows is proposed for the solution of the structural nonlinear problems. Contrary to the previous methods, this algorithm uses the Homotopy approach which is based upon the new mathematical concepts, and has great ability for developing complex load-displacement paths of the structures with multiple degrees of freedom (Allgower and Georg 1980). The equations of presented algorithm are coded in library finite element software. Finally, the ability of the proposed method, particularly for tracing the limit points, is demonstrated by two numerical test cases.

2. Solving nonlinear problem using the Davidenko's equations

In general, the $n \times n$, nonlinear problem of a system can be written as follows

$$\begin{aligned} &f_1(x_1, x_2, x_3, ..., x_n) = 0 \\ &f_2(x_1, x_2, x_3, ..., x_n) = 0 \\ &\cdots\cdots\cdots\cdots \\ &f_n(x_1, x_2, x_3, ..., x_n) = 0 \end{aligned} \tag{1}$$

which can be summarized as

$$F(X) = 0 \tag{2}$$

where

$$\{F\} = \begin{bmatrix} f_1 \\ f_2 \\ . \\ . \\ . \\ f_n \end{bmatrix}, \qquad \{X\} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_n \end{bmatrix}; \qquad \{0\} = \begin{bmatrix} 0 \\ 0 \\ . \\ . \\ . \\ 0 \end{bmatrix} \tag{3}$$

If $f_1, f_2, \ldots, f_n$ are expanded in Taylor series in n dimensions (with respect to the iteration counter k)

$$\begin{aligned} &f_1(x_1^{k+1}, ..., x_n^{k+1}) = f_1(x_1^k, ..., x_n^k) + \frac{\partial f_1(x_1^k, ..., x_n^k)}{\partial x_1} \Delta x_1^k + \frac{\partial f_1(x_1^{k+1}, ..., x_n^{k+1})}{\partial x_2} \Delta x_2^k + ... \\ &\cdots\cdots\cdots \\ &f_n(x_1^{k+1}, ..., x_n^{k+1}) = f_n(x_1^k, ..., x_n^k) + \frac{\partial f_n(x_1^k, ..., x_n^k)}{\partial x_1} \Delta x_1^k + \frac{\partial f_n(x_1^{k+1}, ..., x_n^{k+1})}{\partial x_2} \Delta x_2^k + ... \end{aligned} \tag{4}$$

The equations are expressed according to the Newton's corrections $\Delta x_1^k, \ldots, \Delta x_n^k$ with truncation to the first order derivative terms. Since the right-hand side of Eq. (4) are zero, gives

$$\begin{aligned} &\frac{\partial f_1(x_1^k, .., x_n^k)}{\partial x_1} \Delta x_1^k + \frac{\partial f_1(x_1^{k+1}, .., x_n^{k+1})}{\partial x_2} \Delta x_2^k + ... = -f_1(x_1^k, .., x_n^k) \\ &\cdots\cdots\cdots \\ &\frac{\partial f_n(x_1^k, .., x_n^k)}{\partial x_1} \Delta x_1^k + \frac{\partial f_n(x_1^{k+1}, .., x_n^{k+1})}{\partial x_2} \Delta x_2^k + ... = -f_n(x_1^k, .., x_n^k) \end{aligned} \tag{5}$$

Eq. (5) can be generalized to the $n \times n$ case by writing it in matrix form

$$[J]\{\Delta X\} = \{-F\} \tag{6}$$

where $[J]$ is called the Jacobian matrix. Eq. (6) is Newton's method for a $n \times n$ system. The Jacobian matrix is not constant and therefore must be updated as the iterations proceed; updating at every iteration, may not be required since all that is required is convergence of the iterations. A major difficulty with Newton’s method, Eq. (6) is failure caused by a singular or near singular Jacobian matrix, $[J]$. To overcome this drawback, in this paper, the Davidenko’s equations are used based on a vector homotopy functions. Consider again the 2×2 systems of Eq. (1). Then a vector of homotopy functions can be expressed as follows

$$h_1(x_1, x_2, t) = f_1(x_1, x_2) - e^{-t} f_1(x_1^0, x_2^0) \tag{7a}$$

$$h_2(x_1, x_2, t) = f_2(x_1, x_2) - e^{-t} f_2(x_1^0, x_2^0) \tag{7b}$$

where t is an embedded homotopy continuation parameter. If t is varying over the interval $0 \le t \le 1$, while always correspond to the zero homotopy functions. Therefore, the differentials of the homotopy functions can be written

$$dh_1 = \frac{\partial h_1}{\partial x_1} dx_1 + \frac{\partial h_1}{\partial x_2} dx_2 + \frac{\partial h_1}{\partial t} dt = 0 \tag{8a}$$

$$dh_2 = \frac{\partial h_2}{\partial x_1} dx_1 + \frac{\partial h_2}{\partial x_2} dx_2 + \frac{\partial h_2}{\partial t} dt = 0 \tag{8b}$$

