Structural Engineering and Mechanics, Vol. 35, No. 2 (2010) 191-203 191
DOI: http://dx.doi.org/10.12989/sem.2010.35.2.191

Post-buckling analysis of piles by perturbation method

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(Received July 24, 2008, Accepted January 6, 2010)

Abstract. To investigate the critical buckling load and post-buckling behavior of an axially loaded pile entirely embedded in soil, the non-linear large deflection differential equation for a pinned pile, based on the Winkler-model and the discretionary distribution function of the foundation coefficient along pile shaft, was established by energy method. Assuming that the deflection function was a power series of some perturbation parameter according to the boundary condition and load in the pile, the non-linear large deflection differential equation was transformed to a series of linear differential equations by using perturbation approach. By taking the perturbation parameter at middle deflection, the higher-order asymptotic solution of load-deflection was then found. Effect of ratios of soil depth to pile length, and ratios of pile stiffness to soil stiffness on the critical buckling load and performance of piles (entirely embedded and partially embedded) after flexural buckling were analyzed. Results show that the buckling load capacity increases as the ratios of pile stiffness to soil stiffness increasing. The pile performance will be more stable when ratios of soil depth to pile length, and soil stiffness to pile stiffness decrease.

Keywords: pile buckling load capacity; post-buckling equilibrium; perturbation approach; ratio of soil depth to pile depth; ratio of pile stiffness to soil stiffness.

1. Introduction

When site is of quite weak subsoil, or of large un-embedded length, a slender pile is often going to has buckling failure. Therefore, an analysis on buckling response is paramount important for slender piles. This has been studied by many researchers for quite long time. Davisson and Robinson (1936, 1965) found the solution of buckling load capacity by using analogue computer. Toakley (1965) reported the influence of axial force variation on buckling load capacity for a fully embedded pile with pinned end conditions. Reddy and Valsangkar (1970) analyzed buckling load capacity of fully and partially embedded piles using energy method. Hu (1973) proposed a series of experiential formulas to calculate the equivalent length of buckling piles. Poulos (1980) applied elastic thoery to analyzing the buckling load capacity. Bowles (1987) adopted Finite Element Method to solve this problem. Zhao (1987, 1990, 1996) studied the equivalent length of piles by using energy method taking into account the lateral resistance, and verified the calculated results by comparison to measured data of test model. Heelis and Pavlovic (2004) obtained the solution by

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power series. Rajasekaran (2008) studied buckling of fully or partially embedded beams and columns by method of differential quadrature (DQ) and harmonic differential quadurature (HDQ). Shanker (2007) predicted critical buckling load of pile embedded in liquefiable soil under partial to full loss of lateral support over a portion of pile length.

Nearly all of the researches focused on solving the critical buckling load and associated pile length. However not many attention has been paid on the post-buckling behavior. This is because these studies were based on assumptions: (1) Strain and displacement of piles are so small that they can be neglected. (2) Deflection of pile will reach infinity when the pile reaches buckling. In order to describe realistic behavior of the buckling pile, characteristics of large deflection must be taken into account.

Budkowska and Szymczak (1997) studied the initial post-buckling behavior of piles, but many problems are remained and to be solved. In this study, perturbation method and non-linear large-deflection differential equation were employed to deduce the higher-order asymptotic solution of load-deflection for studying the buckling load capacity and post-buckling behavior of partially embedded pile.

2. Mechanical model

2.1 Soil reaction

Assuming that no lateral forces apply to piles, the large deflection (P-delta effect) induced bending moment on pile shaft will lead to earth-pressure acting on two sides of piles. The simplest method of estimating the ultimate lateral resisting capacity of the pile is to consider the static equilibrium. The exertion of soil reaction depends on not only the buried depth of a pile but also the distribution of deflection, especially for slender piles. Therefore the elastic theory is applicable. In order to simplify the calculation, the following assumptions are made for analyses:

· The active earth-pressure acting on the back of the pile is neglected.

· Winkler elastic model is applicable to the exertion of soil reaction.

