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An exact solution for free vibrations of a non-uniform beam carrying multiple elastic-supported rigid bars

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Abstract. The purpose of this paper is to utilize the numerical assembly method (NAM) to determine the exact natural frequencies and mode shapes of a multi-step beam carrying multiple rigid bars, with each of the rigid bars possessing its own mass and rotary inertia, fixed to the beam at one point and supported by a translational spring and/or a rotational spring at another point. Where the fixed point of each rigid bar with the beam does not coincide with the center of gravity the rigid bar or the supporting point of the springs. The effects of the distance between the "fixed point" of each rigid bar and its center of gravity (i.e., eccentricity), and the distance between the "fixed point" and each linear spring (i.e., offset) are studied. For a beam carrying multiple various concentrated elements, the magnitude of each lumped mass and stiffness of each linear spring are the well-known key parameters affecting the free vibration characteristics of the (loaded) beam in the existing literature, however, the numerical results of this paper reveal that the eccentricity of each rigid bar and the offset of each linear spring are also the predominant parameters.

Keywords: rigid bar; numerical assembly method; exact solution; natural frequency; mode shape; eccentricity; offset.

1. Introduction

The free vibration characteristic of a beam carrying various concentrated elements is an important information for the engineers, thus, a lot of reports were published in this area. Rama Bhat and Wagner (1976) presented the natural frequencies of a uniform cantilever beam carrying a tip mass with its center of gravity different from its attaching point. Gürgöze (1986) and Gürgöze and Batan (1986) researched the natural frequency of a restrained cantilever beam carrying a tip heavy body by using the Dunkerley's and Southwell's methods and the numerical solution of the transcendental frequency equation, respectively. Liu and Huang (1988) studied the vibrations of a constrained beam carrying a heavy tip body with elastic support and eccentricity. Maurizi *et al.* (1990) derived the frequency equations of a uniform beam with one end carrying a tip mass and the other end elastically restrained against rotation or translation. Farghaly (1992) presented the exact natural frequencies and mode shapes of an axially loaded cantilever beam with an elastically mounted end mass of finite length. Zhou (1997) studied the exact natural frequencies and mode shapes of a

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cantilever beam carrying a heavy tip mass supported by a translational spring and a rotational spring. Naguleswaran (2002) found the natural frequencies of an Euler-Bernoulli beam on elastic end supports and with up to three-step changes in cross-sections by equating the fourth order determinant to zero. Wu and Chen (2007a, b) studied the free vibration problem of a non-uniform beam with various boundary conditions and carrying multiple concentrated elements by lumped-mass and continuous-mass transfer matrix methods, respectively, but the attaching point for each set of concentrated elements is at the same station on the beam. Lin (2008) presented the natural frequencies and mode shapes of a multi-span and multi-step beam carrying a number of concentrated elements. From the foregoing literature review one finds that the literature regarding determination of natural frequencies and mode shapes of a non-uniform beam with all concentrated elements attached to the rigid bars is little. Therefore, the objective of this paper is to extend the theory of numerical assembly method (NAM) to investigate the free vibration characteristics of a multi-step beam carrying multiple rigid bars, with each of the rigid bars possessing its own mass and rotary inertia, fixed to the beam at one point and supported by a translational spring and/or a rotational spring at another point. Where the fixed point of each rigid bar with the beam does not coincide with the center of gravity of the rigid bar. For convenience, a rigid bar supported by a translational spring and/or rotational spring is called an elastic-supported rigid bar, in this paper.

2. Equation of motion and displacement function

Fig. 1 shows the sketch of a multi-step pinned-pinned beam with arbitrary simple supports carrying arbitrary elastic-supported rigid bars with each rigid bar fixed on the beam at a station u and possessing its own mass M_u and rotary inertia J_u, and supported by a translational spring k_{Tu} and a rotational spring k_{Ru}. The points corresponding to the locations of simple pinned supports, changes in cross-sections and elastic-supported rigid bars are called "stations". The numbers $1, 2, \ldots, u, \ldots, n$

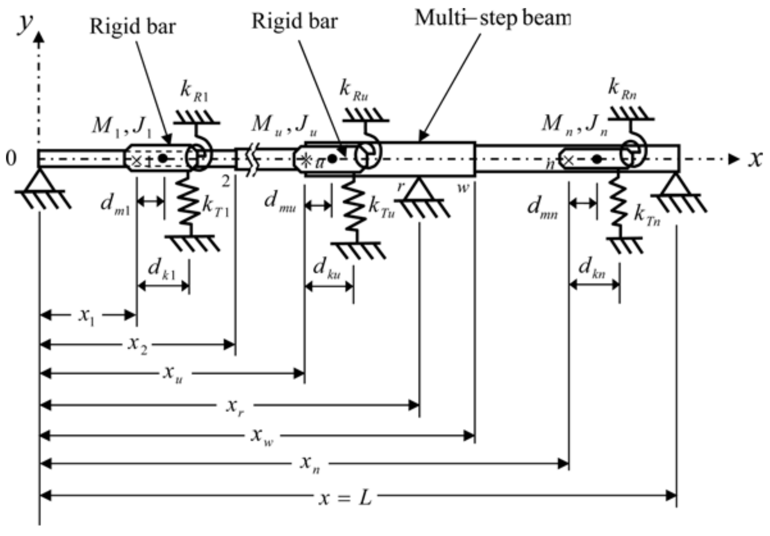

Fig. 1 Sketch for a pinned-pinned beam with intermediate rigid (pinned) supports, multi-step changes in cross-sections and carrying multiple rigid bars

along the x-axis refer to the numberings of the stations. Each of the symbols "×" denotes the fixed point of an elastic-supported rigid bar with the beam and each of the symbols "•" denotes the center of gravity of the rigid bar. Besides, d_{mu} is the distance between the fixed point of the rigid bar and its center of gravity, and d_{ku} is the distance between the fixed point and the translational and rotational springs supporting the rigid body at the uth station.

The differential equation of motion for the *i-th* beam segment is given by

$$EI_i\frac{\partial^4 y_i(x,t)}{\partial x^4}+\overline{m}_i\frac{\partial^2 y_i(x,t)}{\partial t^2} = 0 \qquad i = 1, 2, \ldots \tag{1}$$

where E is Young's modulus, I_i is moment of inertia of cross-sectional area, $\overline{m}_i$ is mass per unit length, and $y_i(x,t)$ is the transverse displacement at position x and time t for the *i-th* beam segment.

For free vibrations, one has

$$y_i(x,t) = Y_i(x)e^{j\omega t} \tag{2}$$

where $Y_i(x)$ is the amplitude of $y_i(x,t)$, ω is the natural frequency of the beam and $j = \sqrt{-1}$.

Substitution of Eq. (2) into Eq. (1) gives

$$Y_i''''(x)-\beta_{v,i}^4 Y_i(x) = 0 \tag{3}$$

where prime (′) denotes the differentiation with respect to x and $\beta_{v,i}$ is the frequency parameter for the *i-th* beam segment corresponding to the *v-th* vibration mode defined by

$$\beta_{v,i}^4 = \frac{\omega_v^2\overline{m}_i}{EI_i} \tag{4}$$

The general solution of Eq. (3) takes the form

$$Y_i(x) = C_{i,1}\sin\beta_{v,i}x + C_{i,2}\cos\beta_{v,i}x + C_{i,3}\sinh\beta_{v,i}x + C_{i,4}\cosh\beta_{v,i}x \tag{5}$$

which is the displacement function for the *i-th* beam segment located at the left side of the *i-th* station.

3. Coefficient matrices for intermediate stations and ends of the beam

In this subsection, the coefficient matrix $[B_u]$ for the station of an intermediate elastic-supported rigid bar (or cross-section change), the coefficient matrix $[B_r]$ for an intermediate rigid support, and the coefficient matrices, $[B_0]$ and $[B_{n+1}]$, for both ends of the entire beam are derived.

