Structural Engineering and Mechanics, Vol. 34, No. 3 (2010) 377-390
DOI: http://dx.doi.org/10.12989/sem.2010.34.3.377

Influence of elastic T-stress on the growth direction of two parallel cracks

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(Received April 18, 2007, Accepted November 16, 2009)

Abstract. This paper studies fracture initiation direction of two parallel non-coplanar cracks of equal length. Using the dislocation pile-up modelling, singular integral equations for two parallel cracks subjected to mixed-mode loading are derived and the crack-tip field including singular and non-singular terms is obtained. The kinking angle is determined by using the maximum hoop stress criterion, or the σ_θ-criterion. Results are presented for simple uniaxial tension and biaxial loading. The biaxiality ratio has a noticeable influence on crack growth direction. For the case of biaxial tension, when neglecting the T-stress the crack branching angle is overestimated for small crack inclination angles relative to the largest applied principal stress direction, and underestimated for large crack inclination angles.

Keywords: crack kinking angle; T-stress; crack growth; biaxiality ratio.

1. Introduction

A lot of experimental evidence showed that in addition to singular stresses near a crack tip, nonsingular stresses near the crack tip play a significant role in determining the path of crack growth in brittle materials and in dominating shape and size of small scale yielding around a crack tip in elastic-plastic materials. The singular stresses of the crack-tip field, denoted as K-field, are usually characterized by stress intensity factor K, while the nonsingular stresses are characterized by the constant-term stress parallel to the crack plane, which is referred to as T-stress (Williams 1957). Most practical situations of our concern are opening cracks. For such cracks, two parameters K and T basically dominate the crack-tip field, denoted as K-T-field, and so a bi-parameter fracture

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criterion has been suggested (Betegon *et al.* 1996).

For a single crack embedded in an infinite elastic medium, the sign and magnitude of *T*-stress can influence the stability of the advance of a crack (Cotterrell and Rice 1980, Melin 2002), as well as fracture initiation directions (Williams 1972, Sladek *et al.* 1997). Evidently, the *K*-*T*-field provides more rich information on the elastic behavior of the crack tip than solely using the singular *K*-field. For example, based on the *K*-*T*-field, a large *T*-stress can induce a crack to be bifurcated symmetrically (Ayatollahi *et al.* 2002), other than expanding along its original crack plane. Similar to stress intensity factors at the crack tip, many useful results of the *T*-stress have been obtained. Fett *et al.* (2006) developed the Cotterell-Rice procedure to give approximately Green's functions for the *T*-stress at the kinked tip. Using the fundamental Westergaard functions, Li and Xu (2007) derived a simple expression for approximately evaluating the *T*-stress at the kinked tip for various cases including opening or closed kinked crack. For other crack configurations such as cruciform cracks (Li 2006), periodic cracks (Chen *et al.* 2009), penny-shaped cracks (Wang 2004) or elliptical cracks (Molla-Abbasi and Schutte 2008), the *T*-stress has been determined analytically respectively. For a crack in compression, the effect of the *T*-stress on fracture initiation angle and on the yielding stress is very notable, and this topic has been addressed in a recent paper (Li *et al.* 2009).

As we know, multiple cracks are often observed in engineering structures (Wang *et al.* 1996) and geological structures (Wu and Pollard 1995, Germanovich and Astakhov 2004), and the interaction of multiple cracks plays a dominant role on crack growth. For example, the presence of a large number of cracks seems not to be understood as the congregation of independent cracks unless they are sufficiently far (Kamaya and Totsuka 2002). Therefore, when cracks are very close, they are more prone to coalesce or deviate (Lunn *et al.* 2008). Such interaction may significantly affect the integrality of a cracked structure. Different from previous studies (relative to two and multiple cracks) where the influences of the singular field on crack propagation were mainly dealt with, this paper analyzes the influence of the *T*-stress on crack kinking angle and further sheds light on coalescence or deviation of two cracks. As a representative example, the interaction of the *T*-stress of two parallel cracks of equal length on crack growth direction is analyzed here.

This paper is organized as follows. In Section 2, a fundamental solution for an edge dislocation is given and then singular integral equations are obtained by superposition of the fundamental solution. Furthermore, an analytical approach for evaluating the *T*-stress at the crack tips of two parallel cracks is described in Section 3. Using the maximum hoop stress criterion, Section 4 is devoted to discussion on the effects of the *T*-stress on the crack kinking angle, with emphasis on the cases of uniaxial and biaxial tension. Obtained results turn out that the *T*-stress plays a significant role in determining the growth direction of cracks.

2. Governing equations

2.1 Fundamental solution

A crack-tip field contains singular and nonsingular stresses, and it can be expressed as

$$\sigma_{mj}(r,\theta) = \frac{K_I}{\sqrt{2\pi r}} f_{mj}^{I}(\theta) + \frac{K_{II}}{\sqrt{2\pi r}} f_{mj}^{II}(\theta) + T\delta_{m1}\delta_{j1} + O(r^{1/2}), \quad (m,j=1,2) \tag{1}$$

where K_I and K_{II} are mode I and II stress intensity factors, T is T-stress, $f_{mj}^{I}(\theta)$ and $f_{mj}^{II}(\theta)$ are two