The Winkler model represents a linear elastic characteristics of soil, and relevant foundation coefficient varying with depth of soil. Assumptions of distribution of foundation coefficient along the pile shaft have been studied for many times by Chang (1937) considering a constant distribution along the whole depths. Terzaghi (1955) presented constant and linear distributions for cohesive soil and cohesionless soil respectively. Palmer and Thompson (1949, 1954) gave the elastic foundation analysis for laterally loaded piles with assumed discretionary order of foundation coefficient functions. All these methods can be classified to three kinds: Constant, linear and nonlinear, which are shown in Fig. 1. In China, there are two kinds of methods often used nowadays, the first one is termed as *m* method, which was introduced into China by K.G. Silin in 1962 assuming the foundation coefficient increases linearly along the depth of pile shafts. The other is termed as *c* method, which was introduced by Kubo Kouichi in 1942 with assuming that the foundation coefficient distribution is a parabolic function of the pile shaft depth.

In this study, the discretionary order of coefficient distribution was taken into account to analyze the buckling load capacity and post-buckling behavior of piles.

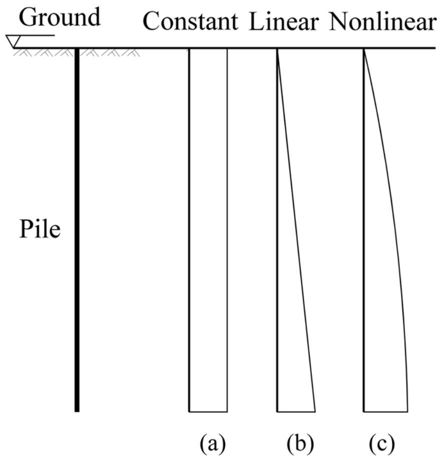

Fig. 1 Sketch map of subsoil coefficient

2.2 Mechanical model for pile-soil system

In recent years, long concrete piles are more and more applied to engineering practical projects. Therefore, study on the buckling load capacity and post-buckling behavior of slender piles is of great significance. To simplify the problem, two-pinned elastic piles were considered. A great number of analyses show that self-weight and side friction do little effect on buckling load capacity. Hence non-load-transfer is assumed in this study. The mechanic model for pile-soil system is shown in Fig. 2.

The total potential energy is a function of bending potential energy, elastic potential energy of soil, potential energy of exterior load, as shown

$$\Pi = \frac{EI}{2}\int_0^l y''^2(1-y'^2)^{-1/2}dx + \frac{kb}{2}\int_0^h (h-x)^m y^2 dx - P\int_0^l 1-\sqrt{1-y'^2}dx \tag{1}$$

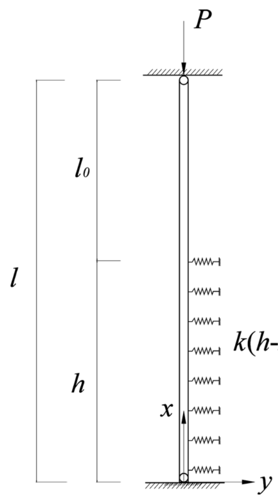

Fig. 2 Mechanical model of the pile

Where EI (N·m^2) is flexural rigidity of a pile; k (N/m^4) is stiffness of subsoil; b (m) is the calculating width of a pile; y is the deflection function of a pile; l (m) is the length of a pile; h (m) is the buried depth of a pile; m (non-dim.) is the power of the distribution function of the foundatio coefficient.

The formula above is the functional of the deflection. The potential energy of the pile-soil system is the minimum when in the stable situation, that is $\delta\Pi = 0$. Euler formula can be used to transform the variation equation into differential form

$$F_y - \frac{d}{dx}F_{y'} + \frac{d^2}{dx^2}F_{y''} - \ldots + (-1)^n \frac{d^n}{dx^n}F_{y^{(n)}} = 0 \tag{2}$$

Where $F = \frac{EI}{2}y''^2(1-y'^2)^{-1/2} + \frac{kb}{2}(h-x)^m y^2 - P(1-\sqrt{1-y'^2})$.