3.1 Coefficient matrix [B_u] for station of an intermediate elastic-supported rigid bar (or cross-section change)

If the station numbering for the fixed point of an intermediate elastic-supported rigid bar is u, then the continuity of deformations and the equilibrium of moments and forces at station u require that

$$Y_u(\xi_u) = Y_{u+1}(\xi_u) \tag{6a}$$

$$Y_u'(\xi_u) = Y_{u+1}'(\xi_u) \tag{6b}$$

$$EI_u \frac{1}{L^2} Y_u''(\xi_u) - (J_u \omega^2 + M_u d_{mu}^2 \omega^2 - k_{Ru} - d_{ku}^2 k_{Tu}) \frac{1}{L} Y_u'(\xi_u) - (M_u d_{mu} \omega^2 - d_{ku} k_{Tu}) Y_u(\xi_u) = EI_{u+1} \frac{1}{L^2} Y_{u+1}''(\xi_u) \tag{6c}$$

$$EI_u \frac{1}{L^3} Y_u'''(\xi_u) + (M_u d_{mu} \omega^2 - d_{ku} k_{Tu}) \frac{1}{L} Y_u'(\xi_u) + (M_u \omega^2 - k_{Tu}) Y_u(\xi_u) = EI_{u+1} \frac{1}{L^3} Y_{u+1}'''(\xi_u) \tag{6d}$$

From Eqs. (5) and (6a)-(6d) one obtains

$$\begin{aligned} &C_{u,1}\sin\Omega_{v,u}\xi_u + C_{u,2}\cos\Omega_{v,u}\xi_u + C_{u,3}\sinh\Omega_{v,u}\xi_u + C_{u,4}\cosh\Omega_{v,u}\xi_u \\ &- C_{u+1,1}\sin\Omega_{v,u+1}\xi_u - C_{u+1,2}\cos\Omega_{v,u+1}\xi_u - C_{u+1,3}\sinh\Omega_{v,u+1}\xi_u - C_{u+1,4}\cosh\Omega_{v,u+1}\xi_u = 0 \end{aligned} \tag{7a}$$

$$\begin{aligned} &\Omega_{v,u}(C_{u,1}\cos\Omega_{v,u}\xi_u - C_{u,2}\sin\Omega_{v,u}\xi_u + C_{u,3}\cosh\Omega_{v,u}\xi_u + C_{u,4}\sinh\Omega_{v,u}\xi_u) \\ &- \Omega_{v,u+1}(C_{u+1,1}\cos\Omega_{v,u+1}\xi_u - C_{u+1,2}\sin\Omega_{v,u+1}\xi_u + C_{u+1,3}\cosh\Omega_{v,u+1}\xi_u + C_{u+1,4}\sinh\Omega_{v,u+1}\xi_u) = 0 \end{aligned} \tag{7b}$$

$$\begin{aligned} &\left[-\Omega_{v,u}^2 - M_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)d_{mu}^*\Omega_{v,u}^4 + k_{Tu}^*\left(\frac{I_1}{I_u}\right)d_{ku}^*\right](C_{u,1}\sin\Omega_{v,u}\xi_u + C_{u,2}\cos\Omega_{v,u}\xi_u) \\ &+\left[\Omega_{v,u}^2 - M_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)d_{mu}^*\Omega_{v,u}^4 + k_{Tu}^*\left(\frac{I_1}{I_u}\right)d_{ku}^*\right](C_{u,3}\sinh\Omega_{v,u}\xi_u + C_{u,4}\cosh\Omega_{v,u}\xi_u) \\ &-\left[J_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)\Omega_{v,u}^5 - k_{Ru}^*\left(\frac{I_1}{I_u}\right)\Omega_{v,u} + M_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)d_{mu}^{*2}\Omega_{v,u}^5 - k_{Tu}^*\left(\frac{I_1}{I_u}\right)d_{ku}^{*2}\Omega_{v,u}\right] \\ &\times(C_{u,1}\cos\Omega_{v,u}\xi_u - C_{u,2}\sin\Omega_{v,u}\xi_u + C_{u,3}\cosh\Omega_{v,u}\xi_u + C_{u,4}\sinh\Omega_{v,u}\xi_u) \\ &= \varepsilon_u\Omega_{v,u+1}^2(-C_{u+1,1}\sin\Omega_{v,u+1}\xi_u - C_{u+1,2}\cos\Omega_{v,u+1}\xi_u + C_{u+1,3}\sinh\Omega_{v,u+1}\xi_u + C_{u+1,4}\cosh\Omega_{v,u+1}\xi_u) \end{aligned} \tag{7c}$$

$$\begin{aligned} &\left[M_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)\Omega_{v,u}^4 - k_{Tu}^*\left(\frac{I_1}{I_u}\right)\right](C_{u,1}\sin\Omega_{v,u}\xi_u + C_{u,2}\cos\Omega_{v,u}\xi_u + C_{u,3}\sinh\Omega_{v,u}\xi_u + C_{u,4}\cosh\Omega_{v,u}\xi_u) \\ &+\left[M_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)d_{mu}^*\Omega_{v,u}^5 - k_{Tu}^*\left(\frac{I_1}{I_u}\right)d_{ku}^*\Omega_{v,u} - \Omega_{v,u}^3\right](C_{u,1}\cos\Omega_{v,u}\xi_u - C_{u,2}\sin\Omega_{v,u}\xi_u) \\ &+\left[M_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)d_{mu}^*\Omega_{v,u}^5 - k_{Tu}^*\left(\frac{I_1}{I_u}\right)d_{ku}^*\Omega_{v,u} + \Omega_{v,u}^3\right](C_{u,3}\cosh\Omega_{v,u}\xi_u + C_{u,4}\sinh\Omega_{v,u}\xi_u) \\ &= \varepsilon_u\Omega_{v,u+1}^3(-C_{u+1,1}\cos\Omega_{v,u+1}\xi_u + C_{u+1,2}\sin\Omega_{v,u+1}\xi_u + C_{u+1,3}\cosh\Omega_{v,u+1}\xi_u + C_{u+1,4}\sinh\Omega_{v,u+1}\xi_u) \end{aligned} \tag{7d}$$

where

$$\xi_u = x_u/L, \quad \Omega_{v,u} = \beta_{v,u}L, \quad \varepsilon_u = \frac{I_{u+1}}{I_u} \tag{8a,b,c}$$

$$M_u^* = \frac{M_u}{\overline{m}_1 L}, \quad J_u^* = \frac{J_u}{\overline{m}_1 L^3}, \quad k_{Ru}^* = \frac{k_{Ru}L}{EI_1}, \quad k_{Tu}^* = \frac{k_{Tu}L^3}{EI_1}, \quad d_{mu}^* = \frac{d_{mu}}{L}, \quad d_{ku}^* = \frac{d_{ku}}{L} \tag{9a,b,c,d,e,f}$$

For convenience, the non-dimensional parameters, M_u^*, J_u^*, k_{Tu}^* and k_{Ru}^*, as shown in Eqs. (9a-d) are introduced. In which, the mass per unit length, $\overline{m}_1$, and moment of inertia, I_1, for the 1st beam segment are used because the diameters of the beam segments composed of the entire beam are different.

Writing Eqs. (7a)-(7d) in matrix form, one has

$$[B_u]\{C_u\} = 0 \tag{10}$$

where

$$\{C_u\} = \{C_{u,1}\ \ C_{u,2}\ \ C_{u,3}\ \ C_{u,4}\ \ C_{u+1,1}\ \ C_{u+1,2}\ \ C_{u+1,3}\ \ C_{u+1,4}\} \tag{11}$$

In the above Eqs. (10) and (11), the symbols, [ ] and { }, denote the rectangular matrix and column vector, respectively. The coefficient matrix $[B_u]$ is placed in Eq. (A1) of Appendix A at the end of this paper. If there exists a "cross-section change" instead of an "elastic-supported rigid bar" at the intermediate station u, then the coefficient matrix $[B_u]$ (associated with the cross-section change) may be obtained from Eq. (A1) by setting all non-dimensional parameters defined by Eq. (9) to be zero.

3.2 Coefficient matrix [B_r] for an intermediate rigid support

Similarly, if the station numbering of an intermediate rigid support is r (cf. Fig. 1), then the continuity of deformations and the equilibrium of moments at station r require that

$$Y_r(\xi_r) = Y_{r+1}(\xi_r) = 0 \tag{12a,b}$$

$$Y_r'(\xi_r) = Y_{r+1}'(\xi_r) \tag{12c}$$

$$Y_r''(\xi_r) = \varepsilon_r Y_{r+1}''(\xi_r) \tag{12d}$$

From Eqs. (5) and (12a)-(12d) one obtains

$$[B_r]\{C_r\} = 0 \tag{13}$$

where $$\{C_r\} = \{C_{r,1}\ \ C_{r,2}\ \ C_{r,3}\ \ C_{r,4}\ \ C_{r+1,1}\ \ C_{r+1,2}\ \ C_{r+1,3}\ \ C_{r+1,4}\} \tag{14}$$

and the coefficient matrix $[B_r]$ is placed in Eq. (B1) of Appendix B at the end of this paper.