These differentials are zero, since $h_1(x_1, x_2, t) = 0$, $h_2(x_1, x_2, t) = 0$. Using Eqs. (7) and (8), it can be rewritten as

$$dh_1 = \frac{\partial f_1}{\partial x_1} dx_1 + \frac{\partial f_1}{\partial x_2} dx_2 + e^{-t} f_1(x_1^0, x_2^0) dt = 0 \tag{9a}$$

$$dh_2 = \frac{\partial f_2}{\partial x_1} dx_1 + \frac{\partial f_2}{\partial x_2} dx_2 + e^{-t} f_2(x_1^0, x_2^0) dt = 0 \tag{9b}$$

or

$$\frac{\partial f_1}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f_1}{\partial x_2}\frac{dx_2}{dt} = e^{-t} f_1(x_1^0, x_2^0) = -f_1(x_1, x_2) \tag{10a}$$

$$\frac{\partial f_2}{\partial x_1}\frac{dx_1}{dt} + \frac{\partial f_2}{\partial x_2}\frac{dx_2}{dt} = e^{-t} f_2(x_1^0, x_2^0) = -f_2(x_1, x_2) \tag{10b}$$

Differential Eq. (10) are usually called the Davidenko’s equations, which can be written in matrix form for the $n \times n$ problem as

$$[J]\frac{dx}{dt} = -\{F\} \tag{11}$$

where again $[J]$ and $\{F\}$ are the Jacobian matrix and function vector of the nonlinear system, respectively, and dx/dt is the derivative of the solution vector with respect to the continuation parameter, t, which is appeared in Eq. (7). From the mathematical point of view, each Davidenko's flow are defined by a perturbation parameter, t, in the nonlinear system of equations governing the problem. The Newton-Raphson iterates, ΔX, of the normal flow algorithm is the unique minimum norm solution of the Davidenko's Eq. (11). This solution may be obtained as (Watson *et al.* 1981)

$$\Delta X = V - \frac{V^T U}{U^T U} U \tag{12}$$

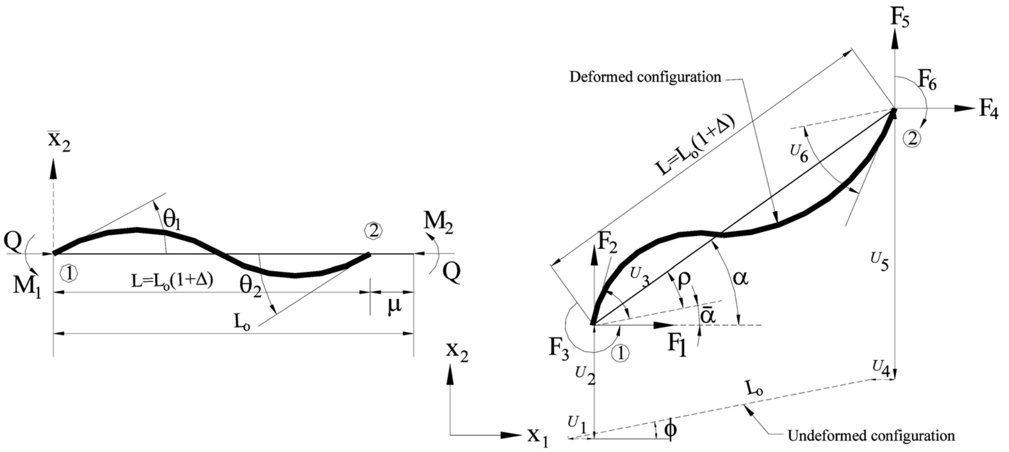
where V is a particular solution can be found by selecting an auxiliary equation and U is any vector in the kernel $[J]$.

3. Large deformation equations for 2D beam-column element

Fig. 1 indicates a beam-column element having cross-sectional area equal to A and modulus of elasticity equal to E. The vectors $\{F\}$ and $\{\delta\}$ show the components of the end forces and end displacements in global coordinates, respectively (Kassimali 1983). Eulerian local coordinate system is used as global coordinate system (Fig. 1a). So, the relationship between the end forces of the member in global and local coordinates is obtained as follows

$$[B]\{S\} = \{F\} \tag{13}$$

in which $\{S\}$ and $\{F\}$ are the internal force vector of the beam-column element in the local and global coordinates systems, respectively, and $\{B\}$ is the transformation matrix that indicates the relationship between the nodal forces of the member in local and global coordinates systems in the following form



