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Time domain identification of multiple cracks in a beam

*Z.Y. He[1] and Z.R. Lu[2a]

[1]*School of Civil and Transportation Engineering, South China University of Technology, Guangzhou 510641, P.R. China*
[2]*School of Engineering, Sun Yat-sen University, Guangzhou 510275, P.R. China*

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Abstract. It is well known that the analytical vibration characteristic of a cracked beam depends largely on the crack model. In the forward analysis, an improved and simplified approach in modeling discrete open cracks in beams is presented. The effective length of the crack zone on both sides of a crack with stiffness reduction is formulated in terms of the crack depth. Both free and forced vibrations of cracked beams are studied in this paper and the results from the proposed modified crack model and other existing models are compared. The modified crack model gives very accurate predictions in the modal frequencies and time responses of the beams particularly with overlaps in the effective lengths with reduced stiffness. In the inverse analysis, the response sensitivity with respect to damage parameters (the location and depth of crack, etc.) is derived. And the dynamic response sensitivity is used to update the damage parameters. The identified results from both numerical simulations and experiment work illustrate the effectiveness of the proposed method.

Keywords: multiple cracks; dynamic response; crack identification; inverse problem.

1. Introduction

The responses of cracked structures to external loading have been extensively studied in the past two decades. A wealth of analytical, numerical and experimental investigations on the problem has been accumulated. Numerous models on a beam with cracks have been developed. These generally include either a crack that is always open or a breathing crack that opens and closes during vibration.

Dimarogonas (1996) and Ostachowicz and Krawczuk (2001) gave comprehensive reviews on the crack models. The simplest one is a reduced stiffness (or increased flexibility) in a finite element to simulate a small crack in the element (Yuan 1985, Pandey *et al*. 1991, Salawu and Williams 1993, Pandey and Biswas 1994, Ratcliffe 1997). Another simple approach is to divide the cracked beam into two beam segments joined by a rotational spring that represents the cracked section (Rizos *et al*. 1990, Ismail *et al*. 1990, Chaudhari and Maiti 2000, Boltezar *et al*. 1998). An improved version of this model (Fernandez-Saez *et al*. 1999) leads to a closed-form solution giving the natural frequencies and mode shapes of the cracked beam directly. Others researchers (Ostachowicz and

*Corresponding author, Associate Professor, E-mail: zhyhe@scut.edu.cn
[a]Associate Professor

Krawczuk 1991, Shen and Taylor 1991, Shifrin and Ruotolo 1999) solved the differential equations with compatible boundary conditions satisfying the crack conditions. Also, two- or three-dimensional finite element meshes may be used for beam-type structures with a crack (Shen and Pierre 1990, Lakshmi and Jebaraj 1999, Krawczuk and Ostachowicz 1993a). But the shortcoming of this complicated model is computationally expensive. Also Krawczuk and Ostachowicz (1993b) and Lee and Chung (2000) have developed the flexibility matrix for a beam element with a crack using the energy method.

Chondros *et al*. (1998) developed a continuous cracked beam vibration theory for the lateral vibration of cracked Euler-Bernoulli beams with single or double-edge cracks. This continuous cracked beam vibration theory was used to predict the dynamic response of a simply supported beam with open surface cracks.

Sinha *et al.* (2002) proposed a simplified open crack model for beam structures with cracks. A linear approximation to the stiffness reduction is used in this modeling approach. This model has the advantage to estimate directly the crack location and damage extent, which reduces the computational burden and improves the localization accuracy.

The model proposed by Sinha *et al.* (2002) is modified in this paper to have more flexibility to model cracks of different depth and with a longitudinal crack zone which is a function of the crack depth. The "overlapping" effect of adjacent cracks is taken into consideration in this paper. Both free and forced vibrations of cracked beams are studied and the results are compared with those from existing models and the experiments. More accurate results are obtained by using the modified crack model in terms of the modal frequency and time history responses under forced excitation.

Many techniques have been proposed making use of the natural frequencies and/or the mode shapes, the dynamic and/or static response of the cracked structures for crack detection in recent years (Zhao and DeWolf 2007, Zhu *et al*. 2005, Noguchi and Harada 2006, Xiang *et al*. 2009).

Response sensitivity respect to the elemental physical parameters (i.e., Young's modulus) has been investigated by the authors (Lu and Law 2007) and the response sensitivity was used to update the unknown physical parameters in the inverse analysis. The shortcoming of the method lies in: when there are a large number of elements in the finite element model, the unknowns converge to the true value slowly, more computation time is needed in the inverse analysis. In the present study, the response sensitivities with respect to damage parameters are derived and these sensitivities are used to update the damage parameters in the inverse analysis. The number of the unknowns reduced much and thus the computational efficiency is improved.

2. Forward analysis

2.1 Crack modeling

A simple beam with multiple cracks along its length is shown schematically in Fig. 1. It is assumed that the cracks have a uniform depth across the width of the beam and they do not change the mass of the beam. The fully open cracks are considered in this study.

It is known that the material in the vicinity of the crack will not be stressed and they will offer only a limited contribution to the stiffness. Christides and Barr (1984) considered the effect of a crack in a continuous beam and proposed an exponential function for the longitudinal distribution of the flexural rigidity $EI(x)$, given by

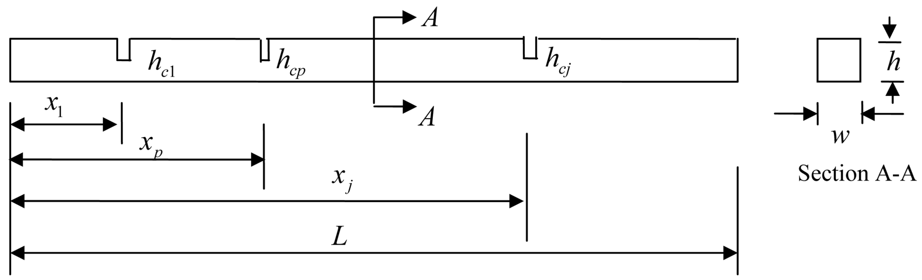

Fig. 1 Sketch of beam with multiple cracks

$$EI(x) = \frac{EI_0}{1 + C\exp(-2\alpha|x - x_j|/h)} \tag{1}$$

where $C = (I_0 - I_{cj})/I_{cj}$. $I_0 = wh^3/12$ and $I_{cj} = w(h - h_{cj})^3/12$ are the second moments of area of the undamaged beam and damaged beam respectively at the jth crack. w and h are the width and height of the undamaged beam, h_{cj} and x_j are the depth and the crack position of the crack respectively. α is a constant that Christides and Barr (1984) estimated from experiments to be 0.667. The shortcoming of this exponential decay function of Christides and Barr (1984) in the finite element model of a structure is that the flexibility is not local to one or two elements, and thus the integration required to produce the stiffness matrix for the beam would have to be performed numerically every time the crack position changes. In addition, for complex structures with non-uniform beams, Eq. (1) would only be approximate.