According to formula (2), the following equation can be obtained

$$\begin{aligned} &EIy^{(4)}(1-y'^2)^{-1/2} + 4EIy'y''y'''(1-y'^2)^{-3/2} \\ &+ EIy''^3(1+3y'^2)(1-y'^2)^{-5/2} + Py''(1-y'^2)^{-3/2} + k(h-x)^m by = 0 \end{aligned} \tag{3}$$

Expand $(1-y'^2)^{-1/2}, (1-y'^2)^{-3/2}$ and $(1-y'^2)^{-5/2}$, the following non-dimensional indices are available

$$\bar{x} = \frac{\pi}{l}x, \quad \bar{y} = \frac{y}{l}, \quad \bar{P} = \frac{Pl^2}{EI\pi^2}$$

Still use the originally symbols, the formula (3) can be transformed to

$$\begin{aligned} &y^{(4)}(1+\pi^2 y'^2 + \pi^4 y'^4 + \pi^6 y'^6 + \ldots) + 4\pi^2 y'y''y'''(1 + 2\pi^2 y'^2 + 3\pi^4 y'^4 + \ldots) \\ &+ \pi^2 y''^3(1 + 6\pi^2 y'^2 + 15\pi^4 y'^4 + \ldots + Py''(1 + 6\pi^2 y'^2 + 15\pi^4 y'^4 + \ldots) + \alpha^{4+m} f(x) y = 0 \end{aligned} \tag{4}$$

Where $f(x) = \begin{cases} 0 & x > \frac{h}{l}\pi \\ \frac{l^4}{\pi^4}\left(h - \frac{xl}{\pi}\right)^m & 0 < x < \frac{h}{l}\pi \end{cases}$, $\alpha = \left(\frac{kb}{EI}\right)^{\frac{1}{4+m}}$

3. Perturbation solution

The perturbation approach was initially applied for celestial mechanics. Furthermore, it was not used for some other fields of natural science until the correctness of asymptotic solution was proved by Poincáre. A developed approach was done and used to analyze post-buckling behavior of plates and shells by Shen (1988, 2002). It also be applied to analyze dynamic elastic local buckling of a pile subjected to an axial impact load (Yang and Ye 2002). This method was employed for researching as shown in this paper.

According to Fig. 2, the boundary condition of a two-pinned pile is

$$\begin{cases} x = 0, \ y(0) = 0, \ y''(0) = 0 \\ x = l, \ y(l) = 0, \ y''(l) = 0 \end{cases} \tag{5}$$

Assuming that Eq. (5) can be rewritten in power series

$$\begin{cases} y = y(x, \varepsilon) = \sum_{i=1} \varepsilon^i y_i(x) \\ P = P(x, \varepsilon) = \sum_{i=0} \varepsilon^i P_i(x) \end{cases} \tag{6}$$

Where ε is a perturbation parameter.

Substituting Eq. (4) into Eq. (2), perturbation formula in each order can be obtained as the following

$O(\varepsilon)$: $y_1^{(4)} + P_0 y_1^{(2)} + \alpha^{4.5} f(x) y_1 = 0$

Considering the boundary condition, the solution is

$$y_1 = A_{10}^{(1)} \sin x \tag{7}$$

Substituting (7) into the first order perturbation formula, it leads to

$$P_0 = 1 + \alpha^{4+m} \tag{8}$$

$O(\varepsilon^2)$:

$$y_1^{(4)} + P_0 y_1^{(2)} + \alpha^{4+m} f(x) y_1 = P_1 A_{10}^{(1)} \sin x \tag{9}$$

Assuming that $y_2 = A_{10}^{(2)} \sin x + A_{20}^{(2)} \sin 2x$

Substitute it into (9) and $P_1 = 0$, $y_2 = 0$ can be got.

$O(\varepsilon^3)$:

$$y_3^{(4)} + P_0 y_3^{(2)} + \alpha^{4.5} f(x) y_3 = P_2 A_{10}^{(1)} \sin x + \pi^2 (A_{10}^{(1)})^3 B_{31} \sin x + \pi^2 (A_{10}^{(1)})^3 B_{33} \sin 3x \tag{10}$$

Assuming that

$$y_3 = A_{10}^{(3)} \sin x + A_{30}^{(3)} \sin 3x \tag{11}$$

Substituting (11) into (10) and this leads to

$$P_2 = -\pi^2 B_{31} (A_{10}^{(1)})^2 \tag{12}$$

$O(\varepsilon^4)$:

$$y_4^{(4)} + P_0 y_4^{(2)} + \alpha^{4+m} f(x) y_4 = P_3 A_{10}^{(1)} \sin x \tag{13}$$

Just like the second order, the $P_3 = 0$ and $y_4 = 0$ is as the same.