3.3 Coefficient matrix [B_0] for the left end of the entire beam

If the left-end support of the beam is pinned as shown in Fig. 1, then the boundary conditions are

$$Y_0(0) = Y_0''(0) = 0 \tag{15a,b}$$

From Eqs. (5), (15a) and (15b) one obtains

$$[B_0]\{C_0\} = 0 \tag{16}$$

where

$$[B_0] = \begin{array}{cc} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} & \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 0 & -1 & 0 & 1 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \end{array} \end{array} \tag{17}$$

$$\{C_0\} = \{C_{0,1}\ \ C_{0,2}\ \ C_{0,3}\ \ C_{0,4}\} \tag{18}$$

Similarly, if the left-end support of the beam is clamped, one obtains the following boundary coefficient matrix

$$[B_0] = \begin{array}{c} \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ 1 \\ 2 \end{array} \tag{19}$$

3.4 Coefficient matrix [B_{n+1}] for the right end of the entire beam

If the right-end support of the beam is pinned as shown in Fig. 1, then the boundary conditions are

$$Y_{n+1}(1) = Y''_{n+1}(1) = 0 \tag{20a,b}$$

where n is the total number of intermediate stations.

From Eqs. (5), (20a) and (20b) one obtains

$$[B_{n+1}]\{C_{n+1}\} = 0 \tag{21}$$

where

$$[B_{n+1}] = \begin{array}{c} \begin{array}{cccc} 4n+1 & 4n+2 & 4n+3 & 4n+4 \end{array} \\ \begin{bmatrix} \sin\Omega_{v,n+1} & \cos\Omega_{v,n+1} & \sinh\Omega_{v,n+1} & \cosh\Omega_{v,n+1} \\ -\sin\Omega_{v,n+1} & -\cos\Omega_{v,n+1} & \sinh\Omega_{v,n+1} & \cosh\Omega_{v,n+1} \end{bmatrix} \end{array} \begin{array}{c} \\ q-1 \\ q \end{array} \tag{22}$$

$$\{C_{n+1}\} = \{C_{n+1,1} \;\; C_{n+1,2} \;\; C_{n+1,3} \;\; C_{n+1,4}\} \tag{23}$$

In Eq. (22), q denotes the total number of equations for the compatibility of deformations and equilibrium of forces and moments given by

$$q = 4(n+1) \tag{24}$$

Similarly, if the right-end support of the beam is free, one obtains the following boundary coefficient matrix

$$[B_{n+1}] = \begin{array}{c} \begin{array}{cccc} 4n+1 & 4n+2 & 4n+3 & 4n+4 \end{array} \\ \begin{bmatrix} -\sin\Omega_{v,n+1} & -\cos\Omega_{v,n+1} & \sinh\Omega_{v,n+1} & \cosh\Omega_{v,n+1} \\ -\cos\Omega_{v,n+1} & \sin\Omega_{v,n+1} & \cosh\Omega_{v,n+1} & \sinh\Omega_{v,n+1} \end{bmatrix} \end{array} \begin{array}{c} \\ q-1 \\ q \end{array} \tag{25}$$

4. Determination of natural frequencies and mode shapes of the beam

The integration constants relating to the left-end and right-end supports of the beam are defined by Eqs. (18) and (23), respectively, while those relating to the intermediate stations are defined by Eqs. (11) and/or (14) depending upon an elastic-support rigid bar (or a cross-section change) or a pinned support being located there. The associated coefficient matrices are given by [B_0] (cf. Eqs. (17) or (19)), [B_u] (cf. Eq. (A1) of Appendix A), [B_r] (cf. Eq. (B1) of Appendix B) and [B_{n+1}]

(cf. Eq. (22) or (25)). From the last equations concerned one may see that the identification number for each element of the last coefficient matrices is shown on the top side and right side of each matrix. Therefore, using the numerical assembly technique as done by the conventional finite element method (FEM) one may obtain a matrix equation for all the integration constants of the entire beam

$$[\bar{B}]\{\bar{C}\} = 0 \tag{26}$$

Non-trivial solution of Eq. (26) requires that its coefficient determinant is equal to zero, i.e.

$$|\bar{B}| = 0 \tag{27}$$

which is the frequency equation for the present problem.

In this paper, the incremental search method is used to find the natural frequencies of the vibrating system, ω_v $(v = 1, 2, \ldots)$. For each natural frequency ω_v, one may obtain the corresponding integration constants from Eq. (26). The substitution of the last integration constants into the displacement functions of the associated beam segments will determine the corresponding mode shape of the entire beam, $Y^{(v)}(\xi)$.

5. Numerical results

In this section, the free vibration analysis of a two-span uniform or multi-step beam carrying multiple elastic-supported rigid bars is performed, the reliability of the theory and the computer program developed for this paper are confirmed by comparing the present results with those obtained from the conventional finite element method (FEM). In FEM, the two-node beam elements are used and the entire beam is subdivided into 40 beam elements. Since each node has two degrees of freedom (DOF's), the total DOF for the entire unconstrained beam is $2(40+1) = 82$. Unless otherwise mentioned, the dimensions of the uniform Euler-Bernoulli beam studied in this paper are as follows: Young's modulus $E = 2.068 \times 10^{11}$ N/m^2, diameter $d_1 = 0.03$ m, mass density $\rho = 7850$ kg/m^3 and total length $L = 2$ m. The reference mass is $\hat{m} = \bar{m}_1 L = \rho(\pi/4)d_1^2 L$ kg, the reference rotary inertia is $\hat{J} = \bar{m}_1 L^3$ kg m^2, the reference stiffness for translational spring is $\hat{k}_T = EI_1/L^3 = E(\pi/64)d_1^4/L^3$ N/m and the reference stiffness for rotational spring is $\hat{k}_R = EI_1/L$ Nm/rad. For convenience, four non-dimensional parameters, M^*, J^*, k_R^* and k_T^*, are introduced, they are defined by: $M^* = M/\hat{m}$, $J^* = J/\hat{J}$, $k_R^* = k_R/\hat{k}_R$ and $k_T^* = k_T/\hat{k}_T$.

5.1 Influence of eccentricity d_{mu} of rigid bar and offset d_{ku} of the supporting spring

The first example is a two-span uniform beam with an intermediate rigid pinned support at $\xi_1 = 0.4$ and carrying an elastic-supported rigid bar fixed on the beam at $\xi_2 = 0.6$ with $k_{T2}^* = 50$, $M_2^* = 0.8$ and $J_2^* = 0.04$ as shown in Fig. 2. Where the beam shown in Fig. 2(a) is pinned-pinned (P-P) and that shown in Fig. 2(b) is clamped-free (C-F). The lowest four natural frequencies of the beam for eight cases are shown in Table 1. Since the symbol d_{m2} denotes the distance between the "fixed point" of the rigid bar and its center of gravity (or the "eccentricity" of the rigid bar) and d_{k2} denotes the distance between the "fixed point" of the rigid bar and its supporting spring (or "offset" of the supporting spring) at station 2, the cases in Table 1 with non-dimensional parameters $d_{m2}^* = d_{m2}/L = 0$ and $d_{k2}^* = d_{k2}/L = 0$ indicate that, in those cases, the concentrate mass M_2 (with

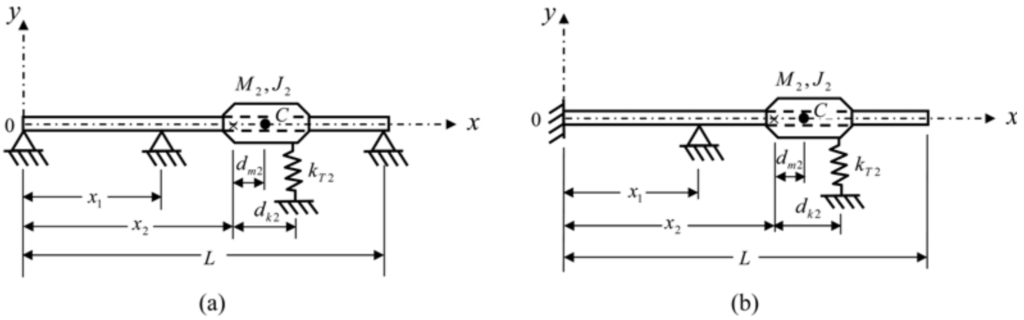

Fig. 2 Sketch for a two-span uniform beam carrying an elastic-supported rigid bar (a) pinned-pinned (P-P), (b) clamped-free (C-F)