Sinha *et al.* (2002) models the crack as a triangular reduction in stiffness close to the crack, and the effective crack zone with stiffness reduction due to the crack is a fraction of the crack depth. This model has an over-simplification of the crack behaviour to trade-in computational efficiency. This model is modified and generalized in this paper to express the stiffness as a power function of the crack location, and a power index p is used such that the model by Sinha can be expressed as a special case of this generalized model.

The flexural rigidity $EI_k(\xi)$ close to the crack is expressed as

$$EI_k(\xi) = \begin{cases} EI_{cj} + E(I_0 - I_{cj})\left(\dfrac{\xi - \xi_j}{\xi_{j1} - \xi_j}\right)^p & \text{if} \quad \xi_{j1} \le \xi \le \xi_j \\ EI_{cj} + E(I_0 - I_{cj})\left(\dfrac{\xi - \xi_j}{\xi_{j2} - \xi_j}\right)^p & \text{if} \quad \xi_j \le \xi \le \xi_{j2} \end{cases} \tag{2}$$

where ξ_j is the location of the jth crack within the kth element and $\xi_{j1} = \xi_j - l_c$ and $\xi_{j2} = \xi_j + l_c$ are the positions on either sides of the crack defining the boundaries of the crack zone. l_c is the effective length of the crack zone. The parameter p characterizes the variation of the flexural rigidity $EI(x)$ in the crack zone, When $p = 1.0$, this model becomes the model of Sinha. When $p = 0.5$, the model is very close to the model of Christides and Barr (1984) as shown in Fig. 2 for a relative crack depth of 0.15.

The effective length of the crack zone along the beam, l_c, is obtained by equating the integrals of the stiffness reduction in Eqs. (1) and (2). We have from Eq. (1)

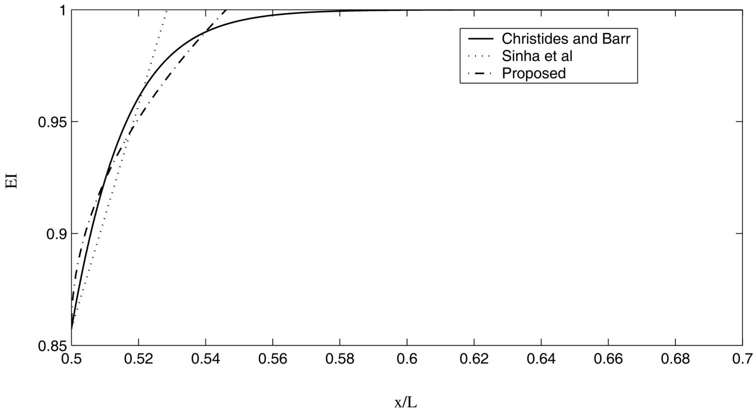

Fig. 2 Comparison in the variation in flexural rigidity near a crack for the different crack models

$$\int_{-\infty}^{+\infty} [EI_0 - EI(x)]dx = EI_0 \frac{h}{\alpha} \log_e(1+C) \tag{3a}$$

and from Eq. (2)

$$\int_{\xi_{j1}}^{\xi_{j2}} [EI_0 - EI_e(x)]d\xi = E(I_0 - I_{cj})\left(\frac{2p}{p+1}\right)l_c \tag{3b}$$

The effective length is therefore given by

$$l_c = \frac{h}{\alpha}\left(\frac{1+p}{2p}\right)\left(\frac{I_0}{I_0 - I_{cj}}\right)\log_e(1+C) \tag{4}$$

Euler-Bernoulli formulation is used to model the beam. It is assumed that the stiffness reduction all falls within a single element. The stiffness matrix of the kth element of the beam may be written as

$$k_{k,\,crack} = k_e - k_{cj} \tag{5}$$

where k_e is the element stiffness matrix for the kth element without crack and k_{cj} is the reduction in the stiffness matrix due to the jth crack. The coefficients of k_{cj} can be expressed as

$$k_{st} = \int_0^{le} [EI_0 - EI_k(\xi)]N''_{es}(\xi)N''_{et}(\xi)d\xi \tag{6}$$

where the shape functions $N_{ei}(\xi)$ are those for a standard Euler-Bernoulli beam element, which are

$$N_{e1} = \left(1 - 3\frac{\xi^2}{l_e^2} + 2\frac{\xi^3}{l_e^3}\right), \quad N_{e2} = \left(\xi - 2\frac{\xi^2}{l_e} + \frac{\xi^3}{l_e^2}\right), \; N_{e3} = \left(3\frac{\xi^2}{l_e^2} - 2\frac{\xi^3}{l_e^3}\right), \quad N_{e4} = \left(-\frac{\xi^2}{l_e} + \frac{\xi^3}{l_e^2}\right) \tag{7}$$

and l_e is the length of the kth element. Using Eqs. (2), (6) and (7), the matrix k_{cj} is obtained as

$$k_{cj} = \begin{bmatrix} k_{11} & k_{12} & -k_{11} & k_{14} \\ k_{12} & k_{22} & -k_{12} & k_{24} \\ -k_{11} & -k_{12} & k_{11} & -k_{14} \\ k_{14} & k_{24} & -k_{14} & k_{44} \end{bmatrix} \tag{8}$$

Different explicit forms of k_{cj} can be obtained from Eq. (8) for different values of p in Eq. (2).