$O(\varepsilon^5)$:

$$\begin{aligned} y_5^{(4)} + P_0 y_5^{(2)} + \alpha^{4+m} f(x) y_5 &= P_4 A_{10}^{(1)} \sin x + \pi^4 (A_{10}^{(1)})^5 B_{41} \sin x \\ &+ \pi^4 (A_{10}^{(1)})^5 B_{43} \sin 3x + \pi^4 (A_{10}^{(1)})^5 B_{45} \sin 5x \end{aligned} \tag{14}$$

Assuming that

$$y_5 = A_{10}^{(5)}\sin x + A_{30}^{(5)}\sin 3x + A_{50}^{(5)}\sin 5x \tag{15}$$

Substitute (15) into (14), the following equation is obtained

$$P_4 = -\pi^4 B_{51}(A_{10}^{(1)})^4 \tag{16}$$

$O(\varepsilon^5)$ ……

So asymptotic solution of (5) can be found

$$y = \varepsilon A_{10}^{(1)}\sin x + \varepsilon^3 A_{30}^{(3)}\sin 3x + \varepsilon^5(A_{30}^{(5)}\sin 3x + A_{50}^{(5)}\sin 5x) + \ldots \tag{17}$$

$$P = 1 + \alpha^{4+m}f(x) + (-\pi^2 B_{31})(A_{10}^{(1)}\varepsilon)^2 + (-\pi^4 B_{51})(A_{10}^{(1)}\varepsilon)^4 + \ldots \tag{18}$$

Where all coefficients can be described as a function of $A_{10}^{(1)}$.

$$A_{10}^{(3)} = 0 \tag{19}$$

$$A_{30}^{(3)} = \frac{\pi^2(A_{10}^{(1)})^3 B_{33}}{81 - 9P_0 + \alpha^{4.5}f(x)} \tag{20}$$

$$A_{30}^{(5)} = \frac{B_{53}\pi^4}{81 - 9P_0 + \alpha^{4.5}f(x)}(A_{10}^{(1)})^5 \tag{21}$$

$$A_{50}^{(5)} = \frac{B_{55}\pi^4}{256 - 16P_0 + \alpha^{4.5}f(x)}(A_{10}^{(1)})^5 \tag{22}$$

$$B_{31} = \frac{3}{8}P_0 - \frac{1}{2} \tag{23}$$

$$B_{33} = \frac{3}{8}P_0 - \frac{3}{2} \tag{24}$$

$$B_{51} = a + \frac{3}{4}b + \frac{5}{8}c \tag{25}$$

$$B_{53} = -\frac{5}{16}c - \frac{b}{4} \tag{26}$$

$$B_{55} = \frac{c}{16} \tag{27}$$

$$a = \left(\frac{99}{2}P_0 - 27B_{31} - 477\right)C_{30}B_{33} + \frac{15}{8}P_0 - \frac{3}{2}B_{31} - 9 \tag{28}$$

$$b = \left(1530 + 36B_{31} - \frac{279}{2}P_0\right)C_{30}B_{33} + \frac{3}{2}B_{31} - \frac{15}{4}P_0 + 24 \tag{29}$$

$$c = (90P_0 - 1080)C_{30}B_{33} + \frac{15}{8}P_0 - 15 \tag{30}$$

$$C_{30} = \frac{1}{81 - 9P_0 + \alpha^{4.5} f(x)} \tag{31}$$

Substituting $x = \pi/2$ into (17), the maximum deflection of the pile can be found

$$y_{\max} = \varepsilon A_{10}^{(1)} + D_1(\varepsilon A_{10}^{(1)})^3 + D_2(\varepsilon A_{10}^{(1)})^5 \ldots \tag{32}$$

Where, $D_1 = -A_{30}^{(3)}\left(\frac{\pi}{2}\right)$;

$D_2 = -A_{50}^{(3)}\left(\frac{\pi}{2}\right) + A_{50}^{(5)}\left(\frac{\pi}{2}\right)$.