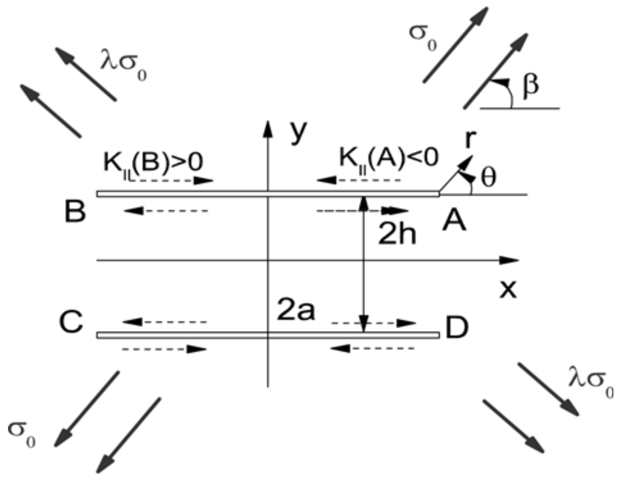


Fig. 1 Two parallel cracks of equal length $2a$ and spaced by $2h$ in an infinite elastic plane

specified angular distribution functions solely dependent on θ, δ_{mj} is the Kronecker symbol, and the last term $O(r^{1/2})$ denotes infinitesimal quantity when r tends to the crack tip, r and θ being local polar coordinates with the origin at the crack tip (Fig. 1). It is worth noting that there are many methods for determining a crack-tip field, in particular the singular elastic field in the immediate vicinity of the crack tip. Here we employ an approach, the dislocation pile-up modelling, to obtain both the singular field and T-stress. Then from obtained results the influence of the T-stress on crack growth is examined. To this end, we first model a crack as the pile-up of infinitesimal continuously distributed dislocations with Burgers vectors $(b_x dx, b_y dx)$ in the region between x and $x+dx$, where

$$b_x = -\frac{d[\![u]\!]}{dx}, \quad b_y = -\frac{d[\![v]\!]}{dx} \tag{2}$$

Here u and v stand for the displacement components along the x- and y-axes, respectively, $[\![u]\!]$ and $[\![v]\!]$ represent the jump of the tangential and normal displacements across the crack surface, respectively.

A basic premise of this method is knowledge of the fundamental solution induced by an edge dislocation. Recalling this solution for a single straight edge dislocation with Burgers vector (b_x, b_y) at the origin in an infinite elastic medium for plane strain, the induced elastic field is (Lardner 1974)

$$\sigma_{xx} = \frac{\mu}{2\pi(1-\nu)}[b_x G_{xxx}(x,y) + b_y G_{yxx}(x,y)] \tag{3}$$

$$\sigma_{yy} = \frac{\mu}{2\pi(1-\nu)}[b_x G_{xyy}(x,y) + b_y G_{yyy}(x,y)] \tag{4}$$

$$\sigma_{xy} = \frac{\mu}{2\pi(1-\nu)}[b_x G_{xxy}(x,y) + b_y G_{yxy}(x,y)] \tag{5}$$

where μ and ν are the shear modulus and Poisson's ratio, respectively, and

$$G_{xxx}(x,y) = \frac{-y(3x^2+y^2)}{(x^2+y^2)^2}, \quad G_{yxx}(x,y) = \frac{x(x^2-y^2)}{(x^2+y^2)^2} \tag{6}$$

$$G_{xyy}(x,y) = \frac{y(x^2-y^2)}{(x^2+y^2)^2}, \quad G_{yyy}(x,y) = \frac{x(x^2+3y^2)}{(x^2+y^2)^2} \tag{7}$$

$$G_{xxy}(x,y) = \frac{x(x^2-y^2)}{(x^2+y^2)^2}, \quad G_{yxy}(x,y) = \frac{y(x^2-y^2)}{(x^2+y^2)^2} \tag{8}$$

2.2 Governing equations for two parallel cracks

In the present study, we consider two parallel non-coplanar cracks of equal length $2a$ and spaced by $2h$. Cartesian coordinate system oxy is chosen such that two cracks are situated symmetrically in the upper and lower half-planes, respectively, as shown in Fig. 1. Under the action of biaxial tension at infinity, we denote σ_0 and $\lambda\sigma_0$ as two principal stresses, where σ_0 is applied in an angle β with respect to the crack plane, and consequently we have

$$\sigma_{xx}^{\infty} = \sigma_0(\cos^2\beta + \lambda\sin^2\beta), \quad \sigma_{yy}^{\infty} = \sigma_0(\sin^2\beta + \lambda\cos^2\beta), \quad \sigma_{xy}^{\infty} = \sigma_0(1-\lambda)\sin\beta\cos\beta \tag{9}$$

This problem can be transformed to two subproblems, one corresponding to a uniform field in the absence of the cracks, and the other corresponding to the disturbed field arising from the presence of two cracks where the loading at the crack surfaces equals to the negative of the counterpart for the former case, which guarantees the crack surfaces to be traction-free when the uniform field of the former case is superposed.

For the latter case, the stress field at point $(x_0, h), (-\infty < x_0 < \infty)$ is composed of two parts. One part originates from the upper crack as the pile-up of infinitesimal distributed dislocations situated at position $(x, h)(|x| < a)$, i.e.

$$\sigma_{xx}^{(1)} = \sigma_{yy}^{(1)} = \frac{\mu}{2\pi(1-\nu)} \int_{-a}^{a} \frac{[\![v'(x)]\!]}{x - x_0} dx \tag{10}$$

$$\sigma_{xy}^{(1)} = \frac{\mu}{2\pi(1-\nu)} \int_{-a}^{a} \frac{[\![u'(x)]\!]}{x - x_0} dx \tag{11}$$

where $[\![u'(x)]\!] = d[u(x,h^+) - u(x,h^-)]/dx$, $[\![v'(x)]\!] = d[v(x,h^+) - v(x,h^-)]/dx$, and the other part is caused by the lower crack $(x, -h)$ $(|x| < a)$. Hence, applying the expressions (3)-(5), the induced elastic field is

$$\sigma_{xx}^{(2)} = -\frac{\mu}{2\pi(1-\nu)} \int_{-a}^{a} \{ [\![\tilde{u}'(x)]\!] \, G_{xxx}(x_0 - x, 2h) + [\![\tilde{v}'(x)]\!] \, G_{yxx}(x_0 - x, 2h) \} dx \tag{12}$$

$$\sigma_{yy}^{(2)} = -\frac{\mu}{2\pi(1-\nu)} \int_{-a}^{a} \{ [\![\tilde{u}'(x)]\!] \, G_{xyy}(x_0 - x, 2h) + [\![\tilde{v}'(x)]\!] \, G_{yyy}(x_0 - x, 2h) \} dx \tag{13}$$

$$\sigma_{xy}^{(2)} = -\frac{\mu}{2\pi(1-\nu)} \int_{-a}^{a} \{ [\![\tilde{u}'(x)]\!] \, G_{xxy}(x_0 - x, 2h) + [\![\tilde{v}'(x)]\!] \, G_{yxy}(x_0 - x, 2h) \} dx \tag{14}$$

where $[\![\tilde{u}(x)]\!]$ and $[\![\tilde{v}(x)]\!]$ are the jump of the derivatives of the tangential and normal displacements across the lower crack, respectively.