Table 1 The lowest four natural frequencies of the two-span uniform P-P and C-F beams (cf. Fig. 2), each with an intermediate rigid pinned support at ξ_1 = 0.4 and carrying an elastic-supported rigid bar fixed on the beam at ξ_2 = 0.6 with k^*_{T2} = 50, M^*_2 = 0.8 and J^*_2 = 0.04

Boundary conditions	Cases	Non-dimensional parameters		Methods	Natural frequencies (rad/sec)			
		d^*_{m2}	d^*_{k2}		$\overline{\omega}_1$ or ω_1	$\overline{\omega}_2$ or ω_2	$\overline{\omega}_3$ or ω_3	$\overline{\omega}_4$ or ω_4
P-P	1	0	0	Present	**156.1807**	**308.2504**	**804.4766**	**992.0400**
				FEM	156.1808	308.2504	804.4782	992.0426
	2	0.1	0	Present [a](%)	129.3294 (-17.19)	365.7199 (18.64)	811.9697 (0.93)	983.2036 (-0.89)
				FEM	129.3295	365.7200	811.9713	983.2062
	3	0	0.15	Present [a](%)	169.7595 (8.69)	304.7648 (-1.13)	804.4166 (-0.01)	983.1870 (0.02)
				FEM	169.7596	304.7648	804.4183	992.2359
	4	0.1	0.15	Present [a](%)	140.6333 (-9.95)	361.5423 (17.29)	811.8406 (0.92)	992.2333 (-0.89)
				FEM	140.6334	361.5425	811.8422	983.1896
C-F	5	0	0	Present	**59.8369**	**282.2685**	**321.4191**	**1162.5393**
				FEM	59.8369	282.2686	321.4192	1162.5441
	6	0.1	0	Present [a](%)	53.2545 (-11.00)	260.5013 (-7.71)	385.0600 (19.80)	1166.9559 (0.38)
				FEM	53.2546	260.5015	385.0601	1166.9607
	7	0	0.15	Present [a](%)	77.8948 (30.18)	286.1619 (1.38)	317.8231 (-1.12)	1162.5222 (0.00)
				FEM	77.8948	286.1620	317.8232	1162.5268
	8	0.1	0.15	Present [a](%)	69.6976 (16.48)	262.7179 (-6.93)	380.7430 (18.46)	1166.9188 (0.38)
				FEM	69.6976	262.7180	380.7431	1166.9235

[a](%)=$[(\omega_v-\overline{\omega}_v)/\overline{\omega}_v]\times 100\%$

rotary inertia J_2) and the linear spring with stiffness k_{T2} are attached to the beam at the same station 2. In other words, the cases in Table 1 with $d_{m2}^* = d_{m2}/L = 0$ and $d_{k2}^* = d_{k2}/L = 0$ are the same as a uniform beam carrying arbitrary sets of point mass and linear spring in the exacting literature Lin (2008). For convenience, the *v-th* natural frequency of such a "loaded" beam is represented by $\overline{\omega}_v$ in this paper. On the contrary, the *v-th* natural frequency of the "loaded" beam with $d_{m2}^* = d_{m2}/L \neq 0$ and/or $d_{k2}^* = d_{k2}/L \neq 0$ (as shown in Fig. 2) is represented by ω_v. In Table 1, case 1 shows the lowest four natural frequencies (ω_v, v = 1 to 4) of the P-P beam with $d_{m2}^* = 0$ and $d_{k2}^* = 0$ and, for highlight, they are represented by the bold-faced digits. However, case 2 shows the lowest four natural frequencies (ω_v, v = 1 to 4) of the same P-P beam carrying an "elastic-supported rigid bar" with $d_{m2}^* = 0.1$ and $d_{k2}^* = 0$, and the percentage differences (%) of the lowest four natural frequencies (ω_v, $v = 1$ to 4) shown in the parentheses are determined by the formula: [a](%) = $[(\omega_v - \overline{\omega}_v)/\overline{\omega}_v] \times 100\%$. From the percentage differences one sees that, for the case of $d_{k2}^* = 0$, the

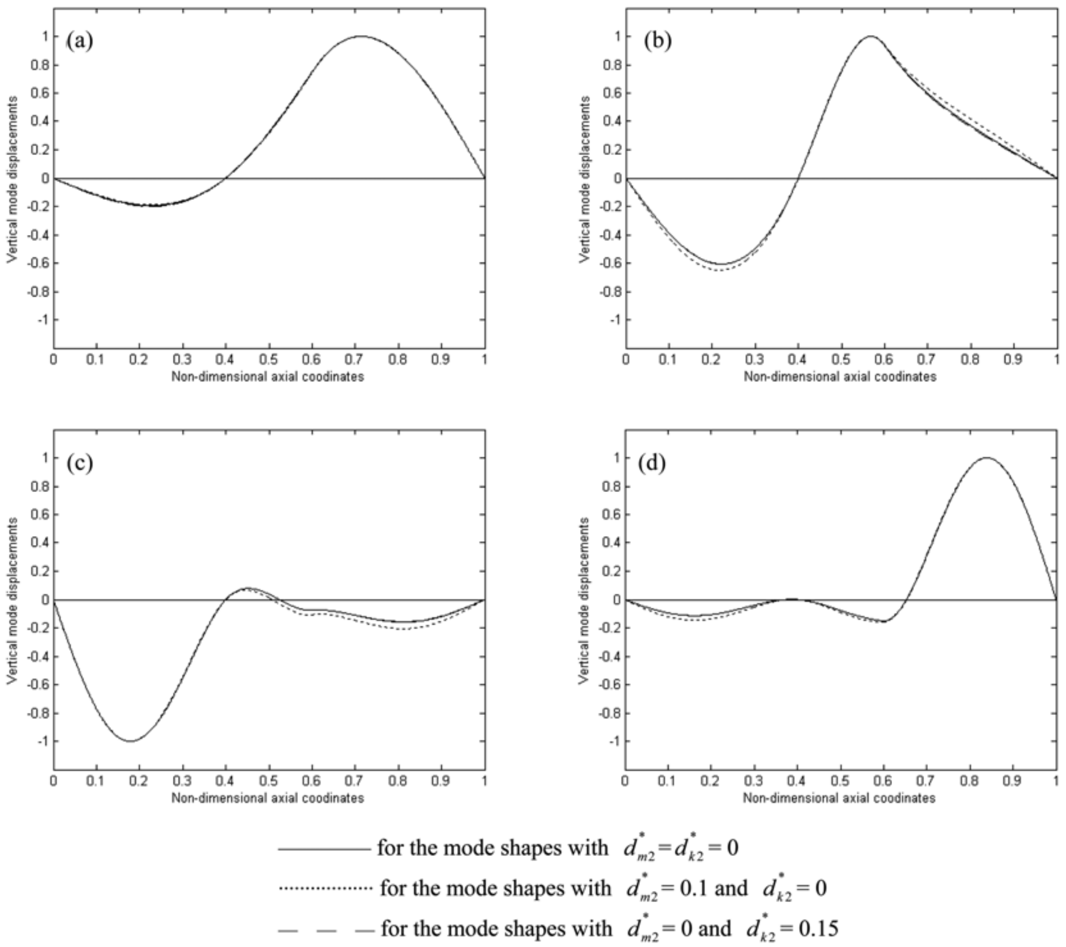

Fig. 3 The lowest four mode shapes of the two-span uniform P-P beam with an intermediate rigid pinned support at $\xi_1 = 0.4$ and carrying an elastic-supported rigid bar fixed on the beam at $\xi_2 = 0.6$ with $M_2^* = 0.8$, $J_2^* = 0.04$ and $k_{T2}^* = 50$ (cf. Fig. 2): (a), (b), (c) and (d) being the 1st, 2nd, 3rd and 4th mode shapes, respectively

eccentricity (d^*_{m2} = 0.1) of the "elastic-supported rigid bar" *reduces* the first natural frequency of the "loaded" beam 17.19% and *raises* the second one 18.64%. In case 3, the eccentricity of the rigid bar is zero (i.e., d^*_{m2} = 0) and the offset of linear spring is d^*_{k2} = 0.15, and the associated percentage differences shown in the parentheses of Table 1 reveal that the effect for the offset of the linear spring (d^*_{k2} = 0.15) raises the first natural frequency of the "loaded beam" 8.69% and reduces the second one 1.13%. Front the foregoing discussions one sees that the effect of eccentricity (d^*_{m2}) of rigid bar is opposite to that of the offset of the linear spring (d^*_{k2}) for the lowest two natural frequencies and the effect of d^*_{m2} is much greater than that of d^*_{k2}. Furthermore, from Table 1 one may see that the effect of either d^*_{m2} or d^*_{k2} on the third and fourth natural frequencies is small. Case 4 is a combination of case 2 and case 3, i.e., d^*_{m2} = 0.1 and d^*_{k2} = 0.15, and from the associated percentage differences shown in the parentheses of Table 1 one sees that the effects of both d^*_{m2} and d^*_{k2} near the combined effects of case 2 (with effect of d^*_{m2} only) and case 3 (with effect of d^*_{k2} only). For example, −17.19%+8.69% = −8.5% is close to −9.95% for the first natural frequency (in case 4); 18.64%-1.13% = 17.51% is very close to 17.29% for the second natural frequency; 0.93%-0.01% = 0.92% is equal to 0.92% for the third natural frequency; and −0.89%+0.02% = −0.87% is very close to −0.89% for the fourth natural frequency.