Then, (32) can be transformed to

$$\varepsilon A_{10}^{(1)} = y_{\max} - D_1 y_{\max}^3 - [D_2 - 3(D_1)^2] y_{\max}^5 + \ldots \tag{33}$$

The formula (18) represents load distribution along the pile shaft to a sine curve deflection. Assuming that the axially load at the pile top end is the mean average of loads distributed along the pile shaft, this leading to

$$\bar{P} = \frac{\int_0^{\pi} P(x)dx}{\pi} \tag{34}$$

Substituting (18) and (33) into(34)

$$\bar{P} = E_0 + E_1(y_{\max})^2 + E_2(y_{\max})^4 + \ldots \tag{35}$$

Where $E_0 = 1 + \frac{\alpha^{4+m}}{\pi} \int_0^{\pi} f(x)dx$

$E_1 = -\pi \int_0^{\pi} B_{31} dx$

$E_2 = 2\pi D_1 \int_0^{\pi} B_{31} dx - \pi^3 \int_0^{\pi} B_{51} dx$

Let $y_{\max} = 0$ in formula (35), then

$$P_{cr} = \left(1 + \frac{\alpha^{4+m}}{\pi} \int_0^{\pi} f(x)dx\right) \frac{EI\pi^2}{l^2} \tag{36}$$

4. Verification

Responses of compressed pile were predicted using the method discussed previously and the method in Ref. 23, then the calculating result is shown in Fig. 3. When there is no deflection, the vertical load P on pile top is 1, showing that the buckling load capacity is the one from classical Eula-solution. For $y_{\max}$ of 0.05, P turns to 1.003, while analytical solution is $P = 1.003$. When $y_{\max}$ is 0.1, the P turns to 1.013, and the result of the analytical solution is $P = 1.012$, and the error is 0.098%; when $y_{\max}$ is 0.2, the result turns to $P = 1.055$, and the result of the analytical solution is $P = 1.057$, and the error is 0.189%. It can be found in Fig. 3 that the perturbation approach is able to predict reliable responses for piles of small deflection.

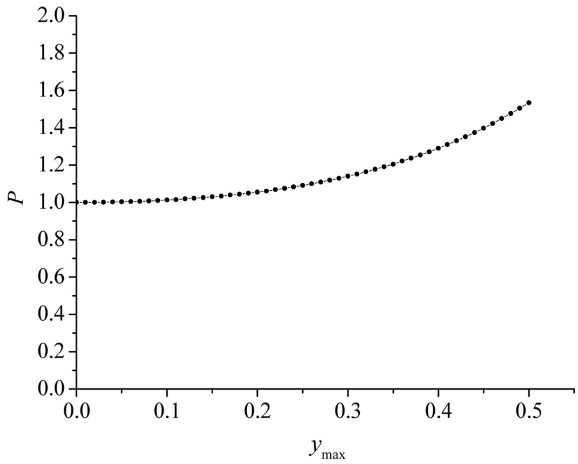

Fig. 3 Post-buckling behavior of pinned pile

5. Analysis on influential factors

Since the generalized solution can solve out the precisely buckling load capacity for entirely embedded piles and post-buckling behavior of piles, it can be applied in analyzing the influential factors of piles in different burying conditions.

5.1 Entirely embedded piles

For different stiffness ratios of pile to soil, the buckling load capacity of piles is different depending on the length of pile, which is shown in Fig. 4. For $k/EI = 10^{-5}$, the buckling load capacity of pile decreases with length of pile, and then approach some constant. For $k/EI = 10^{-4}$ and 10^{-3}, the buckling load capacity of pile curve is descending until the load reach a certain value, then the curvet turns to be ascending, when the $k/EI > 10^{-2}$, the buckling load capacity of pile increasing