Similarly, the stiffness matrix k_{cj} can be established for other element with a crack. And these element matrices are then assembled into the global stiffness matrix for the beam structure. By finite element method, the equation of motion of forced vibration for the beam is

$$[M]\{\ddot{d}\} + [C]\{\dot{d}\} + [K]\{d\} = \{F(t)\} \tag{9}$$

where $[M]$ is the system mass matrix, $[C]$ is the system damping matrix, if Rayleigh damping model is taken, then $[C] = \alpha_0[M] + \alpha_1[K]$, $[K]$ is the system stiffness matrix, and $\{F(t)\}$ is the nodal force vector. $\{d\}, \{\dot{d}\}, \{\ddot{d}\}$ are the vectors of displacement, velocity and acceleration respectively.

2.2 Accuracy of the modified crack model

It is noted that the modified crack model retains the simplicity of Sinha's model. A study on its accuracy in both modal and time domain predictions is made with the following case studies. The relative error of the computed ith modal frequency of the cracked beam from using different models is computed from the following

$$\text{Relative error } \% = \frac{f_{i,\text{model}} - f_{i,\text{exp}}}{f_{i,\text{exp}}} \cdot 100\ \%$$

and the total error is computed as the square root of the summation of the relative error squares of all the modal frequencies taken into consideration.

2.2.1 From free vibration analysis

Case 1: A free-free aluminium beam with a single crack

The first example is a free-free aluminium beam in Sinha *et al.* (2002) the physical dimensions and material properties of which are given in Table 1. The location of the crack is 595 mm from the left end and the depth of the crack varies from 4 mm to 12 mm. A finite element model of the free-free beam is constructed using 27 Euler-Bernoulli beam elements and 56 degrees-of-freedom defining the vibration motion in the horizontal plane of the beam. The index p is taken equal to 1.4 as recommended from Case 3 study shown below. The modal frequencies of the beam obtained with the modified crack model are compared with those from using other existing models and the experiment, and they are tabulated in Table 2. The following observations are made:

Table 1 Properties of beams for the present study

	Case 1	Case 2	Cases 3 and 5	Case4
Boundary conditions	Free-free	Free-free	Free-free	Cantilever
Material	Aluminium	Steel	Steel	Steel
Young's modulus, E	69.79 GN/m^2	203.91 GN/m^2	207 GN/m^2	210 GN/m^2
Mass density, ρ	2600 kg/m^{33}	780 kg/m^3	7832 kg/m^3	7800 kg/m^3
The Possion Ratio, υ	0.33	0.33	0.33	N/A
Beam length, L	1832 mm	1330 mm	2100 mm	800 mm
Beam width, w	50 mm	25.30 mm	25.4 mm	20 mm
Beam depth, h	25 mm	25.30 mm	19 mm	20 mm

Table 2 Modal frequencies (Hz) and the relative error (%) of the aluminium free-free beam with a single crack

Crack States	Models	mode 1	mode 2	mode 3	mode 4	mode 5	Total error (%)
No crack	Experiment	40.000	109.688	215.000	355.000	528.750	
	Analytical	39.789/ -0.527	109.680/ -0.007	214.018/ -0.457	355.440/ 0.124	530.977/ 0.421	0.85
d_{c1} = 4 mm x_1 = 595 mm	Experiment	39.688	109.063	215.000	354.688	527.188	
	Lee *et al.* (2000)	39.698/ 0.025	109.311/ 0.227	214.927/ -0.034	355.028/ 0.096	529.363/ 0.413	0.47
	Sinha *et al.* (2002)	39.379/ -0.779	108.206/ -0.786	214.087/ -0.425	353.107/ -0.446	524.693/ -0.473	1.35
	Proposed model	39.471/ -0.54	108.568/ -0.45	214.179/ -0.38	353.535/ -0.32	526.376/ -0.17	0.89
d_{c1} = 8 mm x_1 = 595 mm	Experiment	39.375	108.125	214.688	353.438	522.812	
	Lee *et al.* (2000)	39.415/ 0.102	108.200/ 0.069	214.654/ -0.016	353.783/ 0.098	524.684/ 0.358	0.52
	Sinha *et al.* (2002)	39.094/ -0.714	107.132/ -0.918	213.825/ -0.402	351.872/ -0.443	520.452/ -0.451	1.38
	Proposed model	39.164/ -0.28	106.396/ -0.67	213.883/ -0.38	352.245/ -0.34	521.761/ -0.21	1.0
d_{c1} = 12 mm x_1 = 595 mm	Experiment	39.063	105.938	214.375	350.625	513.125	
	Lee *et al.* (2000)	38.770/ -0.750	105.850/ -0.083	214.085/ -0.135	351.136/ 0.146	515.507/ 0.464	0.91
	Sinha *et al.* (2002)	38.857/ -0.527	106.278/ 0.321	213.622/ -0.351	350.881/ 0.073	517.219/ 0.798	1.1
	Proposed model	38.604/ -1.1	105.395/ -0.51	213.365/ -0.47	350.005/ -0.17	514.462/ 0.26	1.3

Note: •/• denotes the modal frequency/relative error.

- The computed relative error from the proposed model is very close to the experimental frequencies with a maximum error of 1.1% in the fundamental mode.
- Errors from the proposed model are smaller than those from Sinha *et al*. (2002) in most cases, and are similar to those from Lee and Chung (2000).
- The error in general increases with the depth of crack.
- The total error from all five modes in the study is more or less the same for all the cases studied including the intact state. It is note that part of the relative error for the damage cases arises from the relatively incorrect finite element model used in the analysis.