In Table 1, the loading conditions of the "elastic-supported rigid bar" on the loaded beam for cases 5, 6, 7 and 8 are exactly the same as those for cases 1, 2, 3 and 4, respectively, the only difference is that the former loaded beam is clamped-free (C-F) and the latter one is pinned-pinned

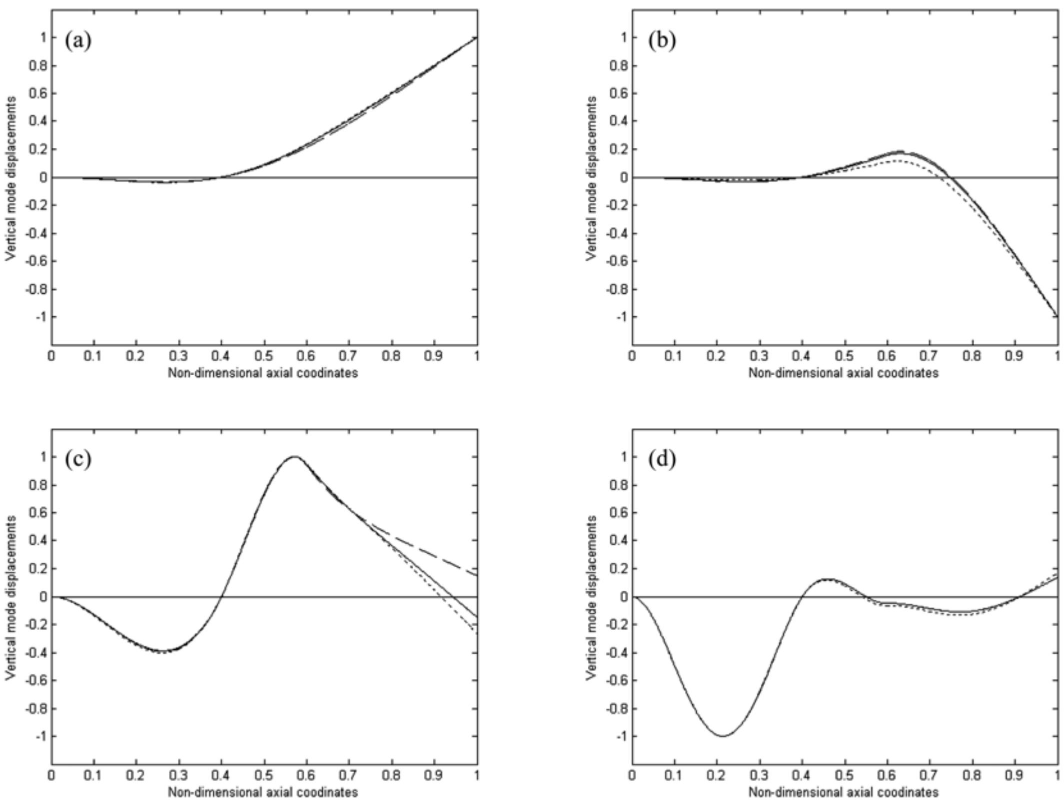

Fig. 4 The legends are the same as Fig. 3 except that the boundary conditions are clamped-free (C-F)

(P-P). From the percentage differences for cases 6 and 7 one sees that, for the C-F beam, the eccentricity (d^*_{m2} = 0.1 in case 6) of the rigid bar has the effect of reducing the lowest two natural frequencies of the loaded beam, but this effect is reverse for the offset (d^*_{k2} = 0.15 in case 7) of the linear spring. Although the effect of d^*_{m2} (or d^*_{k2}) on the C-F beam in case 6 (or 7) is different from that on the P-P beam in case 2 (or 3), the combined effects of both d^*_{m2} and d^*_{k2} on the C-F beam in case 8 are similar to those on the P-P beam in case 4, as one may see from the percentage differences shown in the parentheses of Table 1 for cases 4 and 8. It is noted that the FEM results listed in Table 1 are in good agreement with the corresponding ones obtained from the presented method of this paper.

Based on the non-dimensional parameters for the P-P beam (cases 1, 2 and 3) and C-F beam (cases 5, 6 and 7) shown in Table 1, the lowest four mode shapes are shown in Figs. 3(a)-(d) and Figs. 4(a)-(d), respectively. Where (a), (b), (c) and (d) refer to the 1st, 2nd, 3rd and 4th mode shapes, respectively. Besides, the curves ————, ·················· and — — — denote the mode shapes with $d^*_{m2} = d^*_{k2} = 0$, $d^*_{m2} = 0.1$ and $d^*_{k2} = 0$, and $d^*_{m2} = 0$ and $d^*_{k2} = 0.15$, respectively.

5.2 Free vibration analysis of a three-step beam carrying three elastic-supported rigid bars

The purpose of this subsection is to show the availability of the presented theory and the developed computer program for determining the lowest several natural frequencies and the associated mode shapes of a multi-step beam carrying multiple elastic-supported rigid bars. The example studied is a three-step pinned-pinned (P-P) (cf. Fig. 5) and clamped-free (C-F) beam, each with an intermediate rigid pinned support at $\xi_3 = x_3/L = 0.4$ and carrying three elastic-supported rigid

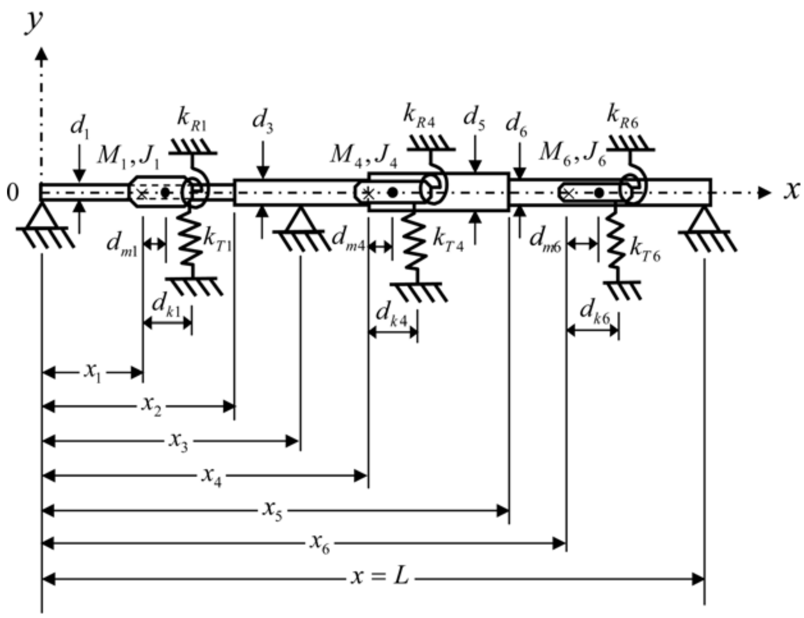

Fig. 5 Sketch for a three-step pinned-pinned (P-P) beam with an intermediate rigid pinned support at $\xi_3 = 0.4$ and carrying three elastic-supported rigid bars at $\xi_1 = 0.15$, $\xi_4 = 0.5$ and $\xi_6 = 0.8$, respectively