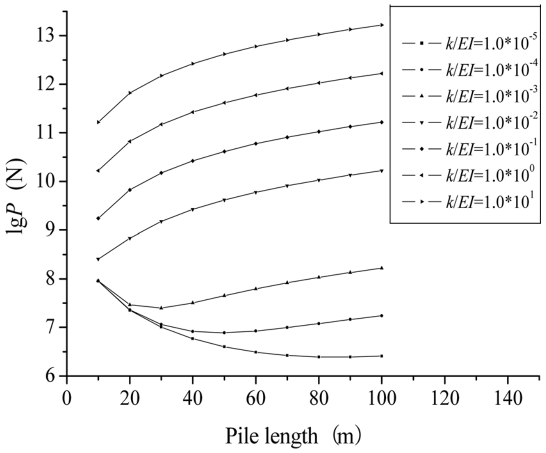

Fig. 4 Effect of pile length ratios on buckling load capacity

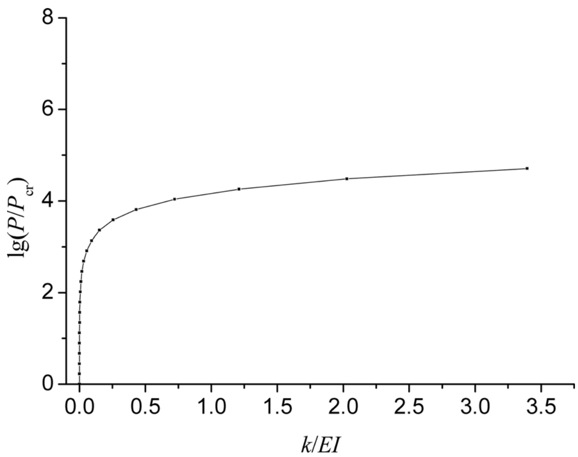

Fig. 5 Effect of stiffness ratios of pile to soil on buckling load capacity

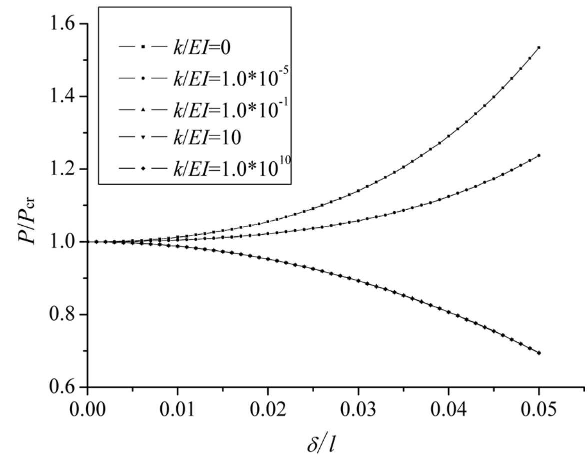

Fig. 6 Effect of stiffness ratios of pile to soil on post-buckling behavior of deflection response

with the length.

Let the bury depth of pile is 30 meters, and calculating width is 1 meters, the slenderness ratio is 30. The curve of non-dimensional buckling load capacity of pile V.S. stiffness ratio is shown in Fig. 5 remarkably varies to stiffness ratios ranging from 0 to 0.25, then being for the rest slenderness ratios.

Considering five stiffness ratios of 0, 1.0×10^{-5}, 1.0×10^{-1}, 1.0×10, 1.0×10^{10} N/m^3, the calculated post-buckling bearing behavior of pile is shown in Fig. 6. Fig. 6 indicates that, when the stiffness ratio is close to zero, the post-buckling bearing behavior of pile can be described as an ascending curve, when the $k/EI = 10^{-5}$, the ascending curve turns to be more gentle, and the post-buckling bearing behavior turns to be descending for k/EI ranging from 10^{-1} to ∞.

5.2 Partially embedded piles

According to Codes for design of highway bridges and culverts (JTJ024-85), the foundation coefficient is shown in Table 1. Elastic Modulus of a pile is shown in Table 2 bury-depth ratio is about 0.4 to 1.0. Assuming that a pile is 30 meters in length with 1 meter in diameter. The stiffness ratios of piles to soil range from 0.326 to 0.75 as shown in Table 1 and Table 2.