(a) The global coordinate system (b) The local coordinate

Fig. 1 The internal forces and deformations in the global and local coordinate systems

$$[B] = \begin{bmatrix} -\frac{n}{L} & -\frac{n}{L} & m \\ \frac{m}{L} & \frac{m}{L} & n \\ 1 & 0 & 0 \\ \frac{n}{L} & \frac{n}{L} & -m \\ -\frac{m}{L} & -\frac{m}{L} & -n \\ 0 & 1 & 1 \end{bmatrix} \tag{14}$$

where $m = \cos\alpha$ and $n = \sin\alpha$ are the cosine directors of the deformed member which, for an arbitrary large nodal displacement. Note that, L_o is the length of the member before deformation, and L is the length of the member after deformation which is obtained by the following relationship

$$L = L_o(1+\Delta) = [(x_1^{(2)} + U_4 - x_1^{(1)} - U_1)^2 + (x_2^{(2)} + U_5 - x_2^{(1)} - U_2)^2]^{1/2} \tag{15}$$

The relative member deformations in local coordinates system can be expressed directly in terms of global displacements by noting that

$$\cos\overline{\alpha} = \frac{x_1^{(2)} - x_1^{(1)}}{L_o} \tag{16}$$

$$\cos\alpha = \frac{x_1^{(2)} + U_4 - x_1^{(1)} - U_1}{L} \tag{17}$$

$$\theta_1 = \delta_3 - \rho \tag{18}$$

$$\theta_2 = \delta_6 - \rho \tag{19}$$

$$\rho = \alpha - \overline{\alpha} \tag{20}$$

in these equations $\overline{\alpha}$ refers to the orientation of the chord in the undeformed configuration as shown in Fig. 1; and ρ = angle of rotation of the chord.

The relationship between relative member deformations, θ_1, θ_2 and u and associated member end forces, M_1, M_2 and Q can be based on the beam-column theory for elastic members thus, (see Fig. 1)

$$M_1 = \frac{EI}{L}(C_1\theta_1 + C_2\theta_2) \tag{21}$$

$$M_2 = \frac{EI}{L}(C_2\theta_1 + C_1\theta_2) \tag{22}$$

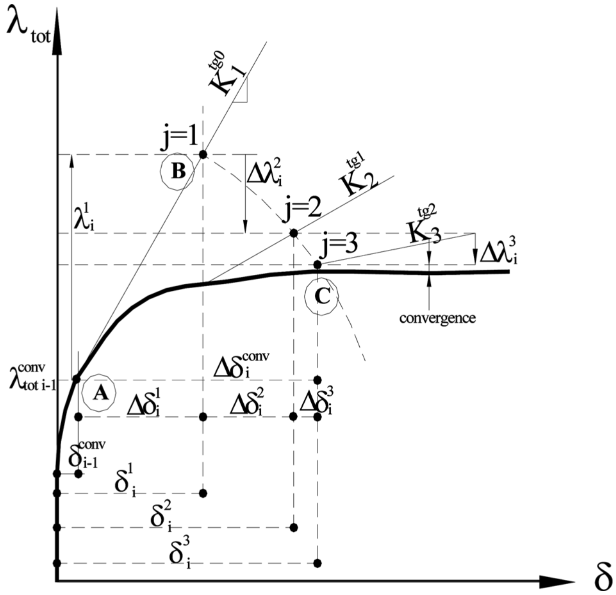


Fig. 2 The iterations along the normal path to the Davidenko's flow

$$Q = EA\left(\frac{\mu}{L_o} - C_b\right) \tag{23}$$

In which A = area of cross section; E = Modulus of elasticity; I = moment of inertia; C_1, C_2 = elastic stability functions; and

$$C_b = b_1(\theta_1 + \theta_2)^2 + b_2(\theta_1 - \theta_2)^2 \tag{24}$$

is the length correction factor due to bowing action (24), with b_1, b_2 = bowing functions.

3.1 The equilibrium equations

Applying external nodal loads, $\{F_{ext}\}$, on the structure causes nodal displacements, $\{\delta_i\}$, stresses and the resultant of the nodal internal loads, $\{f\}$, so, the system of the equilibrium equations of the structure can be stated as follows

$$F(\lambda, \delta) = F_{ext} - f(\delta_1, \delta_2, \ldots, \delta_N) = 0 \qquad i = 1, 2, \ldots, N \tag{25}$$

in which λ is the load factor, N is the number of degrees of freedom of the structure and $F(\lambda, \delta)$ is the vector of the residual forces.

It must be noted that the components of the internal forces of the members are nonlinear functions of the nodal displacements, so the equilibrium Eq. (25) are nonlinear and cannot be solved explicitly. Therefore, in the nonlinear analysis of the structures, the system of the equations is changed into incremental form first, and then is solved through several iterative steps. The incremental form of the system of equations is as follows

$$[\tau]\{\Delta u\} = \{\Delta Q\} \tag{26}$$

where $\{\Delta Q\}$ and $\{\Delta u\}$ are the internal forces increment and the displacement increment in the local coordinates system, respectively, and $[\tau]$ is known as tangential stiffness matrix.