Case 2: A free-free steel beam with a single crack

The second example is a free-free steel beam in Sinha *et al.* (2002) the physical dimensions and material properties of which are also listed in Table 1. The location of the crack is 430 mm from

Table 3 Modal frequencies (Hz) and the relative error (%) of the steel free-free beam with one crack

Crack States	Models	Mode 1	Mode 2	Mode 3	Mode 4	Total error (%)
No Crack	Experiment	75.313	207.188	406.250	667.813	
	Analytical	75.171/ -0.189	207.212/ 0.012	406.225/ -0.006	671.536/ 0.558	0.59
d_{c1} = 4 mm x_1 = 430 mm	Experiment	74.688	205.625	405.625	666.250	
	Lee *et al.* (2000)	74.938/ 0.335	206.262/ 0.310	405.974/ 0.086	670.550/ 0.645	0.79
	Sinha *et al.* (2002)	74.406/ -0.378	204.183/ -0.701	405.368/ -0.063	668.429/ 0.327	0.86
	Proposed method	74.656/ -0.04	205.171/ -0.22	405.686/ -0.02	669.565/ 0.50	0.55
d_{c1} = 8 mm x_1 = 430 mm	Experiment	74.063	202.500	404.688	662.813	
	Lee *et al.* (2000)	74.224/ 0.217	203.458/ 0.473	405.235/ 0.135	667.615/ 0.725	0.90
	Sinha *et al.* (2002)	73.628/ -0.587	201.283/ -0.601	404.557/ -0.032	665.356/ 0.384	0.92
	Proposed method	73. 839/ -0.3	202.079/ -0.2	404.832/ -0.04	666. 403/ 0.54	0.65
d_{c1} = 12 mm x_1 = 430 mm	Experiment	72.813	197.188	403.125	655.938	
	Lee *et al.* (2000)	72.634/ -0.246	197.764/ 0.292	403.770/ 0.160	661.635/ 0.869	0.96
	Sinha *et al.* (2002)	72.958/ 0.199	198.928/ 0.882	403.916/ 0.196	662.874/ 1.057	1.4
	Proposed method	72.2944/ -0.71	196.802/ -0.19	403.298/ 0.04	660.776/ 0.74	1.0

Note: •/• denotes the modal frequency/relative error.

the left end and the depth of the crack varies from 4 mm to 12 mm. A finite element model of the free-free beam is constructed using 20 Euler-Bernoulli beam elements and 42 degrees-of-freedom defining the vibration motion in the horizontal plane of the beam. The index p is again taken equal to 1.4 as recommended from Case 3 study below. The modal frequencies of the beam obtained with the modified crack model are compared with those from using the crack model by Sinha and the experiment as shown in Table 3. The following observations are made:

- Results from the proposed model are better than those from Lee and Chung (2000) and Sinha *et al.* (2002) except in the fundamental modal frequency for the second and third crack states. The maximum relative error is 0.74%.
- The performance of the proposed model is similar to that for Case 1 given the smaller error in the predicted modal frequencies at the intact state.

Case 3: Laboratory tests on a free-free steel beam with single and multiple cracks

The third example is a free-free steel beam the physical dimensions and material properties of which are shown in Table 1. A total of 8 cases with single and two cracks in close proximity and at different depth are studied as described in Table 4. The crack depth takes up values of 3, 6, 9 and 12 mm. These two cracks form one group. Overlap of the crack zone from adjacent cracks occurs within the group at large crack depth. The cracks are created using a machine saw with 0.6 mm thick circular cutting blade. The finite element model of the free-free beam is constructed using 20 Euler-Bernoulli beam elements and 42 degrees-of-freedom defining the vibration motion in the horizontal plane of the beam as shown in Fig. 4. The beam is suspended at its two ends by fine nylon lines as shown in Fig. 4 to simulate the free-free boundary condition. The modal frequencies are obtained with an impact hammer model B & K 8202 applying excitation at 1200 mm from the left free end and an accelerometer B & K 4370 placed at the middle of the beam at node 11. The

Table 4 Crack cases for the Cases 3 and 5 experimental study

Crack Cases	Number of crack	Crack Location (mm)	Crack Depth (mm)
a	1	1720	3
b	1	1720	6
c	1	1720	9
d	2	$x_1 = 1720$, $x_2 = 1660$	$h_{c1} = 9$, $h_{c2} = 3$
e	2	$x_1 = 1720$, $x_2 = 1660$	$h_{c1} = 9$, $h_{c2} = 6$
f	2	$x_1 = 1720$, $x_2 = 1660$	$h_{c1} = 9$, $h_{c2} = 9$
g	2	$x_1 = 1720$, $x_2 = 1660$	$h_{c1} = 12$, $h_{c2} = 9$
h	2	$x_1 = 1720$, $x_2 = 1660$	$h_{c1} = 12$, $h_{c2} = 12$

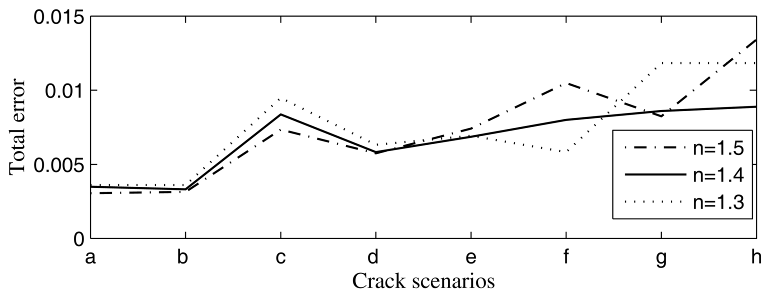

Fig. 3 Total error for each crack scenarios

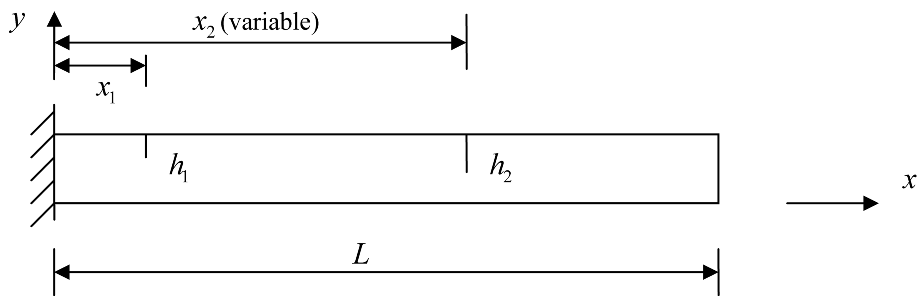

Fig. 4 A cantilevered beam with two cracks in Tikhonov (1963)

Table 5 Natural frequencies (Hz) and the relative error (%) of the steel free-free beam with single and multiple cracks