Therefore, adding the stress fields marked with superscripts (1) and (2) to the uniform elastic field, one gets the total stress field at (x_0, h) as follows

$$\sigma_{xx} = \sigma_{xx}^{(1)} + \sigma_{xx}^{(2)} + \sigma_{xx}^{\infty} \tag{15}$$

$$\sigma_{yy} = \sigma_{yy}^{(1)} + \sigma_{yy}^{(2)} + \sigma_{yy}^{\infty} \tag{16}$$

$$\sigma_{xy} = \sigma_{xy}^{(1)} + \sigma_{xy}^{(2)} + \sigma_{xy}^{\infty} \tag{17}$$

Accordingly, in order for the crack surfaces to be traction-free, substituting (10) and (11) in conjunction with (13) and (14) into (16) and (17) yields a system of singular integral equations with Cauchy kernel

$$\frac{\mu}{2(1-\nu)\pi}\int_{-a}^{a}\left\{\frac{[\![v'(x)]\!]}{x-x_0} - G_{yyy}(x_0-x, 2h)\,[\![\tilde{v}'(x)]\!] - G_{xyy}(x_0-x, 2h)\,[\![\tilde{u}'(x)]\!]\right\}dx = -\sigma_{yy}^{\infty} \tag{18}$$

$$\frac{\mu}{2(1-\nu)\pi}\int_{-a}^{a}\left\{\frac{[\![u'(x)]\!]}{x-x_0} - G_{xxy}(x_0-x, 2h)\,[\![\tilde{u}'(x)]\!] - G_{yxy}(x_0-x, 2h)\,[\![\tilde{v}'(x)]\!]\right\}dx = -\sigma_{xy}^{\infty} \tag{19}$$

over $-a < x_0 < a$. Similarly, for the other crack lying the lower half-plane, two singular integral equations similar to Eqs. (18) and (19) may be deduced. All of these equations are coupled.

For the purpose of numerical computation, we denote normalized variables as follows

$$\bar{x} = \frac{x}{a}, \quad \bar{x}_0 = \frac{x_0}{a} \tag{20}$$

and as usual the dislocation density functions can be assumed to take the following forms

$$[\![v'(x)]\!] = \frac{2(1-\nu)}{\mu}\frac{\Omega(\bar{x})}{\sqrt{1-\bar{x}^2}}, \quad [\![u'(x)]\!] = \frac{2(1-\nu)}{\mu}\frac{\Lambda(\bar{x})}{\sqrt{1-\bar{x}^2}} \tag{21}$$

$$[\![\tilde{v}'(x)]\!] = \frac{2(1-\nu)}{\mu}\frac{\tilde{\Omega}(\bar{x})}{\sqrt{1-\bar{x}^2}}, \quad [\![\tilde{u}'(x)]\!] = \frac{2(1-\nu)}{\mu}\frac{\tilde{\Lambda}(\bar{x})}{\sqrt{1-\bar{x}^2}} \tag{22}$$

where $\Omega(\bar{x}), \tilde{\Omega}(\bar{x})$ and $\Lambda(\bar{x}), \tilde{\Lambda}(\bar{x})$ are continuous functions over [–1,1], and the factor $2(1-\nu)/\mu$ is introduced for convenience. According to these normalized variables, the resulting singular integral equations become

$$\frac{1}{\pi}\int_{-1}^{1}\left\{\frac{\Omega(\bar{x})}{\bar{x}-\bar{x}_0} - aG_{yyy}(x_0-x, 2h)\tilde{\Omega}(\bar{x}) - aG_{xyy}(x_0-x, 2h)\tilde{\Lambda}(\bar{x})\right\}\frac{d\bar{x}}{\sqrt{1-\bar{x}^2}} = -\sigma_{yy}^{\infty} \tag{23}$$

$$\frac{1}{\pi}\int_{-1}^{1}\left\{\frac{\Lambda(\bar{x})}{\bar{x}-\bar{x}_0} - aG_{xxy}(x_0-x, 2h)\tilde{\Lambda}(\bar{x}) - aG_{yxy}(x_0-x, 2h)\tilde{\Omega}(\bar{x})\right\}\frac{d\bar{x}}{\sqrt{1-\bar{x}^2}} = -\sigma_{xy}^{\infty} \tag{24}$$

$$\frac{1}{\pi}\int_{-1}^{1}\left\{\frac{\tilde{\Omega}(\bar{x})}{\bar{x}-\bar{x}_0}-aG_{yyy}(x_0-x,-2h)\Omega(\bar{x})-aG_{xyy}(x_0-x,-2h)\Lambda(\bar{x})\right\}\frac{d\bar{x}}{\sqrt{1-\bar{x}^2}} = -\sigma_{yy}^{\infty} \tag{25}$$

$$\frac{1}{\pi}\int_{-1}^{1}\left\{\frac{\tilde{\Lambda}(\bar{x})}{\bar{x}-\bar{x}_0}-aG_{xxy}(x_0-x,-2h)\Lambda(\bar{x})-aG_{yxy}(x_0-x,-2h)\Omega(\bar{x})\right\}\frac{d\bar{x}}{\sqrt{1-\bar{x}^2}} = -\sigma_{xy}^{\infty} \tag{26}$$

for $|\bar{x}_0|<1$. To obtain a unique solution of the system of equations, it is necessary to supplement necessary constraint conditions. Physically, the displacements must fulfill single-value conditions at the crack tips, i.e.