3.2 The tangential stiffness matrix of the member

Using the abovementioned subjects and the tangential stiffness matrix concept, the incremental load-displacement relationship in global coordinate system is in the following form

$$[K^{tg}]\{\Delta\delta\} = \{\Delta F_{ext}\} \tag{27}$$

where $[K^{tg}]$ is the tangential stiffness matrix of the two-dimensional frame in global coordinate system which can be determined as follows

$$[K^{tg}] = [B][\tau][B]^T + \sum_{k=1}^{3} S_k[g^{(k)}] \tag{28}$$

where $[g]$ is the geometric matrix which can be found in appendix A and $S_1 = M_1$; $S_2 = M_2$, $S_3 = Q$ (see Fig. 1). It is noted that, from the mathematical point of view, differential Eq. (11) is defined by the nonlinear system of equations governing the problem such as Eq. (27). So, the Newton-Raphson iterates, $\Delta\delta$, of the normal flow algorithm is the unique minimum norm solution of the Davidenko's Eq. (27).

3.3 The iteration process in the first step of proposed algorithm

Every step of proposed algorithm consists of two phases; anticipation and modifying iterations. If i is the number of the step and j is the number of the modifying iteration, the vector of external load applying on the nodes, $\{F_{ext}\}_i^j$, will be stated as follows

$$\{F_{ext}\}_i^j = \lambda_{toti}^j\{F_I\} \tag{29}$$

in which, λ_{toti}^j is the factor of total external load and $\{F_I\}$ is the reference external load vector.

If the point A in Fig. 2 is concerning the converged point in the step $i-1$ of the load-displacement path, then the following relationship will exist for the step i.

$$\lambda_{toti}^j = \lambda_{toti-1}^{conv} + \lambda_i^1 + \sum_{j=2} \Delta\lambda_i^j \tag{30}$$

where λ_{toti-1}^{conv} is the factor of the total external load at the end of the previous step in which the solution has been converged, λ_i^1 is the assumed load increment at the beginning of the calculations and $\Delta\lambda_i^j$ is the increment of the load level calculated in each iteration. In anticipation phase, calculating the tangential stiffness matrix $[K^{tg}]_i^{j=0}$ at point A, the tangential displacement $\{\delta_I\}_i^{j=1}$ is determined first through the following relationship

$$[K^{tg}]_i^{j=0}\{\delta_I\}_i^{j=1} = \{F_I\} \tag{31}$$

Then in the first iteration $j = 1$, the displacement increment $\{\Delta\delta\}_i^{j=1}$ is found using the following relationship

$$\{\Delta\delta\}_i^{j=1} = \lambda_i^1\{\delta_I\}_i^{j=1} \tag{32}$$

Thus, the total displacement or, the approximate response of the equilibrium path at point B is updated as follows

$$\{\delta\}_i^{j=1} = \{\delta\}_{i-1}^{conv} + \{\Delta\delta\}_i^{j=1} \tag{33}$$

In the next stage, computing the Newton-Raphson step in each iteration j, the solution is obtained and is modified along the real equilibrium path, (see Fig. 2). In proposed algorithm, the Newton-Raphson step size is the minimum solution of the system of Eq. (27). This solution can be found through two steps:

Step 1: Select an auxiliary equation in the following form and solve it together with Eq. (25), the particular solution V will be obtained

$$[K^{tg}]_i^{j-1}\{V\} = \Delta\lambda_i^j\{F_I\} - \{\psi\}_i^{j-1} \tag{34}$$

$$\Delta\lambda_i^j = \frac{[\{\delta_I\}_i^j]^T\{\Delta\delta_R\}_i^j}{[\{\delta_I\}_i^j]^T\{\delta_I\}_i^j} \tag{35}$$

where $\{\psi\}_i^{j-1}$ is the vector of the residual of internal forces (in other words, unbalanced forces) and $\{\Delta\delta_R\}_i^j$ is the vector of unbalanced displacement such that

$$\{\psi\}_i^{j-1} = \{F_{\text{int}}\}_i^{j-1} - \left(\lambda_{toti-1}^{conv} + \lambda_i^{j=1} + \sum_{j=2}^{j}\Delta\lambda_i^{j-1}\right)\{F_I\} \tag{36}$$

in which F_{int} is the vector of resultant internal forces at the nodes. The vector of unbalanced displacement $\{\Delta\delta_R\}$ is computed by the following system of equations

$$[K^{tg}]_i^{j-1}\{\Delta\delta_R\}_i^j = -\{\psi\}_i^{j-1} \tag{37}$$

It is noted that in this study, the auxiliary equation presented in the method of minimum unbalanced displacement has been used (Saffari *et al*. 2008).