Crack Cases	Models	Mode					Total error (%)
		1	2	3	4	5	
No crack	Experiment	22.868	62.763	123.049	203.236	303.452	
	Analytical	22.827/ -0.179	62.744/ -0.030	123.047/ -0.002	203.003/ -0.115	302.856/ -0.196	0.29
a	Experiment	22.797	62.622	122.559	202.271	302.490	
	Sinha *et al*. (2002)	22.756/ -0.18	62.627/ -0.01	122.575/ 0.01	202.670/ 0.2	303.369/ 0.29	0.4
	Proposed method	22.754/ -0.19	62. 602/ -0.03	122.490/ -0.06	202.535/ 0.13	303.266/ 0.26	0.35
b	Experiment	22.766	62.378	121.704	201.050	301.514	
	Sinha *et al*. (2002)	22.739/ -0.12	62.434/ 0.09	121.916/ 0.17	201.634/ 0.29	302.591/ 0.36	0.51
	Proposed method	22.733/ -0.14	62.369/ -0.01	121.692/ -0.01	201.292/ 0.12	302.337/ 0.27	0.33
c	Experiment	22.766	61.890	119.995	198.486	299.500	
	Sinha *et al*. (2002)	21.712/ -0.24	62.134/ 0.39	120.909/ 0.76	200.114/ 0.82	301.466/ 0.66	1.38
	Proposed method	22.700/ -0.29	61.988/ 0.16	120.428/ 0.36	199.413/ 0.47	300.959/ 0.49	0.84
d	Experiment	22.736	61.768	119.751	198.486	299.500	
	Sinha *et al*. (2002)	22.671/ -0.28	61.727/ -0.07	119.845/ 0.08	199.088/ 0.3	301.205/ 0.57	0.71
	Proposed method	22.670/ -0.28	61. 709/ -0.09	119.722/ -0.02	198.834/ 0.15	300.856/ 0.45	0.56
e	Experiment	22. 675	61.401	118.897	197.266	299.408	
	Sinha *et al*. (2002)	22.640/ -0.15	61.427/ 0.04	119.121/ 0.19	198.482/ 0.62	301.147/ 0.58	0.88
	Proposed method	22.626/ -0.22	61.272/ -0.21	118.645/ -0.21	197.825/ 0.28	300.660/ 0.41	0.63
f	Experiment	22.583	60.791	117.431	196.289	299.316	
	Sinha *et al*. (2002)	22.613/ 0.13	61.177/ 0.63	118.516/ 0.92	197.952/ 0.85	301.041/ 0.58	1.52
	Proposed method	22. 543/ -0.18	60. 493/ -0.49	116.846/ -0.50	196.314/ 0.01	300.340/ 0.34	0.80
g	Experiment	22.461	59.692	114.136	192.261	296.387	
	Sinha *et al*. (2002)	22.545/ 0.38	60.450/ 1.27	116.467/ 2.04	195.366/ 1.62	299.169/ 0.94	3.07
	Proposed method	22.454/ -0.03	59.523/ -0.22	114.392/ 0.22	193.347/ 0.56	298.083/ 0.57	0.86
h	Experiment	22.339	57.987	111.206	190.307	296.143	
	Sinha *et al*. (2002)	22.525/ 0.84	60.266/ 3.9	116.064/ 4.37	195.051/ 3.75	299.111/ 1.00	6.51
	Proposed method	22.247/ -0.41	57. 795/ -0.33	111.245/ 0.04	191.375/ 0.56	297.816/ 0.57	0.96

Note: •/• denotes the modal frequency/relative error.

index p is assumed to take up the value of 1.3, 1.4 and 1.5 allowing for overlap in the crack zone from the adjacent cracks. The total error is calculated for all the cases, and they are plotted in Fig. 3. It is found that when p equals 1.4, the total errors from all the 8 cases are consistently smaller. It is therefore recommended that $p = 1.4$ is used for all cases under study in this paper.

The modal frequencies of the beam obtained with the modified crack model are compared with those from the crack model by Sinha (2002) and the experiment as shown in Table 5. It should be noted that overlap in the crack zone from two adjacent cracks occurs in Cases (f) to (h), and the overlap is more significant with large crack depth. The following observations are made:

- Both the proposed model and the Sinha model give similar errors for the first three modes in Cases (a) to (g) which mainly involve single crack or two cracks without small overlap in the crack zone.
- The proposed model gives more accurate predictions than those from Sinha for all the other modal frequencies.
- The total error is less than 1% for Cases (a) to (h) which involves a single crack or a group of two cracks at different depth.
- The above observations indicate that the proposed method is accurate in the prediction of the modal frequencies, particularly for the cases where overlaps of crack zone from adjacent cracks in close proximity.

Case 4: A cantilevered beam with two cracks

Fig. 5 shows a cantilevered beam with two cracks. The same geometrical properties and material properties of the beam as in Refs. (Shifrin and Ruotolo 1999, Zheng and Fan 2001) are used, i.e., length $L = 0.8$, rectangular cross-section has width $w = 0.02$ m and height $h = 0.02$ m. The first crack is at fixed location $x_1 = 0.12$ m and has a depth $h_1 = 2$ mm. The location of the second crack varies from the left end to the right end of the beam and its depth h_2 also varies from 2 mm to 6 mm. The results obtained from the proposed method and those from Refs. (Shifrin and Ruotolo 1999, Zheng and Fan 2001) are shown in Table 6. Good agreements are observed; this further verified the correctness of the proposed method.

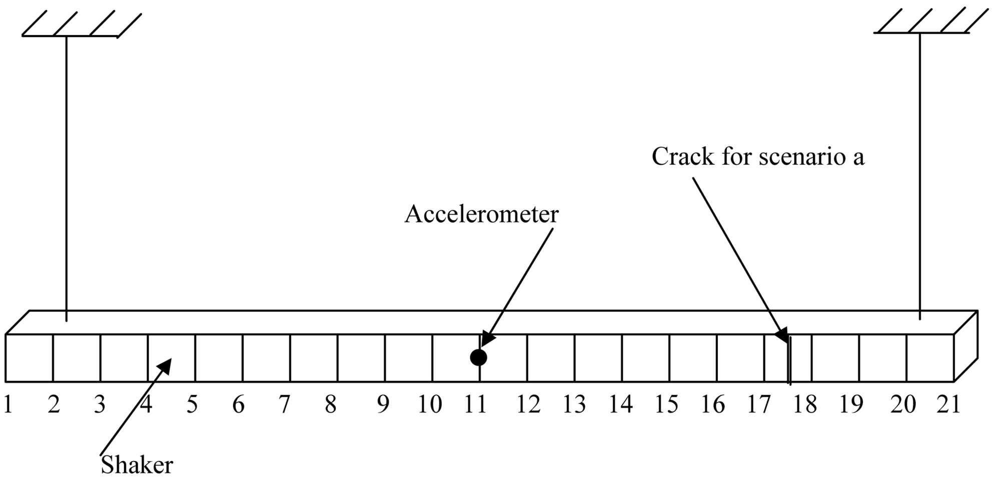

Fig. 5 Forced vibration testing of a free-free beam

Table 6 Comparison on the effect of the crack on the first three natural frequencies of the beam