Table 2 The lowest four natural frequencies of the three-step (a) P-P beam and (b) C-F beam with an intermediate rigid pinned support and carrying three elastic-supported rigid bars (cf. Fig. 5)

(a) for P-P beam

Locations of rigid bars $\xi = x/L$	M^*	J^*	k_T^*	k_R^*	d_m^*	d_k^*	Methods	$\bar{\omega}_1$ or ω_1	$\bar{\omega}_2$ or ω_2	$\bar{\omega}_3$ or ω_3	$\bar{\omega}_4$ or ω_4
0.15	0.3	0.01	10	5	0	0	Present	270.6295	522.1405	552.8593	661.0960
0.50	0.4	0.02	20	10	0	0					
0.80	0.5	0.03	15	5	0	0	FEM	270.6296	522.1408	552.8595	661.0963
0.15	0.3	0.01	10	5	0.08	0	Present	267.7969	488.6296	549.7853	618.3787
0.50	0.4	0.02	20	10	0.10	0					
0.80	0.5	0.03	15	5	0.08	0	FEM	267.7970	488.6298	549.7855	618.3790
0.15	0.3	0.01	10	5	0	0.10	Present	271.6456	521.3903	553.8057	662.2269
0.50	0.4	0.02	20	10	0	0.15					
0.80	0.5	0.03	15	5	0	0.10	FEM	271.6456	521.3905	553.8057	662.2272
0.15	0.3	0.01	10	5	0.08	0.10	Present	268.9676	489.4594	550.5421	617.6240
0.50	0.4	0.02	20	10	0.10	0.15					
0.80	0.5	0.03	15	5	0.08	0.10	FEM	268.9678	489.4596	550.5422	617.6243

(b) for C-F beam

Locations of rigid bars $\xi = x/L$	M^*	J^*	k_T^*	k_R^*	d_m^*	d_k^*	Methods	$\bar{\omega}_1$ or ω_1	$\bar{\omega}_2$ or ω_2	$\bar{\omega}_3$ or ω_3	$\bar{\omega}_4$ or ω_4
0.15	0.3	0.01	10	5	0	0	Present	99.1628	325.9989	633.7047	759.2218
0.50	0.4	0.02	20	10	0	0					
0.80	0.5	0.03	15	5	0	0	FEM	99.1628	325.9990	633.7048	759.2224
0.15	0.3	0.01	10	5	0.08	0	Present	91.1194	318.9297	554.3213	861.8099
0.50	0.4	0.02	20	10	0.10	0					
0.80	0.5	0.03	15	5	0.08	0	FEM	91.1195	318.9299	554.3215	861.8106
0.15	0.3	0.01	10	5	0	0.10	Present	103.1221	326.8376	634.5717	758.9101
0.50	0.4	0.02	20	10	0	0.15					
0.80	0.5	0.03	15	5	0	0.10	FEM	103.1221	326.8377	634.5718	758.9106
0.15	0.3	0.01	10	5	0.08	0.10	Present	94.7732	319.8867	555.2973	861.0975
0.50	0.4	0.02	20	10	0.10	0.15					
0.80	0.5	0.03	15	5	0.08	0.10	FEM	94.7732	319.8868	555.2973	861.0983

bars at $\xi_1 = 0.15$, $\xi_4 = 0.5$ and $\xi_6 = 0.8$, respectively. The dimensions of the three-step beam are as follows: diameters $d_1 = d_2 = 0.03$ m, $d_3 = d_4 = 0.04$ m, $d_5 = 0.05$ m and $d_6 = d_7 = 0.04$ m; total length $L = 2.0$ m. The locations for the step changes in cross-sections are $\xi_2 = 0.3$, $\xi_4 = 0.5$ and $\xi_5 = 0.7$. The lowest four natural frequencies and non-dimensional parameters for the beam are shown in Table 2. From the table one sees that the results of the present paper are in excellent agreement with those of FEM. Based on the values of the non-dimensional parameters, M_u^*, J_u^*, k_{Tu}^* and k_{Ru}^* (for u = 1, 4, 6), as shown in Table 2, the lowest four mode shapes of the P-P beam and C-F beam are shown in Figs. 5(a)-(d) and Figs. 6(a)-(d), respectively, with legends the same as that of Fig. 3. In which, the curve ———— denotes the mode shapes with $d_{mu}^* = 0$ and $d_{ku}^* = 0$ (for $u = 1, 4, 6$), the

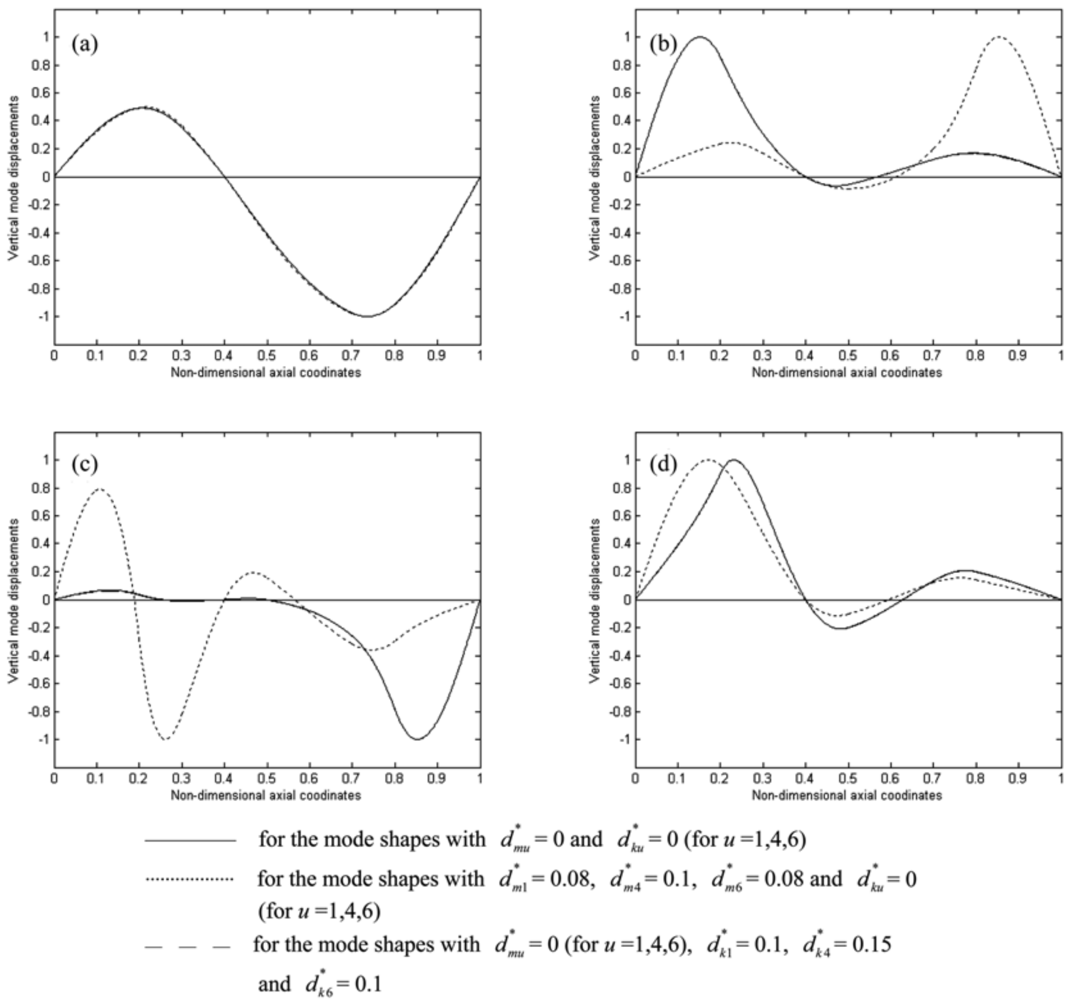

Fig. 6 The lowest four mode shapes of the three-step P-P beam with an intermediate rigid pinned support and carrying three elastic-supported rigid bars (cf. Fig. 5): (a), (b), (c) and (d) being the 1st, 2nd, 3rd and 4th mode shapes, respectively

curve ················ denotes the mode shapes with $d^*_{m1} = 0.08$, $d^*_{m4} = 0.1$, $d^*_{m6} = 0.08$ and $d^*_{ku} = 0$ (for u = 1, 4, 6), and the curve — — — denotes the mode shapes with $d^*_{mu} = 0$ (for u = 1, 4, 6), $d^*_{k1} = 0.1$, $d^*_{k4} = 0.15$ and $d^*_{k6} = 0.1$.