Take the stiffness ratio 0.326 and 0.75 to analyze the post-buckling behavior of pile, then compare

Table 1 Foundation coefficient

No. of layers	Soil category	k (kN/m^4)	Note
1	Clay or sullage	3000-5000	$I_L \geq 1$
2	Clay or thin sand	5000-10000	$1 > I_L \geq 0.5$
3	Clay, Thin sand medium sand	10000-20000	$0.5 > I_L \geq 0$
4	Hard, semi-hard clay thick sand	20000-30000	$I_L < 0$
5	Gravel	30000-80000	
6	Densified scree with thick sand	80000-120000	

Note: Where I_L refers to liquid limit of clay

Table 2 Elastic modulus of concrete (10^4 MPa)

Concrete Grade	10	15	20	25	30
Elastic Modulus	1.85	2.30	2.60	2.85	3.00

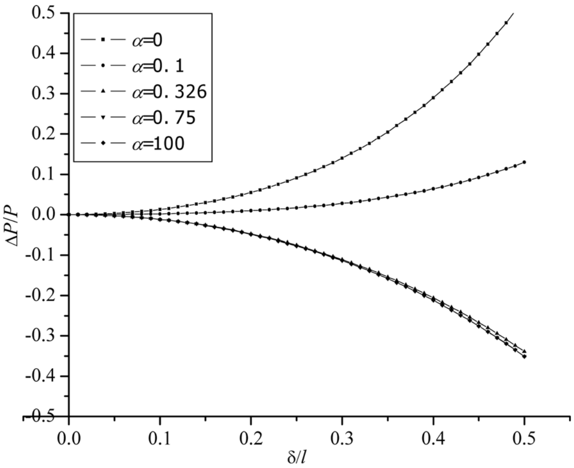

Fig. 7 Effect of stiffness ratios of pile shaft to soil on behavior of deflection response for $l_0 = 0.6l$

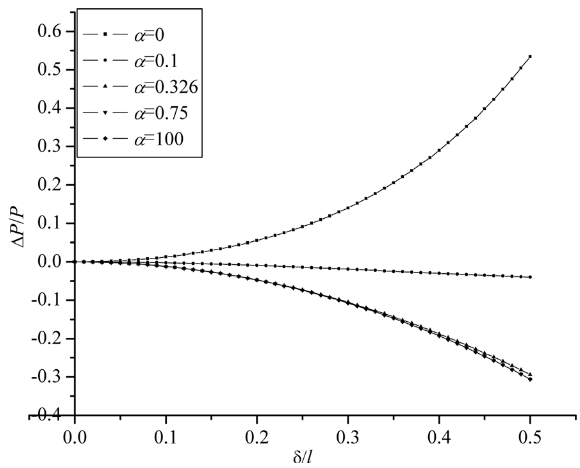

Fig. 8 Effect of stiffness ratios of pile shaft to soil on behavior of deflection response for $l_0 = 0.35l$

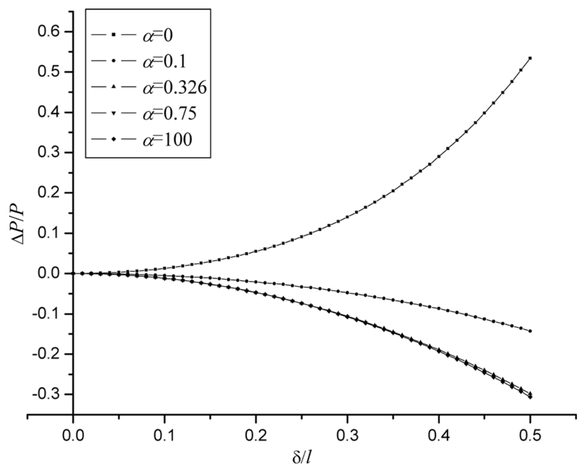

Fig. 9 Effect of stiffness ratios of pile shaft to soil on behavior of deflection response for $l_0 = 0.1l$

it to the results when stiffness ratio is 0, 0.1, 100. Fig. 7 show the situation when bury-depth ratio is 0.4, as the stiffness ratio increasing, the ascending curve turns to be descending gradually, when do analysis according to stiffness ratio recommended, the results are almost as the same as the situation the stiffness ratio reach the infinity. Fig. 8 and Fig. 9 respectively show the post-buckling behavior of pile with bury-depth ratio of 0.65 and 0.9, and they are similar to curves in Fig. 3. The difference is that as the bury-depth ratio increasing, the ascending curve turns to be descending.