Crack States	Models	Mode 1	Mode 2	Mode 3
$y_{c2} = 0.1$	Zheng and Fan (2001)	0.9942/0.9857/0.9725	0.9991/0.9968/0.9942	1.0/1.0/1.0
	Shifrin and Ruotolo(1999)	0.9941/0.9857/0.9725	0.9991/0.9970/0.9932	1.0/1.0/1.0
	Proposed method	0.9939/0.9859/0.9728	0.9993/0.9967/0.9931	1.0/1.0/1.0
$y_{c2} = 0.2$	Zheng and Fan (2001)	0.9949/0.9888/0.9802	0.9995/0.9993/0.9991	0.9986/0.9947/0.9887
	Shifrin and Ruotolo(1999)	0.9947/0.9887/0.980	0.9995/0.9993/0.9991	0.9988/0.9948/0.9886
	Proposed method	0.9947/0.9887/0.9803	0.9994/0.9993/0.9992	0.9989/0.9945/0.9885
$y_{c2} = 0.3$	Zheng and Fan (2001)	0.9952/0.9927/0.9868	0.9984/0.9950/0.9889	0.9989/0.9948/0.9889
	Shifrin and Ruotolo(1999)	0.9952/0.9926/0.9868	0.9984/0.9950/0.9889	0.9990/0.9950/0.9889
	Proposed method	0.9950/0.9928/0.9869	0.9986/0.9953/0.9888	0.9988/0.9949/0.9890
$y_{c2} = 0.4$	Zheng and Fan (2001)	0.9955/0.9947/0.9922	0.9973/0.9913/0.9810	1.0/1.0/1.0
	Shifrin and Ruotolo(1999)	0.9955/0.9947/0.9921	0.9973/0.9911/0.9805	1.0/1.0/1.0
	Proposed method	0.9957/0.9946/0.9924	0.9970/0.9914/0.9812	1.0/1.0/1.0
$y_{c2} = 0.5$	Zheng and Fan (2001)	0.9958/0.9950/0.9935	0.9981/0.9923/0.9848	0.9981/0.9931/0.9859
	Shifrin and Ruotolo(1999)	0.9957/0.9948/0.9935	0.9980/0.9921/0.9848	0.9981/0.9929/0.9859
	Proposed method	0.9955/0.9943/0.9931	0.9982/0.9924/0.9845	0.9984/0.9935/0.9856
$y_{c2} = 0.6$	Zheng and Fan (2001)	0.9959/0.9958/0.9946	0.9991/0.9968/0.9942	0.9977/0.9923/0.9806
	Shifrin and Ruotolo(1999)	0.9958/0.9958/0.9944	0.9991/0.9970/0.9932	0.9978/0.9924/0.9806
	Proposed method	0.9956/0.9954/0.9944	0.9994/0.9966/0.9933	0.9980/0.9926/0.9808
$y_{c2} = 0.7$	Zheng and Fan (2001)	0.9960/0.9960/0.9960	0.9995/0.9993/0.9991	0.9993/0.9977/0.9957
	Shifrin and Ruotolo(1999)	0.9960/0.9959/0.9960	0.9995/0.9993/0.9991	0.9993/0.9977/0.9957
	Proposed method	0.9958/0.9958/0.9959	0.9996/0.9991/0.9994	0.9996/0.9979/0.9955

Note: •/•/• denotes the frequency ratio for crack depth 2 mm, 4 mm and 6 mm.

2.2.2 From forced vibration analysis

Case 5: Laboratory tests on a free-free steel beam with single and multiple cracks

The accuracy of the modified crack model proposed in this paper is further studied in the time domain with a forced vibration experiment on a cracked steel beam. The same beam as for Case 3 was used. The test was applied to the beam for each of the crack cases listed in Table 4. Fig. 4 shows the experimental setup for the test. A sinusoidal force was applied through a Ling Dynamic LDS V450 shaker at node 5 with an amplitude varying from 1 N, 2.5 N and 5 N at a forcing frequency equal to one half of the first modal frequency measured for that particular crack case. The response was obtained with a B & K 4370 accelerometer placed at node 11 which is at the middle of the beam. Newmark method was used for the numerical solution of the system dynamic equations. The time step used was 0.0005 second. Rayleigh damping model is adopted and the damping ratios are taken as 0.007 and 0.01 for the first two modes. Figs. 6(a) and 6(b) show the experimental acceleration responses, and Figs. 6(c) and 6(d) show the computed acceleration responses at the middle of the cracked beam for Case 3(d) with 5 N excitation force. They are very