Step 2: Using the following equation, the minimum solution of the norm is calculated

$$\{\Delta\delta\}_i^j = \{V\} - \frac{\{V\}^T\cdot\{\delta_I\}_i^j}{\|\delta_I\|}\cdot\{\delta_I\}_i^j \tag{38}$$

in which $\{\delta_I\}_i^j$ is the vector of tangential displacement in the converged point in $i-1$ step.

As mentioned before, this solution is equal to the step size of the Newton-Raphson iterations, S_i^j. Since in this paper the method of controlling the displacement has been used, the solution is equal to the vector of displacement increment. The vector $\{\delta_I\}_i^j$ is the same as the tangential displacement vector which is converged in step $i-1$.

3.4 The iterations process for achieving the equilibrium path

After calculating the vector of increment displacement, $\{\Delta\delta\}_i^j$, the vector of total displacement is updated in jth iteration as follows

$$\{\delta\}_i^j = \{\delta\}_i^{j-1} + \{\Delta\delta\}_i^j \tag{39}$$

On the other hand, in any iteration the load level changes by the amount of $\Delta\lambda_i^j$ which is obtained from the auxiliary Eq. (35) and is updated by (30). The process is continued until the solution at point C is converged to the equilibrium path of the structure with acceptable accuracy.

3.5 Selecting the amount of load increment for steps i > 1

For steps $i > 1$, after obtaining the converged point at the first step $i = 1$, the load level must be modified selecting suitable load increment and the process is repeated similar to the first step. In this work the direct method of updating has been used. In direct method of updating, the load increment is related to the number of iterations and the sign of the determinant of the tangential stiffness matrix of the previous step and can be computed through the following relationship.

$$\lambda_{i+1}^j = \pm\lambda_i^j\left(\frac{J_D}{J_M}\right)^\gamma, \quad \gamma = 0.5 \tag{40}$$

where J_D is the assumed number of iterations at the beginning of the calculations and J_M is the number of iterations in the previous step and, λ_{i+1}^j will be negative if the determinant of the stiffness matrix of the previous step is negative. Therefore, in the presented algorithm, if the structure has linear behavior, the number of iterations of the previous step is low and so, the load increment will be increased based upon (40) which affect the speed of the problem solution significantly. On the other hand, if the determinant of the tangential stiffness matrix is negative or equal to zero, it implies tracing the limit point. Also, the negative sign indicates the negative slope of the equilibrium path of the structure. Sticking to this process, the presented algorithm affords the possibility of tracing the complex equilibrium paths such as shown in Fig. 3.

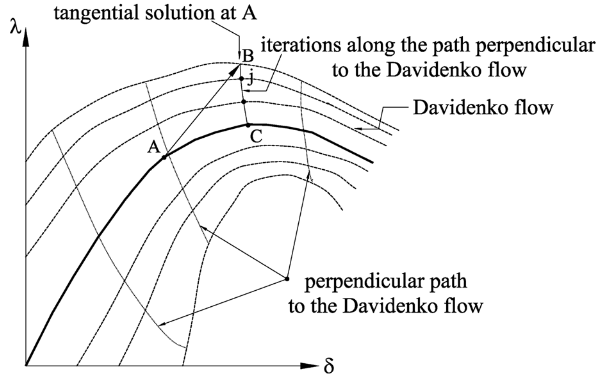


Fig. 3 The normal path to the Davidenko's flow algorithm (Saffari *et al*. 2008)

4. Numerical studies

In this section two test cases concern geometrical nonlinear problems is examined using proposed algorithm.

4.1 Test case 1: The Lee frame

Fig. 4 shows the Lee frame under external concentrated loading. This structure has large displacement and instabilities. This structure has been the subject of previous investigations (Levy and Spillers 2003). The equilibrium path of such structures involved with snap-back response. It must be noted that the load increment iterative of the Newton-Raphson method are weak in tracing this limit point (Sheng and Khaliq 2002). The characteristics of this frame are: the cross-sectional area of the members $A = 6\ \text{cm}^2$, the moment of inertia $I = 2\ \text{cm}^4$ and the modulus of elasticity $E = 720\ \text{kN/cm}^2$, Also, the following parameters are assumed: $J_D = 5$, $J_{\max} = 15$ and tolerance for convergence norm $\varepsilon_c = 10^{-5}$. The load parameters are: an initial load F of 0.1 kN, a final load F of 2.0 kN, a minimum step of 0.01 and a maximum step of 10.