$$\int_{-a}^{a}\left\{\begin{matrix}[\![u'(x)]\!]\\ [\![\tilde{u}'(x)]\!]\end{matrix}\right\}dx = 0, \quad \int_{-a}^{a}\left\{\begin{matrix}[\![v'(x)]\!]\\ [\![\tilde{v}'(x)]\!]\end{matrix}\right\}dx = 0 \tag{27}$$

or

$$\int_{-1}^{1}\frac{1}{\sqrt{1-\bar{x}^2}}\left\{\begin{matrix}\Omega(\bar{x})\\ \tilde{\Omega}(\bar{x})\end{matrix}\right\}d\bar{x} = 0, \quad \int_{-1}^{1}\frac{1}{\sqrt{1-\bar{x}^2}}\left\{\begin{matrix}\Lambda(\bar{x})\\ \tilde{\Lambda}(\bar{x})\end{matrix}\right\}d\bar{x} = 0 \tag{28}$$

3. *T*-stress at the crack tips

Once $\Omega(\bar{x}), \tilde{\Omega}(\bar{x})$ and $\Lambda(\bar{x}), \tilde{\Lambda}(\bar{x})$ are determined numerically, all the quantities of interest may be obtained. This can be done by expressing the stress fields as follows

$$\sigma_{xx}(x_0,h) = \frac{1}{\pi}\int_{-1}^{1}\left\{\frac{\Omega(\bar{x})}{\bar{x}-\bar{x}_0}-aG_{xxx}(x_0-x,2h)\tilde{\Lambda}(\bar{x})-aG_{yxx}(x_0-x,2h)\tilde{\Omega}(\bar{x})\right\}\frac{d\bar{x}}{\sqrt{1-\bar{x}^2}}+\sigma_{xx}^{\infty} \tag{29}$$

$$\sigma_{yy}(x_0,h) = \frac{1}{\pi}\int_{-1}^{1}\left\{\frac{\Omega(\bar{x})}{\bar{x}-\bar{x}_0}-aG_{xyy}(x_0-x,2h)\tilde{\Lambda}(\bar{x})-aG_{yyy}(x_0-x,2h)\tilde{\Omega}(\bar{x})\right\}\frac{d\bar{x}}{\sqrt{1-\bar{x}^2}}+\sigma_{yy}^{\infty} \tag{30}$$

$$\sigma_{xy}(x_0,h) = \frac{1}{\pi}\int_{-1}^{1}\left\{\frac{\Lambda(\bar{x})}{\bar{x}-\bar{x}_0}-aG_{xxy}(x_0-x,2h)\tilde{\Lambda}(\bar{x})-aG_{yxyy}(x_0-x,2h)\tilde{\Omega}(\bar{x})\right\}\frac{d\bar{x}}{\sqrt{1-\bar{x}^2}}+\sigma_{xy}^{\infty} \tag{31}$$

for an arbitrary position $\bar{x}_0 = x_0/a, -\infty<\bar{x}_0<\infty$. To obtain asymptotic stress fields at the crack tip $A(a,h)$, it suffices to focus our attention on the region near this crack tip $(\bar{x}_0 = 1)$. Recalling the known result

$$\frac{1}{\pi}\int_{-1}^{1}\frac{1}{\bar{x}-\bar{x}_0}\frac{d\bar{x}}{\sqrt{1-\bar{x}^2}} = \begin{cases}0, & |\bar{x}_0|<1\\ -\dfrac{\operatorname{sgn}(\bar{x}_0)}{\sqrt{\bar{x}_0^2-1}}, & |\bar{x}_0|>1\end{cases} \tag{32}$$

where $\operatorname{sgn}(\bar{x}_0) = 1$ for $\bar{x}_0>0$ and $\operatorname{sgn}(\bar{x}_0) = -1$ for $\bar{x}_0<0$, we can obtain

$$\sigma_{yy}(x_0,h) = 0, \ \sigma_{xy}(x_0,h) = 0, \ |\bar{x}_0|<1 \tag{33}$$

as expected. However, ahead of the upper crack, e.g., for $x_0 > 1$ we can write

$$\begin{Bmatrix} \sigma_{xx}(x_0, h) \\ \sigma_{yy}(x_0, h) \\ \sigma_{xy}(x_0, h) \end{Bmatrix} = -\frac{1}{\sqrt{\bar{x}_0^2 - 1}} \begin{Bmatrix} \Omega(\bar{x}_0) \\ \Omega(\bar{x}_0) \\ \Lambda(\bar{x}_0) \end{Bmatrix} + \begin{Bmatrix} T_{xx}(x_0, h) \\ T_{yy}(x_0, h) \\ T_{xy}(x_0, h) \end{Bmatrix} \tag{34}$$

where

$$T_{mj}(x_0, h) = \frac{\delta_{mj}}{\pi}\int_{-1}^{1} \frac{\tilde{\Omega}(\bar{x}) - \tilde{\Omega}(\bar{x}_0)}{\bar{x} - \bar{x}_0} \frac{d\bar{x}}{\sqrt{1-\bar{x}^2}} + \frac{\delta_{m1}\delta_{j2}}{\pi}\int_{-1}^{1} \frac{\tilde{\Lambda}(\bar{x}) - \tilde{\Lambda}(\bar{x}_0)}{\bar{x} - \bar{x}_0} \frac{d\bar{x}}{\sqrt{1-\bar{x}^2}}$$
$$-\frac{a}{\pi}\int_{-1}^{1} [G_{xmj}(x_0 - x, 2h)\tilde{\Lambda}(\bar{x}) + G_{ymj}(x_0 - x, 2h)\tilde{\Omega}(\bar{x})] \frac{d\bar{x}}{\sqrt{1-\bar{x}^2}} + \sigma_{mj}^{\infty}$$