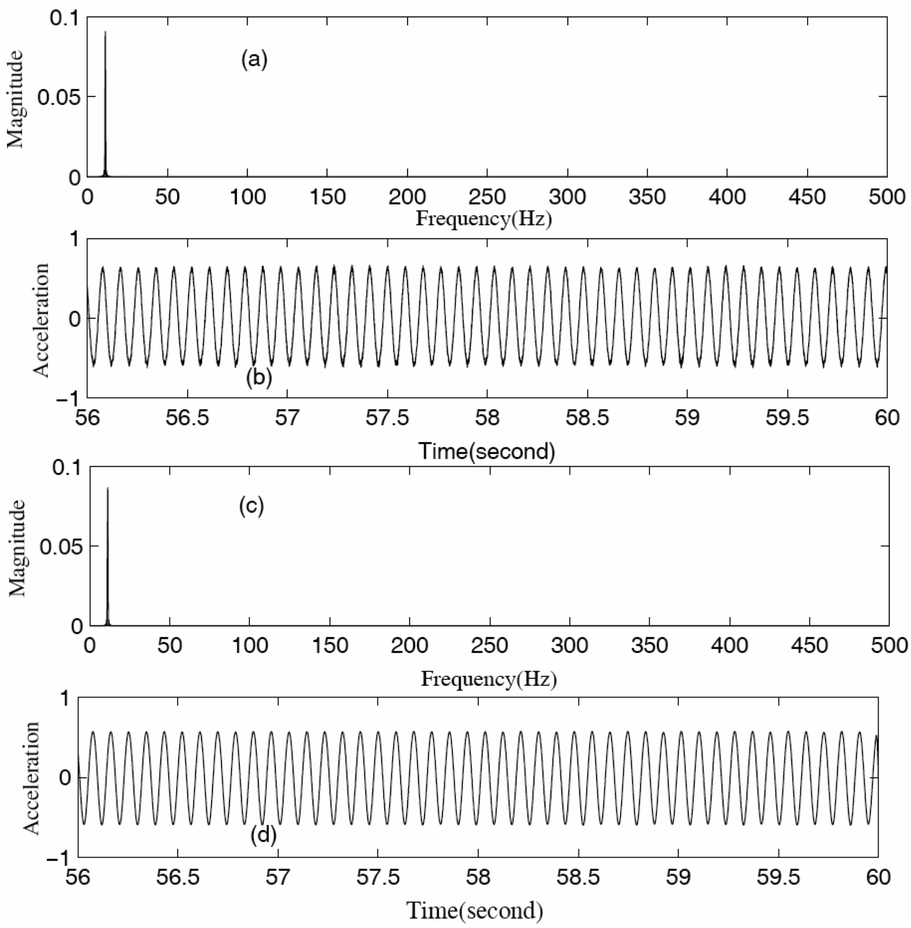

Fig. 6 Experimental and theoretical dynamic response of a free-free cracked beam for Case 3(d)

close to each other with only a small difference in the amplitude, and are typical of the responses for all the other cases under study.

The proposed model for an open crack has been shown valid for all the cases with single and two cracks, and the experimental behaviour of the beam is linear which can be predicted with the proposed linear crack model.

3. Inverse analysis

3.1 Theory

In the inverse analysis, the crack locations, depths and power index p are estimated using model updating. The penalty function method (Friswell and Mottershead 1995), based on measured accelerations, is used. The vector of updating parameters is expressed as $\vartheta = [x\ h\ p]^T$, where $x = [x_1, x_2, \ldots, x_m], h = [h_1, h_2, \ldots, h_m]$, and $p = [p_1, p_2, \ldots, p_m]$ are the vectors of locations, crack depths and corresponding power index of the m cracks. The measurement vector consists of the data of r accelerometers locate at the different places of the structure from time t_0 to t_f. Taken for example, if only one accelerometer is used, the measurement vector will be $a_e = [a_e(t_0), a_e(t_1), \ldots, a_e(t_f)]^T$, and the corresponding calculated acceleration from the FE model is $a_c = [a_c(t_0), a_c(t_1), \ldots, a_c(t_f)]^T$.

The acceleration response may be written as a first order truncated Taylor series expansion in terms of the updating parameters, giving the error vector ε, as follows

$$\varepsilon = \delta a - S\delta\vartheta \tag{10}$$

where $\delta\vartheta$ is the vectors of perturbations in the updating parameters and $\delta a = a_e - a_c$ is the error between the measured and calculated accelerations. The sensitivity matrix S is the first derivatives with respect to the updating parameters, these sensitivities can be obtained from the equation of motion, by taking first derivatives with respect to the updating parameters on both sides of Eq. (9), we have

$$[M]\left\{\frac{\partial\ddot{d}}{\partial x^i}\right\} + [C]\left\{\frac{\partial\dot{d}}{\partial x^i}\right\} + [K]\left\{\frac{\partial d}{\partial x^i}\right\} = -\left[\frac{\partial K}{\partial x^i}\right]\{d\} - c_2\left[\frac{\partial K}{\partial x^i}\right]\{\dot{d}\} \quad (i = 1, 2, \ldots, m) \tag{11}$$

$$[M]\left\{\frac{\partial\ddot{d}}{\partial h^i}\right\} + [C]\left\{\frac{\partial\dot{d}}{\partial h^i}\right\} + [K]\left\{\frac{\partial d}{\partial h^i}\right\} = -\left[\frac{\partial K}{\partial h^i}\right]\{d\} - c_2\left[\frac{\partial K}{\partial h^i}\right]\{\dot{d}\} \quad (i = 1, 2, \ldots, m) \tag{12}$$

$$[M]\left\{\frac{\partial\ddot{d}}{\partial p^i}\right\} + [C]\left\{\frac{\partial\dot{d}}{\partial p^i}\right\} + [K]\left\{\frac{\partial d}{\partial p^i}\right\} = -\left[\frac{\partial K}{\partial p^i}\right]\{d\} - c_2\left[\frac{\partial K}{\partial p^i}\right]\{\dot{d}\} \quad (i = 1, 2, \ldots, m) \tag{13}$$

Where $\{\partial d/\partial\bullet\}, \{\partial\dot{d}/\partial\bullet\}, \{\partial\ddot{d}/\partial\bullet\}$ are displacement sensitivity, velocity sensitivity and acceleration sensitivity with respect to different unknown parameters. These response sensitivities can be calculated from Eqs. (11) to (13) by direct integration method, say, Newmark method. And the response sensitivity matrix S can be constructed from these dynamic response sensitivities, in this paper, the acceleration response sensitivity is used in the inverse analysis.

As the response sensitivities have been obtained from Eqs. (11) to (13), Eq. (10) can be solved by the damped least-squares method (Tikhonov 1963) with bounds to the solution

$$\delta\vartheta = (S^T S + \lambda I)^{-1} S^T \delta a \tag{14}$$

Where λ is the non-negative damping (regularization) coefficient governing the participation of least-squares error in the solution. When the parameter λ approaches to zero, the estimated vector $\{\delta\vartheta\}$ approaches to the solution obtained from the simple least-squares method. It is note that Eq. (10) is a linear approximation, iterative algorithm is adopted to update the unknown crack parameters. Once a new model with the updated crack locations and corresponding depths and power index is generated, then the revised calculated responses and new sensitivity matrix can be obtained. This iteration process continues until a converged solution is obtained. Since the model updating minimizes a non-linear function by using an iterative algorithm, a local rather than a global minimum may be found. This may be checked by trying a number of different initial values for the unknown parameters.