This structure has been analyzed using the method proposed in this paper and the equilibrium path has been drawn as shown in Fig. 5. Also, the result obtained using the advanced iterative methods of Crisfield and Chan are indicated in Fig. 5 (Crisfield 1983, Chan 1988).

Furthermore, the total number of increments, the cumulative sum of total iterations and the CPU time used by each method is shown in Table 1. As it can be seen, in advanced iterative methods which use the arc-length factor, the number of loads steps and consequently the number of iterations is more than the number of iterations of the method of this paper. Therefore, it can be concluded that in the case of uniform slope and for hardening branch with high slope, the normal path to the Davidenko's flows algorithm can analyze the system more quickly in addition to having the ability of tracing the equilibrium path of the structure.

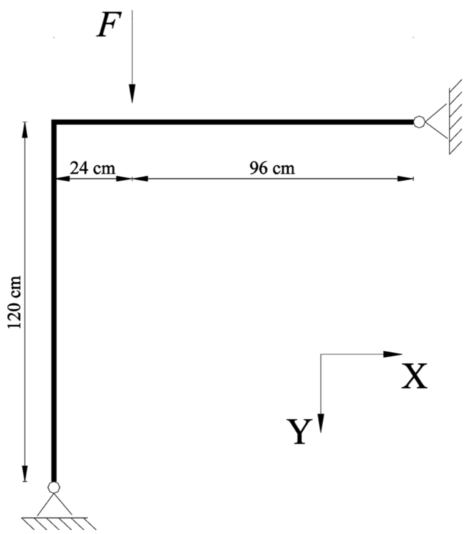


Fig. 4 The Lee frame (Test Case 1)

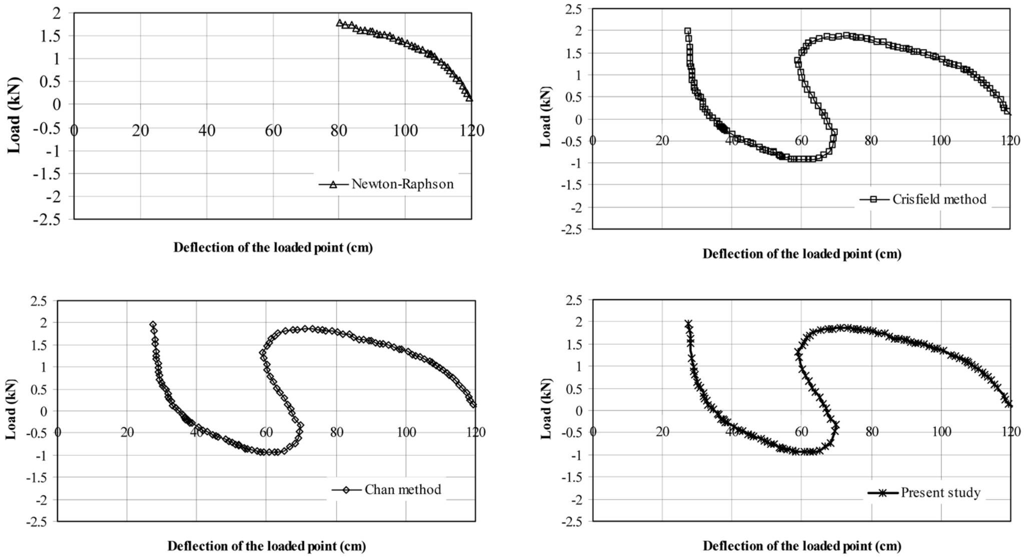


Fig. 5 The equilibrium path of the Lee frame (Test Case 1)

Table 1 Performance and iteration for test case 1

Method	Number of increments	Total iterations	Time (sec)
Newton-Raphson	failed	failed	-
Crisfield (1983)	512	2590	20.58
Chan (1988)	506	2530	20.05
Present study	498	2489	18.72

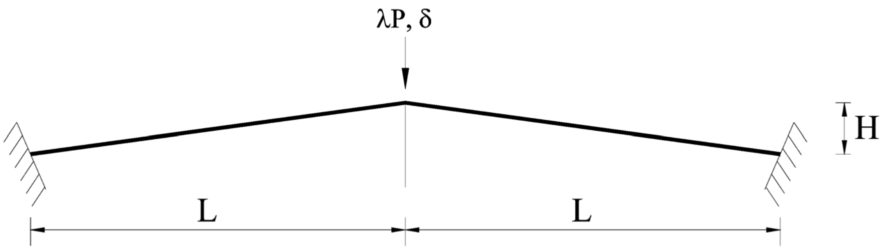


Fig. 6 The William's Toggle frame (Test Case 2)