Taking into account the fact that the nonsingular stress $T_{mj}(x_0, h)$ are continuous with respect to variable x_0 and $\sigma_{yy}(x_0, h) = 0$ for $|x| < 1$, we get

$$\lim_{x_0 \to a^+} T_{yy}(x_0, h) = \lim_{x_0 \to a^-} T_{yy}(x_0, h) = 0, \quad \lim_{x_0 \to a^+} T_{xy}(x_0, h) = \lim_{x_0 \to a^-} T_{xy}(x_0, h) = 0 \tag{35}$$

and then obtain

$$T = T_{xx} - T_{yy} = \frac{8\bar{h}}{\pi}\int_{-1}^{1} \frac{(1-\bar{x})[(1-\bar{x})\tilde{\Lambda}(\bar{x}) + 2\bar{h}\tilde{\Omega}(\bar{x})]}{[(1-\bar{x})^2 + 4\bar{h}^2]^2\sqrt{1-\bar{x}^2}} d\bar{x} + \sigma_{xx}^{\infty} - \sigma_{yy}^{\infty} \tag{36}$$

where $\bar{h} = h/a$. Remembering (21), the above result can be represented as

$$T = \frac{4h\mu}{\pi(1-\nu)}\int_{-a}^{a} \frac{(a-x)\,[\![\tilde{u}'(x)]\!] + 2h\,[\![\tilde{v}'(x)]\!]}{[(a-x)^2 + 4h^2]^2} (a-x)dx + \sigma_{xx}^{\infty} - \sigma_{yy}^{\infty} \tag{37}$$

Clearly, this result permits us to evaluate the *T*-stress analytically only if the solution of the resulting singular integral Eqs. (23)-(26) is determined. From (37), we make some observations. The above expressions are composed of two parts. The integral-free part (i.e., $\sigma_{xx}^{\infty} - \sigma_{yy}^{\infty}$) corresponds to that for a single crack embedded in an infinite elastic plane subjected to biaxial tension, and the integral term gives the contribution of the other crack on the *T*-stress, depending upon not only the crack opening displacement but also the crack sliding displacement of the other crack. As a check, if letting h be large enough, the first integral gradually tends to vanish, thus the above result collapses to the well-known result $T = \sigma_{xx}^{\infty} - \sigma_{yy}^{\infty}$. This is attributed to the fact that the interaction is negligible when the distance between two cracks is sufficiently large. For a finite (small) value of h compared to a, the interaction between two cracks cannot be ignored.

4. Results and discussion

Once the solution is determined numerically, keeping (34) in conjunction with (32) in mind, one readily determines the stress intensity factors at the crack tip $A(a, h)$ as

$$K_I = -\sqrt{\pi a}\,\Omega(1), \quad K_{II} = -\sqrt{\pi a}\,\Lambda(1) \tag{38}$$

Table 1 Values of k_I, k_{II}, and t at the tip A(a, h) when $\sigma_{yy}^{\infty} = \sigma_0$

h/a	k_I	k_I(*)	k_{II}	t
∞	1.0000	1.0000	0.0000	−1.0000
5	0.9858	0.9855	−0.0014	−0.9817
2.5	0.9505	0.9508	−0.0094	−0.9429
1.25	0.8722	0.8727	−0.0431	−0.9001
1	0.8431	0.8319	−0.0611	−0.9061
0.8	0.8166	0.8037	−0.0803	−0.9267
0.5	0.7734	0.7569	−0.1165	−1.0001
0.2	0.7215	0.6962	−0.1633	−1.2089
0.1	0.7000	0.6651	−0.1839	−1.4273

*data from Murakami (1987)

Using (37), the elastic T-stress at the crack tip $\mathrm{A}(a,h)$ can be evaluated by

$$T = \frac{8\bar{h}}{\pi}\int_{-1}^{1}\sqrt{\frac{1-\bar{x}}{1+\bar{x}}}\frac{2\bar{h}\tilde{\Omega}(\bar{x})+(1-\bar{x})\tilde{\Lambda}(\bar{x})}{[(1-\bar{x})^2+4\bar{h}^2]^2}d\bar{x}+\sigma_{xx}^{\infty}-\sigma_{yy}^{\infty} \tag{39}$$

In order to describe quantitatively the dependence of the T-stress, we must solve (23)-(26) together with (28). Here we invoke the Lobatto-Chebyshev collocation method (Theocaris and Ioakimidis 1977), and a detailed procedure is omitted here. By defining the normalized stress intensity factors and the normalized T-stress as

$$k_I = \frac{K_I}{\sqrt{\pi a}\,\sigma_0}, \quad k_{II} = \frac{K_{II}}{\sqrt{\pi a}\,\sigma_0}, \quad t = \frac{T}{\sigma_0} \tag{40}$$

for the case of uniaxial tension, i.e., $\lambda = 0$, evaluated results of k_I, k_{II}, and t at the tip A are listed in Table 1 for $\beta = 90^{\circ}$. From Table 1, besides k_I, one can find that k_{II} and t are also induced simultaneously. For comparison, previously obtained results of k_I (Murakami 1987) are also tabulated in Table 1. Clearly, high accuracy can be achieved for the range of $h/a > 1$. Due to symmetry of the problem, it is readily found that all k_I and t values are the same at four tips. However for k_{II} we have $k_{II}(A) = -k_{II}(B) = k_{II}(C) = -k_{II}(D)$.