Table 7 The crack parameters for the free-free beam example (initial parameter estimates: $x_1 = 400$ mm, $h_1 = 2$ mm, $p_1 = 1$)

	Case 1a		Case 1b		Case 1c	
	True	Estimate (%) error	True	Estimate (%) error	True	Estimate (%) error
Crack location x_1 (mm)	275	273.38 (-0.59)	275	273.96 (-0.38)	275	276.56 (0.57)
Crack depth h_1 (mm)	4	3.94 (-1.5)	8	8.08 (1.0)	12	12.14 (1.17)
Power index p_1	1.4	1.38 (-1.43)	1.4	1.38 (-1.43)	1.4	1.41 (0.71)
Number of iterations required	10		13		18	
Optima regularization parameter	0.039		0.047		0.063	

3.2 Computation simulation

A free-free aluminium beam with a single crack

The first example is a free-free aluminium beam used in *Study Case 1* in the forward problem. The location of the crack is 595 mm from the left end and the depth of the crack varies from 4mm to 12 mm. A finite element model of the beam is constructed using 27 Euler-Bernoulli beam elements and 56 degrees-of-freedom defining the vibration motion in the horizontal plane of the beam. A sinusoidal force $F(t) = 10*\sin(8\pi t)N$ horizontally acted at the 5th node corresponding to the finite element of the beam, one accelerometer locates at the 15th node was used to record the acceleration response of the beam, which was used to identify the parameters of the crack. The sampling rate is 1000 Hz and time during is 2 seconds. Rayleigh damping model is adopted and the damping ratio is taken as 0.01 for the first two modes, Newmark method is adopted for calculating the response and response sensitivity with respect to crack parameters. Time response of the first two seconds is used in the identification, i.e., 2000 data points are used. Table 7 shows the identified results. The required step for convergence and the optimal regularization parameter for each case are also listed in Table 7. From this table one can see, the identified results match the target values very well. And they are all converged to the true values in several steps. This shows the presented method is efficient for crack identification.

3.3 Experiment verification

The proposed method for crack identification is further studied with a forced vibration experiment on a cracked steel beam. To identify the parameters of the crack, the measured acceleration response and input excitation force are needed and also, like many other model updating methods, an initial estimate of the updating parameters are required.

The same beam as for Case 3 in the forward problem was used. A sinusoidal force was applied through a Ling Dynamic LDS V450 shaker at node 5 corresponding to the finite element model with an amplitude 5 N at a forcing frequency equal to one half of the first modal frequency measured for that particular crack case. The acceleration response was obtained with a B & K 4370 accelerometer placed at node 11 which is at the middle of the beam which was used to identify the

Table 8 Single crack identification for the test beam (initial parameter estimates: $x_1 = 1300$ mm, $h_1 = 2$ mm, $p_1 = 1$)

	Case a		Case b		Case c	
	True	Estimate (%) error	True	Estimate (%) error	True	Estimate (%) error
Crack location x_1 (mm)	1720	1713.43 (-0.38)	1720	1726.26 (0.36)	1720	1708.12 (-0.69)
Crack depth h_1 (mm)	3	2.91 (-3.0)	6	6.08 (1.33)	9	9.22 (2.44)
Power index p_1	1.4	1.37 (-2.14)	1.4	1.37 (-2.14)	1.4	1.37 (-2.14)
Number of iterations required	10		13		15	
Optima regularization parameter	0.019		0.033		0.049	

Table 9 Two cracks identification for the test beam (initial parameter estimates: $x_1 = 1300$ mm, $h_1 = 2$ mm, $p_1 = 1$, $x_2 = 1200$ mm, $h_2 = 2$ mm, $p_2 = 1$)

	Crack one			Crack two		
	x_1 (mm)	h_1 (mm)	p_1	x_2 (mm)	h_2 (mm)	p_2
True	1720	9	1.4	1660	9	1.4
Estimated (%) error	1728.41 (0.49)	8.72 (-3.11)	1.35 (-3.57)	1645.31 (-0.88)	8.85 (-1.67)	1.36 (-2.86)
Iteration number	22					
Regularization parameter	0.054					

parameters of the crack. Time history of the input excitation force is also recorded as the input force for calculating the numerical response of the beam. Newmark method was used to calculate the dynamic responses of the beam. Rayleigh damping model is adopted and the damping ratios are taken as 0.007 and 0.01 for the first two modes. The time step used was 0.0005 second. The acceleration response data of the first three seconds was used for the crack parameters identification.

It is known that the finite modeling error in the undamaged structure has significant effect on the accuracy of identified result, and most often, two-stage identification approach is adopted, the initial finite element is updated in the first stage to obtain a good representation of the intact structure, then damage detection is performed in the second stage. From Table 5 one can see, the frequencies of the intact from eigenvalue analysis match the experimental frequencies very well, this shows the finite element of the beam is good enough and thus it can be used for crack identification.

Four crack cases: case a to case c, case f, corresponding to single crack and two cracks in the beam, are studied as listed in Table 4. Table 8 shows the identified results for case a to case c. The required step for convergence and the optimal regularization parameter for each study case are also listed in Table 8. From this table one can see, the identified results match the target values well. And they are all converged to the true values in a few steps except a larger relative percentage in crack depth. Table 9 shows the identified results for case f. The required step for convergence and the optimal regularization parameter for each study case are also listed in Table 9. From this table one can see, the match between the identified results and the target values is still good, although the identified results are not so good as the single crack case. The number of required steps for convergence is a little more than the single crack case.

4. Conclusions

A linear modified crack model is presented for an open crack in a beam with an expression for the effective length of the crack zone which is a function of the crack depth. The crack model is validated with both free and forced vibration analysis using Euler-Bernoulli beam elements. More accurate results are obtained from using the proposed model when compared with those obtained from other existing models particularly when overlap of crack zones from adjacent cracks occurs. A new approach for crack identification is proposed based on finite element model updating from response sensitivity with respect to crack parameters. Computation simulation and experimental work show that the proposed method is efficient for crack identification with high accuracy.

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