4.2 Test case 2: The William's Toggle frame

The William's toggle frame indicated in Fig. 6 is another test case that has been considered in the previous studies (Papadrakakis 1981). The characteristics of this structure are: the cross-sectional area of the members $A = 1.18$ cm^2, the moment of inertia $I = 0.037$ cm^4, the modulus of elasticity $E = 7200$ kN/cm^2, $L = 33$ cm and $H = 0.8$ cm. Also, the following parameters are assumed for this

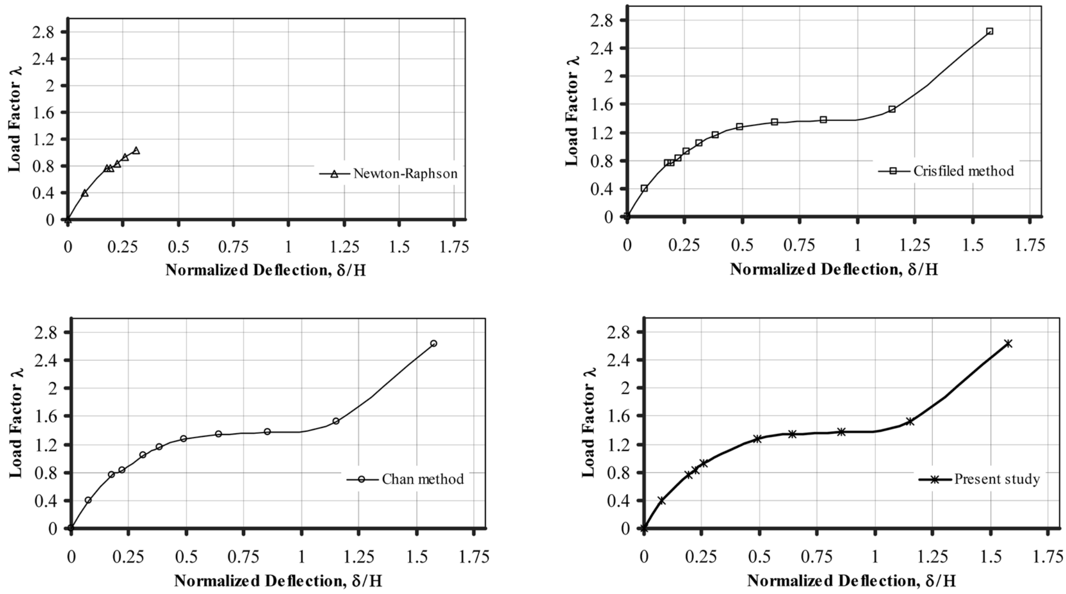


Fig. 7 The equilibrium path of the William's Toggle frame (Test Case 2)

Table 2 Performance and iteration for test case 2

Method	Number of increments	Total iterations	Time (sec)
Newton-Raphson	failed	failed	-
Crisfield 1983	82	304	12.74
Chan 1988	62	220	12.15
Present study	58	203	10.84

structure: P = 0.09 kN, $\Delta\lambda_1^1 = 0.003$, $\lambda_{\max} = 3$, $J_D = 2$, $J_{\max} = 15$ and $\varepsilon_c = 10^{-4}$.

The load-displacement path for this structure is shown in Fig. 7 and the results are compared to the results of the methods of Crisfield and Chan (Crisfield 1983, Chan 1988). Note that, as in previous study, the iterative methods with load increment factor are weak in tracing the limit point (the snap-through) branch. As can be seen, the method of the present paper has passed the equilibrium path having returning load and returning displacement branches.

Table 2 summarizes the number of increments, total iterations and the CPU time. In all the advanced iterative methods with arc-length factor, the number of iterations is almost the same, but in this study, the normal path to the Davidenko's flows algorithm has the minimum number of iterations.

5. Conclusions

In structural mechanism, the nonlinear equations are often solved using the incremental Newton-

Raphson method. The frame structures have highly nonlinear behavior regarding the geometry and the level of load applied. It is seen that the simple iterative and incremental Newton-Raphson method are failed to trace the limit points of snap-through or snap-back of the equilibrium path. On the other hand, in all the advanced iterative methods with arc-length factor, the number of iterations is high and almost the same. In the proposed algorithm, based upon the Homotopy functions concepts, the auxiliary equation is transmitted to the normal path to the Davidenko's flows first, and then by incremental iterative method, the equilibrium path of the structure is traced in fewer steps. It has been shown by the numerical studies that the method developed in this paper which uses the normal path to the Davidenko's flows algorithm reduces the time and the cost for nonlinear analysis of high degree of freedom frames in addition to having the capability of tracing the complex equilibrium paths.

References

Allgower, E.L. and Georg, K. (1980), "Homotopy methods for approximating several solutions to nonlinear systems of equations", Ed. Forster, W., *Numerical Solution of Highly Nonlinear Problems*, Amsterdam, North-Holland, New York.