Next we examine the influence of the T-stress on the crack kinking angle. In existing fracture criteria, the simplest one is the maximum hoop stress criterion (Erdogan and Sih 1963). Using this criterion, from expression (1) when the higher terms are ignored except the singular term and constant term, the asymptotic hoop stress can be expressed as

$$\sigma_{\theta\theta} = \frac{1}{\sqrt{2\pi r}}\cos^2\frac{\theta}{2}\left(K_I\cos\frac{\theta}{2}-3K_{II}\sin\frac{\theta}{2}\right)+T\sin^2\theta \tag{41}$$

Then the direction of the onset of fracture is determined by demanding $d\sigma_{\theta\theta}/d\theta = 0$. After manipulation, the crack kinking angle θ_k satisfies

$$K_I\sin\theta - K_{II}(1-3\cos\theta)-\frac{16}{3}\sqrt{2\pi r_c}\,T\sin\frac{\theta}{2}\cos\theta = 0 \tag{42}$$

where r_c is a critical distance.

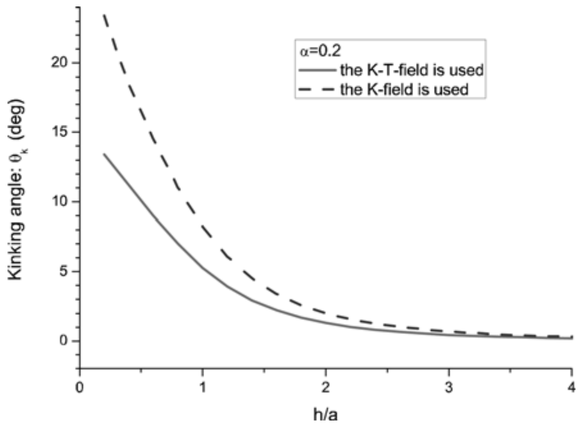


Fig. 2 The influence of the crack spacing on the kinking angle for the case of uniaxial mode I tensile loading

Solving Eq. (42) after substituting numerical results of K_I, K_{II} and T into (42), one can give the dependence of the crack kinking angle θ_k. Fig. 2 shows the kinking angle θ_k versus the ratio of the crack spacing $2h$ to the crack length $2a$ with $\alpha = 0.2$, where $\alpha = \sqrt{2r_c/a}$. It is seen from Fig. 2 that when two cracks are very close, tensile mode I loading drives two cracks to deviate away from each other. However, with the spacing h increasing, the angle of deviation becomes progressively small. When the distance of two cracks is large enough, two cracks will propagate in parallel. In this case, two cracks may be understood as two independent single cracks since their interaction is very weak. In particular, when the effects of the *T*-stress is taken into account, it is obvious that the angle of deviation of two cracks is smaller. For example, when taking $\alpha = 0.2$, $h/a = 0.2$, the kinking angle is about 13.4° when the *T*-stress is included and about 23.4° when the *T*-stress is neglected, respectively. It indicates that the *T*-stress in this case has a tendency to impede two cracks to deviate away.

For a general case of uniaxial tension, $\beta \neq 90^\circ$, two cracks are actually loaded by mixed mode. Here we only consider two special crack spacings, $h/a = 4$ and $h/a = 0.4$. Similarly, we first compute the corresponding k_I, k_{II}, and t values at two tips A and D, which are listed in Tables 2 and 3, respectively. In particular, from Table 3 it is seen that for small crack spacing $h/a = 0.4$, k_I at the tip A may be negative when the crack angle β is small enough (e.g., $\beta = 5^\circ$). This is attributed to the interaction of two close cracks. However, k_I at the tip D is still positive. Although $k_I < 0$ cannot

Table 2 Values of k_I, k_{II}, and t for uniaxial tension $\sigma_{yy}^{\infty} = \sigma_0$ in an angle β ($h/a = 4$)

β (deg)	Tip A(a, h)			Tip D(a, $-h$)		
	k_I	k_{II}	t	k_I	k_{II}	t
5	0.0007	0.0874	0.9856	0.0008	0.0875	0.9844
15	0.0649	0.2516	0.8696	0.0662	0.2520	0.8661
30	0.2434	0.4354	0.5099	0.2458	0.4368	0.5037
45	0.4878	0.5022	0.0172	0.4905	0.5049	0.0101
60	0.7326	0.4341	−0.4765	0.7349	0.4381	−0.4826
75	0.9121	0.2493	−0.8388	0.9135	0.2543	−0.8424
90	0.9783	−0.0003	−0.9728	0.9783	0.0003	−0.9728

Table 3 Values of k_I, k_{II}, and t for uniaxial tension $\sigma_{yy}^{\infty} = \sigma_0$ in an angle β (h/a = 0.4)

β (deg)	Tip A(a, h)			Tip D(a, $-h$)		
	k_I	k_{II}	t	k_I	k_{II}	t
5	−0.0202	0.0802	1.0006	0.0317	0.0822	0.9683
15	−0.0239	0.2251	0.9095	0.1254	0.2426	0.8166
30	0.0601	0.3725	0.5694	0.3186	0.4377	0.4085
45	0.2295	0.4025	0.0708	0.5280	0.5330	−0.1149
60	0.4389	0.3072	−0.4527	0.6974	0.5029	−0.6135
75	0.6322	0.1121	−0.8608	0.7814	0.3556	−0.9537
90	0.7575	−0.1305	−1.0442	0.7575	0.1305	−1.0442