6. Conclusions

From this study the following concluding remarks can be made:

1. For a multi-step beam with an intermediate pinned support and carrying multiple rigid bars with each of the rigid bars possessing its own mass and rotary inertia and fixed to the beam at one point, and supported by a (linear) translational spring and a rotational spring at another point, it

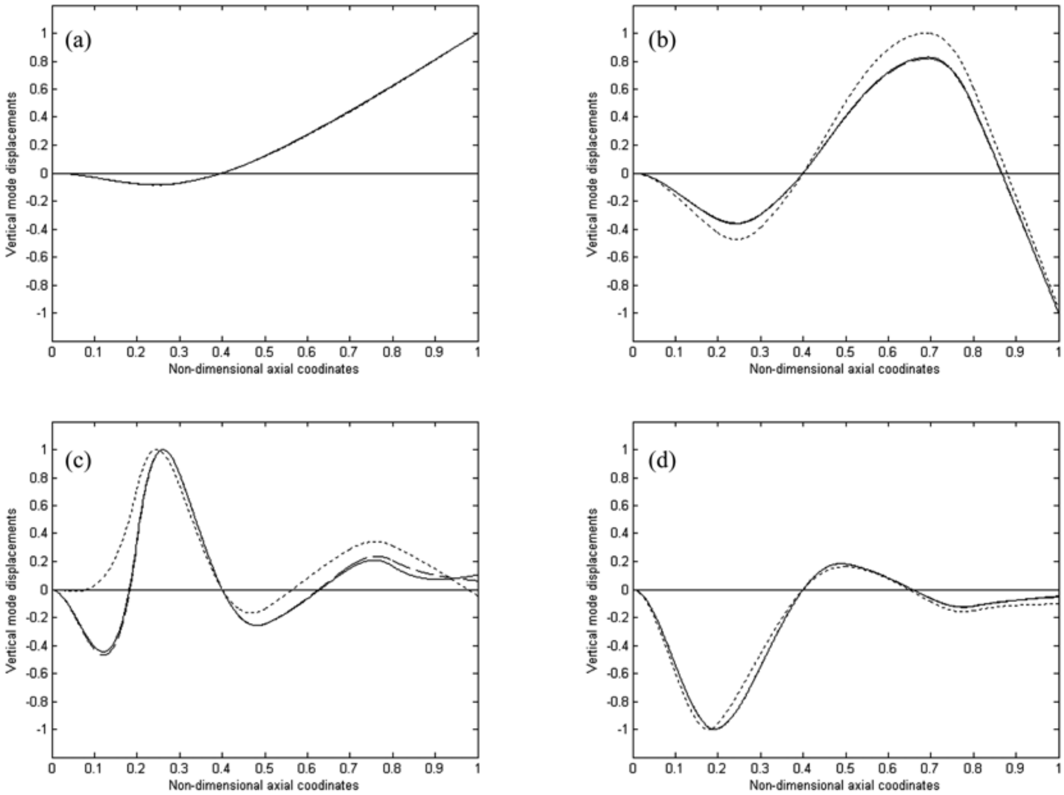

Fig. 7 The legends are the same as Fig. 6 except that the boundary conditions are clamped-free (C-F)

has been found that one can easily obtained its lowest several “exact” natural frequencies and corresponding mode shapes by using the numerical assembly method. The presented method can easily be extended to determine the exact natural frequencies and mode shapes of a multi-step beam carrying multiple elastic-supported rigid bars with each rigid bar carrying multiple translational and rotational springs at another point.

2. If a beam carrying any number of concentrated masses and linear springs is called the “loaded beam”, then it is well-known that one may change the natural frequencies and mode shapes of the loaded beam by changing the magnitude of the concentrated masses and the stiffness of the linear springs as one may see from the exiting literature. However, from the numerical examples presented in this paper, one finds that adjusting the eccentricity (d_{mu}) of each rigid bar and the offset (d_{ku}) of each linear spring supporting the rigid bar is also a very simple and very effective technique for change the natural frequencies and mode shapes of a loaded beam.

References

Farghaly, S.H. (1992), “Bending vibrations of an axially loaded cantilever beam with an elastically mounted end mass of finite length”, *J. Sound Vib.*, **156**, 373-380.

Gürgöze, M. (1986a), “On the approximate determination of the fundamental frequency of restrained cantilever

beam carrying a tip heavy body", *J. Sound Vib.*, **105**, 443-449.
Gürgöze, M. and Batan, H. (1986b), "A note on the vibrations of a restrained cantilever beam carrying a tip heavy body", *J. Sound Vib.*, **106**, 533-536.
Lin, H.Y. (2008), "On the natural frequencies and mode shapes of a multi-span and multi-step beam carrying a number of concentrated elements", *Struct. Eng. Mech.*, **29**(5), 531-550.
Liu, W.H. and Huang, C.C. (1988), "Vibrations of constrained beam carrying a heavy tip body", *J. Sound Vib.*, **123**, 15-29.
Maurizi, M.J., Belles, P. and Rosales, M. (1990), "A note on free vibrations of a constrained cantilever beam with a tip mass of finite length", *J. Sound Vib.*, **138**, 170-172.
Naguleswaran, S. (2002), "Vibration of an Euler-Bernoulli beam on elastic end supports and with up to three step changes in cross-section", *Int. J. Mech. Sci.*, **44**, 2541-2555.
Rama Bhat, B. and Wagner, H. (1976), "Natural frequencies of a uniform cantilever with a tip mass slender in the axial direction", *J. Sound Vib.*, **45**, 304-307.
Wu, J.S. and Chen, C.T. (2007a), "A lumped-mass TMM for free vibration analysis of a multiple-step beam carrying eccentric tip masses with rotary inertias", *J. Sound Vib.*, **301**, 878-897.
Wu, J.S. and Chen, C.T. (2007b), "A continuous-mass TMM for free vibration analysis of a non- uniform beam with various boundary conditions and carrying multiple concentrated elements", *J. Sound Vib.*, **311**, 1420-1430.
Zhou, D. (1997), "The vibrations of a cantilever beam carrying a heavy tip mass with elastic supports", *J. Sound Vib.*, **206**, 275-279.

Notations

d_i : diameter of the *i-th* beam segment
d_{mu} : distance between the fixed point and the center of gravity of the rigid bar (or "eccentricity" for the center of gravity of the rigid bar) at the uth station
d_{ku} : distance between the fixed point and the springs supporting the rigid bar (or "offset" for the springs supporting the rigid bar) at the uth station
E : Young's modulus
i : numbering for the *i-th* beam segment
I_i : moment of inertia of cross-sectional area of the *i-th* beam segment
j : $\sqrt{-1}$
J_u : rotary inertia of the rigid bar itself at the uth station
k_{Ru} : spring constant of rotational spring supporting the rigid bar at the uth station
k_{Tu} : spring constant of translational spring supporting the rigid bar at the uth station
L : total length of the entire beam
L_i : length of the i-th beam segment
$\overline{m}_i$: mass per unit length of the *i-th* beam segment
M_u : mass of the rigid bar itself at the uth station
n : total number of intermediate stations
q : total number of equations for compatibility of deformations and equilibrium of forces and moments
v : the *v-th* vibration mode
x_u : coordinate of station u
$y_i(x, t)$: transverse displacement at position x and time t for the *i-th* beam segment
$Y_i(x)$: amplitude function of $y_i(x, t)$
$\beta_{v,i}$: frequency parameter for the *i-th* beam segment corresponding to the *v-th* vibration mode
ε_u : ratio of moment of inertia of cross-sectional area at right side to that at left side of the uth station ($=I_{u+1}/I_u$)
$\Omega_{v,i}$: non-dimensional frequency parameter for the *i-th* beam segment corresponding to the *v-th* vibration mode
ω_v : the *v-th* natural frequencies of the beam carrying an elastic-supported rigid bar with fixed point of rigid bar to be different from the center of gravity of the rigid bar and/or the supporting point of the springs
$\overline{\omega}_v$: the *v-th* natural frequencies of the beam carrying an elastic-supported rigid bar with fixed point of rigid bar to be coincident with center of gravity of the rigid bar and/or the supporting point of the springs
ξ_i : non-dimensional coordinate of *i-th* station ($=x_i/L$)