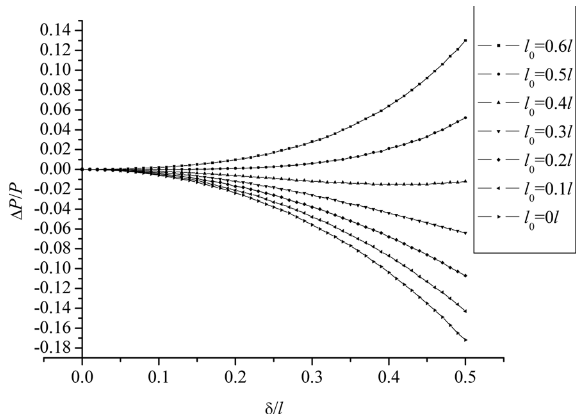

Fig. 10 Effect of depth ratios on behavior of deflection response for $\alpha = 0.1$

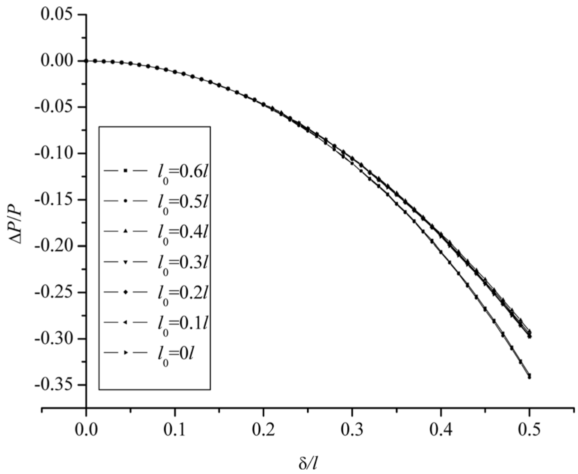

Fig. 11 Effect of depth ratios on behavior of deflection response for $\alpha = 0.307$

Fig. 10 and Fig. 11 show the post-buckling behavior of pile with stiffness ratio of 0.1 and 0.326. When $\alpha = 0.1$ ($\alpha = (kb/EI)^{1/4+m}$), as the increasing of bury-depth ratio, the ascending post-buckling behavior curve turns to be descending. When $\alpha = 0.326$, as the bury-depth ratio increasing, the post-buckling behavior curve all are descending, and curve with larger bury-depth ratio is more obvious in descending.

As we all know, ascending curve indicates that the structural will damage ductile, when the structural reach the ultimate limit state, it will keep on absorbing energy, instead of ruining immediately, but the descending one is on the contrast. For the pile with large bury-depth ratio, although the post-buckling behavior is unstable, the buckling load is still large, and its behavior is better in common time, whenas the pile will be acted by huge instantaneous load and absorb huge energy, so the problem is meaningful in dynamic buckling. Besides, in the region of weak subsoil, higher grade concrete is supposed to be applied, in this way, the buckling load can be enhanced, as well as ductility in post-buckling.

6. Conclusions

From the above study, some conclusions can be drawn as follows:

(1) Based on energy method and discretionary distribution function of foundation coefficients along pile shaft, the generalized nonlinear differential equation for large deflection vertical loaded pile was established firstly. Then, the perturbation parameter was introduced to transform the equation to a series of linear differential equations to be solved, by substituting the perturbation parameters, and take the deflection function according with the boundary condition into account. Finally, the generalized nonlinear higher-order asymptotic solution of post-buckling behavior of a pile was obtained.

(2) The entirely embedded pile was analyzed by this method. It is found that the larger a bury-depth ratio is, higher the buckling load capacity of pile is, more unstable the pile is to be in post-

buckling state, and the critical buckling load of pile significantly varies for stiffness ratios ranging from 0 to 0.25.

(3) For partially embedded piles, the buckling load capacity of pile increases with the bury-depth ratio increasing. The post-buckling behavior of pile turns to be unstable.

(4) The critical buckling load increases with the stiffness of soil increasing, but the pile may ruin as brittleness, thus, in the region where buckling behavior of pile must be considered, the high grade concrete is supposed to be applied, and it will be of significant to study dynamic buckling behavior of pile.

Acknowlegements

The work presented is supported by National Natural Science Foundation of China (No. 50908024).

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