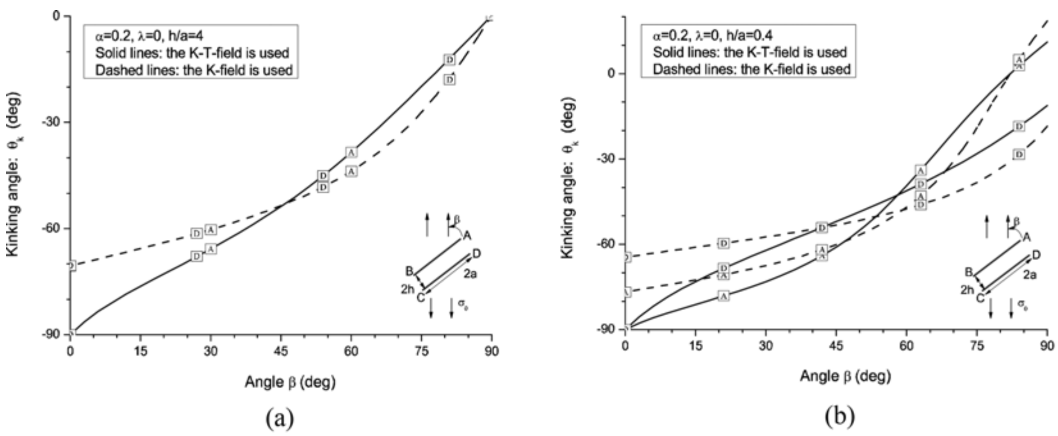


Fig. 3 Kinking angle at the crack tips A(a, h) and D(a, $-h$) versus the crack inclination angle β for two parallel cracks under simple uniaxial tension; (a) h/a = 4, (b) h/a = 0.4

occur in practice since the crack is closed in this case, it only implies a shielding effect for very small β angles at the tip A. Similarly, the upper crack has also a shielding effect at the tip C. With β rising, the shielding effects disappear. Furthermore, the kinking angle as a function of the crack angle β is displayed in Figs. 3(a,b), respectively, where solid lines correspond to those with the T-stress, i.e., the K-T-field is used, whereas dashed lines correspond to those without T-stress, i.e., only K-field is used. Obviously, the crack kinking angle is sensitive to the T-stress. When the distance between two cracks is large enough (h/a = 4), the interaction of two cracks is negligible, moreover in this case the kinking angles coincide with those for a single crack subjected to the same loading (Williams and Ewing 1972). However, when the distance between two cracks is small (h/a = 0.4), the interaction of two cracks is significant and cannot be ignored. Or rather, from Fig. 3(b) when $\beta = 90^{\circ}$, the crack kinking angle is larger than zero for h/a = 0.4 ($\theta_k = 11.2^{\circ}$ when using the bi-parameter criterion if taking $\alpha = 0.2$, and $\theta_k = 18.5^{\circ}$ when using the uni-parameter criterion), manifesting that two cracks have a tendency to inhibit two cracks to deviate away. This result turns out the strong interaction of two cracks. While for two cracks with large spacing h/a = 4, two cracks will advance along the individual original crack planes if $\beta = 90^{\circ}$, which can be clearly observed in Fig. 3(a). Therefore, for uniaxial tension, the classical kinking angle is overestimated for larger β angles and underestimated for smaller β angles, respectively.

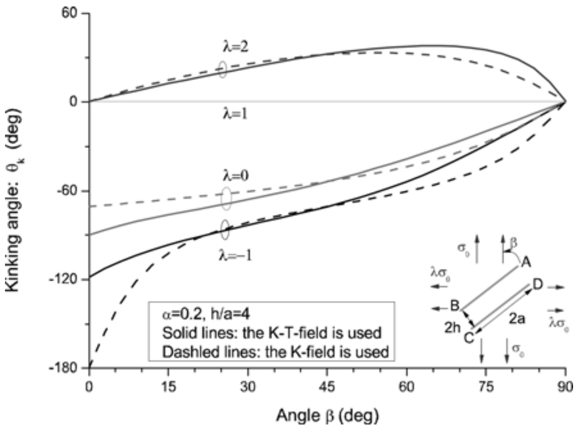
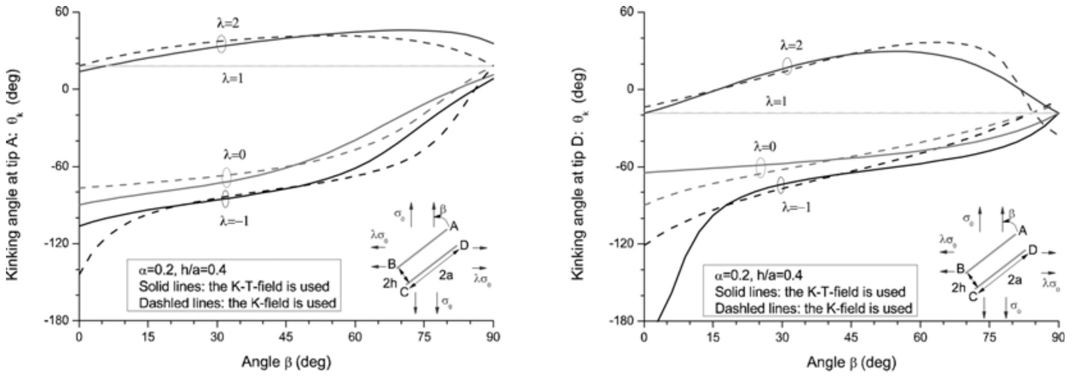


Fig. 4 Kinking angle at the crack tips A(a, h) and D(a, $-h$) versus the crack inclination angle β for two parallel cracks with large spacing for different biaxiality ratios



(a) (b)

Fig. 5 Kinking angle versus crack inclination angle β for two parallel cracks with smalle spacing for different biaxiality ratios; (a) tip A(a, h), (b) tip D(a, $-h$)