Appendix A

The coefficient matrix $[B_u]$ for Eq. (14) is given by

$$
[B_u]=\begin{array}{c}
\begin{array}{cccccccc}
4u-3 & 4u-2 & 4u-1 & 4u & 4u+1 & 4u+2 & 4u+3 & 4u+4
\end{array}\\
\left[\begin{array}{cccccccc}
\mathrm{s}\theta_u & \mathrm{c}\theta_u & \mathrm{sh}\theta_u & \mathrm{ch}\theta_u & -\mathrm{s}\theta_{u+1} & -\mathrm{c}\theta_{u+1} & -\mathrm{sh}\theta_{u+1} & -\mathrm{ch}\theta_{u+1}\\
\Omega_{v,u}\mathrm{c}\theta_u & -\Omega_{v,u}\mathrm{s}\theta_u & \Omega_{v,u}\mathrm{ch}\theta_u & \Omega_{v,u}\mathrm{sh}\theta_u & -\Omega_{v,u+1}\mathrm{c}\theta_{u+1} & \Omega_{v,u+1}\mathrm{s}\theta_{u+1} & -\Omega_{v,u+1}\mathrm{ch}\theta_{u+1} & -\Omega_{v,u+1}\mathrm{sh}\theta_{u+1}\\
\alpha_{u1}\mathrm{s}\theta_u-\eta_u\mathrm{c}\theta_u & \alpha_{u1}\mathrm{c}\theta_u+\eta_u\mathrm{s}\theta_u & \alpha_{u2}\mathrm{sh}\theta_u-\eta_u\mathrm{ch}\theta_u & \alpha_{u2}\mathrm{ch}\theta_u-\eta_u\mathrm{sh}\theta_u & \varepsilon_u\Omega_{v,u+1}^2\mathrm{s}\theta_{u+1} & \varepsilon_u\Omega_{v,u+1}^2\mathrm{c}\theta_{u+1} & -\varepsilon_u\Omega_{v,u+1}^2\mathrm{sh}\theta_{u+1} & -\varepsilon_u\Omega_{v,u+1}^2\mathrm{ch}\theta_{u+1}\\
\delta_u\mathrm{s}\theta_u+\kappa_{u1}\mathrm{c}\theta_u & \delta_u\mathrm{c}\theta_u-\kappa_{u1}\mathrm{s}\theta_u & \delta_u\mathrm{sh}\theta_u+\kappa_{u2}\mathrm{ch}\theta_u & \delta_u\mathrm{ch}\theta_u+\kappa_{u2}\mathrm{sh}\theta_u & \varepsilon_u\Omega_{v,u+1}^3\mathrm{c}\theta_{u+1} & -\varepsilon_u\Omega_{v,u+1}^3\mathrm{s}\theta_{u+1} & -\varepsilon_u\Omega_{v,u+1}^3\mathrm{ch}\theta_{u+1} & -\varepsilon_u\Omega_{v,u+1}^3\mathrm{sh}\theta_{u+1}
\end{array}\right]
\end{array}
\begin{array}{l}
\\ 4u-1\\ 4u\\ 4u+1\\ 4u+2
\end{array}
\tag{A1}
$$

$$
\alpha_{u1}=-\Omega_{v,u}^2-M_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)d_{mu}^*\Omega_{v,u}^4+k_{Tu}^*\left(\frac{I_1}{I_u}\right)d_{ku}^*,\ \ \alpha_{u2}=\Omega_{v,u}^2-M_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)d_{mu}^*\Omega_{v,u}^4+k_{Tu}^*\left(\frac{I_1}{I_u}\right)d_{ku}^*
$$

$$
\eta_u=J_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)\Omega_{v,u}^5-k_{Ru}^*\left(\frac{I_1}{I_u}\right)\Omega_{v,u}+M_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)d_{mu}^{*\,2}\Omega_{v,u}^5-k_{Tu}^*\left(\frac{I_1}{I_u}\right)d_{ku}^{*\,2}\Omega_{v,u},\ \ \delta_u=M_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)\Omega_{v,u}^4-k_{Tu}^*\left(\frac{I_1}{I_u}\right)
$$

$$
\kappa_{u1}=-\Omega_{v,u}^3+M_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)d_{mu}^*\Omega_{v,u}^5-k_{Tu}^*\left(\frac{I_1}{I_u}\right)d_{ku}^*\Omega_{v,u},\ \ \kappa_{u2}=\Omega_{v,u}^3+M_u^*\left(\frac{\overline{m}_1}{\overline{m}_u}\right)d_{mu}^*\Omega_{v,u}^5-k_{Tu}^*\left(\frac{I_1}{I_u}\right)d_{ku}^*\Omega_{v,u} \tag{A2a,b,c,d,e,f}
$$

$$
\mathrm{s}\theta_u=\sin\Omega_{v,u}\xi_u,\ \ \mathrm{c}\theta_u=\cos\Omega_{v,u}\xi_u,\ \ \mathrm{sh}\theta_u=\sinh\Omega_{v,u}\xi_u,\ \ \mathrm{ch}\theta_u=\cosh\Omega_{v,u}\xi_u \tag{A3a,b,c,d}
$$

$$
\mathrm{s}\theta_{u+1}=\sin\Omega_{v,u+1}\xi_u,\ \ \mathrm{c}\theta_{u+1}=\cos\Omega_{v,u+1}\xi_u,\ \ \mathrm{sh}\theta_{u+1}=\sinh\Omega_{v,u+1}\xi_u,\ \ \mathrm{ch}\theta_{u+1}=\cosh\Omega_{v,u+1}\xi_u \tag{A4a,b,c,d}
$$

Appendix B

The coefficient matrix $[B_r]$ for Eq. (18) is given by

$$
[B_r] = \begin{array}{c}
\begin{array}{cccccccc} 4r-3 & 4r-2 & 4r-1 & 4r & 4r+1 & 4r+2 & 4r+3 & 4r+4 \end{array} \\
\left[\begin{array}{cccccccc}
\mathrm{s}\theta_r & \mathrm{c}\theta_r & \mathrm{sh}\theta_r & \mathrm{ch}\theta_r & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & \mathrm{s}\theta_{r+1} & \mathrm{c}\theta_{r+1} & \mathrm{sh}\theta_{r+1} & \mathrm{ch}\theta_{r+1} \\
\Omega_{v,r}\mathrm{c}\theta_r & -\Omega_{v,r}\mathrm{s}\theta_r & \Omega_{v,r}\mathrm{ch}\theta_r & \Omega_{v,r}\mathrm{sh}\theta_r & -\Omega_{v,r+1}\mathrm{c}\theta_{r+1} & \Omega_{v,r+1}\mathrm{s}\theta_{r+1} & -\Omega_{v,r+1}\mathrm{ch}\theta_{r+1} & -\Omega_{v,r+1}\mathrm{sh}\theta_{r+1} \\
-\Omega_{v,r}^2\mathrm{s}\theta_r & -\Omega_{v,r}^2\mathrm{c}\theta_r & \Omega_{v,r}^2\mathrm{sh}\theta_r & \Omega_{v,r}^2\mathrm{ch}\theta_r & \varepsilon_r\Omega_{v,r+1}^2\mathrm{s}\theta_{r+1} & \varepsilon_r\Omega_{v,r+1}^2\mathrm{c}\theta_{r+1} & -\varepsilon_r\Omega_{v,r+1}^2\mathrm{sh}\theta_{r+1} & -\varepsilon_r\Omega_{v,r+1}^2\mathrm{ch}\theta_{r+1}
\end{array}\right]
\end{array}
\begin{array}{l} \\ 4r-1 \\ 4r \\ 4r+1 \\ 4r+2 \end{array}
\tag{B1}
$$

where

$$
\mathrm{s}\theta_r = \sin\Omega_{v,r}\xi_r, \quad \mathrm{c}\theta_r = \cos\Omega_{v,r}\xi_r, \quad \mathrm{sh}\theta_r = \sinh\Omega_{v,r}\xi_r, \quad \mathrm{ch}\theta_r = \cosh\Omega_{v,r}\xi_r \tag{B2a,b,c,d}
$$

$$
\mathrm{s}\theta_{r+1} = \sin\Omega_{v,r+1}\xi_r, \quad \mathrm{c}\theta_{r+1} = \cos\Omega_{v,r+1}\xi_r, \quad \mathrm{sh}\theta_{r+1} = \sinh\Omega_{v,r+1}\xi_r, \quad \mathrm{ch}\theta_{r+1} = \cosh\Omega_{v,r+1}\xi_r \tag{B3a,b,c,d}
$$