Al-Rasby, S.N. (1991), "Solution techniques in nonlinear structural analysis", *Comput. Struc*t., **40**(4), 985-993.

Baltoz, J.L. and Dhatt, G. (1979), "Incremental displacement algorithms for nonlinear problems", *Int. J. Numer. Meth. Eng.*, **14**, 1262-1267.

Bashir-Ahmad, M. and Xiao-zu, S. (2004), "Arc-length technique for nonlinear finite element analysis", *J. Zhej. Uni. Sci.*, **5**(5), 618-628.

Chan, S.L. (1988), "Geometric and material nonlinear analysis of beam columns and frames using the minimum residual displacement method", *Int. J. Numer. Meth. Eng.*, **26**, 2657-2669.

Crisfield, M.A. (1981), "A fast incremental/iterative solution procedure that "Handles Snap-Through"", *Comput. Struct.*, **13**, 55-62.

Crisfield, M.A. (1983), "An arc-length method including line searches and accelerations", *Int. J. Numer. Meth. Eng.*, **19**, 1269-1289.

Fujii, F. and E. Ramm, E. (1997), "Computational bifurcation theory: path-tracing, pinpointing and path-switching", *Eng. Struct.*, **19**(5), 385-392.

Jagannathan, D.S., Epstein, H.I. and Christiana, P. (1975), "Nonlinear analysis of reticulated space trusses", *J. Struct. Div.-ASCE*, **101**(12), 2641-2658.

Kassimali, A. (1983), "Large deformation analysis of elastic-plastic frame", *J. Struct. Eng.-ASCE*, **109**(8), 1869-1885.

Kassimali, A. and Abbassnia, R. (1991), "Large deformation analysis of elastic space frames", *J. Struct. Eng.*-ASCE, **117**(7).

Levy, R. and Spillers, W.R. (2003), *Analysis of Geometrically Nonlinear Structures*, 2nd Edition, Dordrecht, Kluwer.

Mallardo, V. and Alessandri, C. (2004), "Arc-length procedures with BEM in physically nonlinear problems", *Eng. Anal. Boun. Elem.*, **28**(6), 547-559.

Papadrakakis, M. (1981), "Post-Buckling analysis of spatial structures by vector iteration methods", *Comput. Struct.*, **14**(5-6), 393-402.

Pecknold, D.A., Ghaboussi, J. and Healey, T.J. (1985), "Snap-through and bifurcation in a sample structure", *J. Eng. Mech.-ASCE*, **111**(7), 909-922.

Ramm, E. (1981), "Strategies for tracing the nonlinear response near limit points", Eds. Wunderlich, W., Stein, E. and Bathe, K.J., *Nonlinear Finite Element Analysis in Structural Mechanics*, Springer-Verlag, Berlin, 63-89.

Riks, E. (1979), "An incremental approach to the solution of snapping and buckling problems", *Int. J. Sol. Struct.*, **15**, 529-551.

Saffari, H., Fadaee, M.J. and Tabatabaei, R. (2008), "Non-linear analysis of space trusses using modified normal

flow algorithm", *J. Struct. Eng.-ASCE*, **134**(6), 998-1005.
Sheng, Q. and Khaliq, A.Q.M. (2002), "Modified arc-length adaptive algorithms for degenerate reaction-diffusion equations", *Appl. Math. Comput.*, **126**(2-3), 279-297.
Toklu, Y.C. (2004), "Nonlinear analysis of trusses through energy minimization", *Comput. Struct.*, **82**(20-21), 1581-1589.
Watson, L.T., Holzer, S.M. and Hansen, M.C. (1981), "Tracking nonlinear equilibrium paths by a homotopy method", Technical Reports Num., CS81001-R, Virginia Poly. Institute & Blacksburg Uni., USA.
Wempner, G.A. (1971), "Discrete approximation related to nonlinear theories of solids", *Int. J. Sol. Struct.*, **7**, 1581-1599.

Appendix A

$$[g^{(1)}]=[g^{(2)}]=\frac{1}{L^2}\begin{bmatrix} -2mn & m^2-n^2 & 0 & 2mn & -(m^2-n^2) & 0 \\ m^2-n^2 & 2mn & 0 & -(m^2-n^2) & -2mn & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 2mn & -(m^2-n^2) & 0 & -2mn & m^2-n^2 & 0 \\ -(m^2-n^2) & -2mn & 0 & m^2-n^2 & 2mn & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$$[g^{(3)}]=\frac{1}{L}\begin{bmatrix} -n^2 & mn & 0 & n^2 & -mn & 0 \\ mn & -m^2 & 0 & -mn & m^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ n^2 & -mn & 0 & -n^2 & mn & 0 \\ -mn & m^2 & 0 & mn & -m^2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$