In the following, an analysis of the effects of the *T*-stress on crack initiation angle is made for two parallel cracks under different biaxiality ratios. Fig. 4 displays the kinking angle θ_k versus the crack inclination angle β for crack spacing h/a = 4. For this case, the interaction between two cracks is weak, so the curves shown in Fig. 4 in fact approximately give the corresponding dependence of an isolated crack. From Fig. 4, it is seen that the kinking angle $\theta_k < 0$ for $\lambda < 1$, and $\theta_k > 0$ for $\lambda > 1$. In particular, when $\lambda = 1$, meaning that tensile principal stresses are equal, two cracks expand along their individual crack plane. This is easily understood since $\sigma_{xy}^{\infty} = 0$ in this case. Additionally, for $\lambda \geq 0$, meaning that the cracked plate is subjected to uniaxial or biaxial tension, one can find that the solid lines in Fig. 4 usually lie above the dashed lines for large β angles, while they lie below the dashed lines for small β angles. This reveals the contribution of the biaxiality ratio on the kinking angle. In other words, if $\lambda > 1$, the classical kinking angle is overestimated for β larger than about 45° and underestimated for β less than about 45°, whereas the trend is reversed for $0 \leq \lambda < 1$. This is not surprising because σ_0 in place of $\lambda\sigma_0$ becomes a larger tensile principal stress. Furthermore, for tensile-compressive loading, the situation becomes more complicated, as seen from the curves with $\lambda = -1$. Since two cracks have large spacing, the tips A and D almost have a

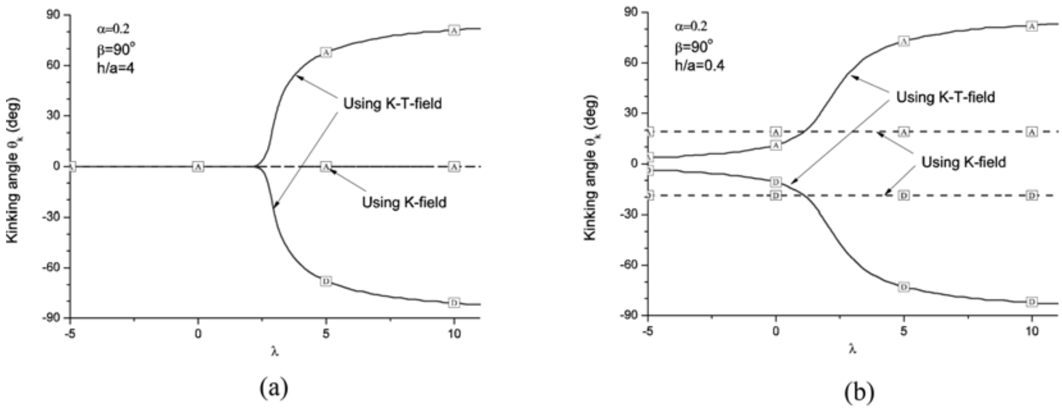


Fig. 6 Effect of the biaxiality ratio λ on the kinking angle at the crack tips A(a, h) and D(a, $-h$) versus λ with $\beta = 90^{\circ}$, $\alpha = 0.2$; (a) h/a = 4, (b) h/a = 0.4

common feature. For two close parallel cracks, Figs. 5(a,b) show the variation of the kinking angle at the tips A and D versus the crack inclination angle β with different biaxiality ratios. The trends observed in Fig. 4 are still retained for two near cracks in Figs. 5(a,b), and the magnitude of the kinking angle at the tip A is however noticeably different from that at the tip D. For example, for the case of equal-biaxiality ratio, the crack deviates away at the tip A in a positive branching angle, and in a negative branching angle at the tip D. The branching angle at the tips A and D depends on β save the case of $\lambda = 1$.

To further demonstrate the effects of the *T*-stress on the crack kinking angle, Figs. 6(a,b) display the dependence of the kinking angle on the biaxial ratio λ for h/a = 4,0.4 with $\beta = 90^{\circ}$, respectively. It is evident that for h/a = 4, α = 0.2, the kinking angle exhibits a nearly symmetrical characteristic with respect to the crack plane, which basically agrees with that for a single crack in an infinite plane (Ayatollahi *et al.* 2002). The reason is that the crack spacing is large, and so their interaction is negligible. As a result, two cracks branch in nearly identical kinking angle. Moreover, the higher the value of λ, the larger the kinking angle. However, if the *T*-stress is neglected, the classical kinking angle implies that two cracks advance straightly ahead, and do not kink or branch, irrespective of the value of λ. For two parallel cracks with small crack spacing h/a = 0.4, the above feature is no longer retained. In this case, the kinking angle at the tips A and D versus λ is illustrated in Fig. 6(b). Since the interaction of two cracks is remarkable, it is observed in Fig. 6(b) that one crack kinks away from the other, even for the classical result. Nevertheless, the classical result without the *T*-stress does not vary with λ, implying two cracks deviate in an identical angle. Opposite to the above, when the nonsingular stress is included a large positive λ increases the kinking angle, and a negative λ decreases the kinking angle. In particular, when λ = 1, the kinking angle based on the *K*-*T*-field is almost equal to the one based on the *K*-field, while the classical kinking angle is severely underestimated for $\lambda > 1$, and overestimated for $\lambda < 1$, respectively.

5. Conclusions

Using the fundamental solution of the elastic field induced by an edge dislocation, singular integral equations for two parallel cracks have been derived. The stress intensity factors and elastic

T-stress were evaluated by numerical solutions of the resulting equations. Results were presented for two parallel cracks under the action of uniaxial and biaxial tension. The effects of the *T*-stress on the crack kinking angle was discussed. It was found that the *T*-stress plays a strong role in dominating crack growth behavior, in particular for two cracks with small spacing. Therefore, in engineering design an efficient approach for changing the crack growth direction is to suitably adjust crack orientation and biaxiality ratio. Main conclusions are drawn as follows:

- There is a shielding effect when two parallel cracks are very close for small β angles between the cracks and loading direction.
- For uniaxial tension, the classical kinking angle is overestimated for larger β angles and underestimated for smaller β angles, respectively.
- For the case of biaxial loading, the classical kinking angle is overestimated for $\lambda < 1$ and underestimated for $\lambda > 1$, respectively.

Acknowledgements

This work was supported by the National Natural Science Foundation of China (Grant No. 10672189).

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