Structural Engineering and Mechanics, Vol. 33, No. 6 (2009) 675-696
DOI: http://dx.doi.org/10.12989/sem.2009.33.6.675

Solution method for the classical beam theory using differential quadrature

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(Received February 29, 2008, Accepted September 29, 2009)

Abstract. In this paper, a unified solution method is presented for the classical beam theory. In Strength of Materials approach, the geometry, material properties and load system are known and related with the unknowns of forces, moments, slopes and deformations by applying a classical differential analysis in addition to equilibrium, constitutive, and kinematic laws. All these relations are expressed in a unified formulation for the classical beam theory. In the special case of simple beams, a system of four linear ordinary differential equations of first order represents the general mechanical behaviour of a straight beam. These equations are solved using the numerical differential quadrature method (DQM). The application of DQM has the advantages of mathematical consistency and conceptual simplicity. The numerical procedure is simple and gives clear understanding. This systematic way of obtaining influence line, bending moment, shear force diagrams and deformed shape for the beams with geometric and load discontinuities has been discussed in this paper. Buckling loads and natural frequencies of any beam prismatic or non-prismatic with any type of support conditions can be evaluated with ease.

Keywords: differential quadrature; buckling load; natural frequency; boundary conditions; constitutive law; equilibrium.

1. Introduction

There is much literature treating classical beam theory (Timoshenko 1938) going bottom up through tension, torsion, bending, compound stresses, deflections in statically determinate beams to hyperstatic structures and end up with energy methods, buckling and considering even curved beams (Rajasekaran and Padmanabhan 1989, Gimena *et al.* 2008). In this paper, first we will present the general equations of the general problem of a straight beam. The system of differential equations that relate forces, moments, slopes and deflections presented in a compact format, offers certain advantages, when deriving solutions for special cases. Equilibrium, constitutive relationships,

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compatibility equations are derived for a straight beam. All of these equations will be combined into a single unique system of twelve equations with twelve unknowns. This novel unified formulation has several advantages. It permits ready identification and all data needed to solve structural problem. The theory does not distinguish a priori between different kinds of structural elements, as compression, torsion, bending of beams all of which are special cases of the general formulation. In this paper, differential quadrature method DQM is employed as an effective numerical tool for solving the above mentioned domain problems having any form of discontinuity in geometry materials, loading, deformations and boundary conditions. Concentrated stress resultants are considered as discontinuity in loading and influence line for stress resultants are drawn by considering discontinuity in deformations. The DQM can be used to solve prismatic/non-prismatic straight or curved beams subjected to any type of loading and also used to draw the influence lines for various stress resultants at any point involving fewer grid points yet achieving acceptable accuracy. Buckling loads and natural frequencies of any beam prismatic or non-prismatic with any type of support conditions can be evaluated with ease.

2. Unified formula for straight beam

In this section of the paper, a unified formulation for the analysis of a straight beam will be offered to apply the DQM for obtaining accurate results. This new approach for considering the classical beam theory, permits solving the problem taken into account the different boundary conditions at the same time. Arbitrary curved beams in local or global coordinates have been also investigated by the authors (Gimena 2008).

2.1 Assumptions and limitations

1.- It is assumed that the elastic principle established by Hooke and generalized further by Cauchy applies.
2.- The material behaviour is assumed to be linear homogeneous and isotropic.
3.- Second order effects like the change of forces and moments as a result of elastic deflection are not considered.
4.- Principle of superposition applies because of linearity of the problem.
5.- Saint-Venant's principle permits the representation of internal stresses by stress resultants acting on the cross section centroid.
6.- Similarly external loads can be represented by forces and moments applied to the section centroid.
7.- Navier-Bernoulli's hypothesis is assumed.

2.2 Known quantities

Based on the assumptions, concerning the member shape, the known quantities of the problem should be related in two consecutive points separated by an infinitesimal segment of length dx. In general, longitudinal E and transversal G, Young's moduli are functions of x. Likewise, all geometric section properties, cross sectional area A, shear factors $\alpha_y, \alpha_z, \alpha_{yz}$ and moments of inertia I_y, I_z, I_{yz} and torsional inertia I_t are also functions of x. External load consists of forces q_x, q_y, q_z and moments m_x, m_y, m_z rotations $\theta_x, \theta_y, \theta_z$ and displacements $\Delta_x, \Delta_y, \Delta_z$ defined in right hand coordinate system O_{xyz}. These are the known quantities needed to evaluate the unknown internal forces, moments, rotations and displacements.

2.3 Unknown quantities

The objective of the beam analysis problem is to determine the internal state of stresses and displacements produced by externally applied loads and deformations. These quantities may be expressed for each point P of the line element as the sum of internal stresses or strains developed in the element. The internal forces and moments can be expressed as (see Fig. 1)

$$N = \int_A \sigma dA; \quad V_y = \int_A \tau_y dA; \quad V_z = \int_A \tau_z dA$$

$$T = \int_A (\tau_z y - \tau_y z) dA; \quad M_y = \int_A \sigma z dA; \quad M_z = -\int_A \sigma y dA \tag{1}$$

The remaining unknowns are the rotations ϕ, θ_y, θ_z about x, y and z axes respectively, and displacement u, v and w in the coordinate axes at the point P on the member centre line, respectively.

2.4 Formulation of the beam analysis problem

Once all the known and unknown quantities have been identified, the beam analysis problem can be formulated using the equations of equilibrium, constitutive relationship and compatibility equations.

Equilibrium equations

The internal forces acting on the infinitesimal beam segment must equilibrate the loads applied to the segment. Force equilibrium in a tangent direction x in normal direction y and normal direction z

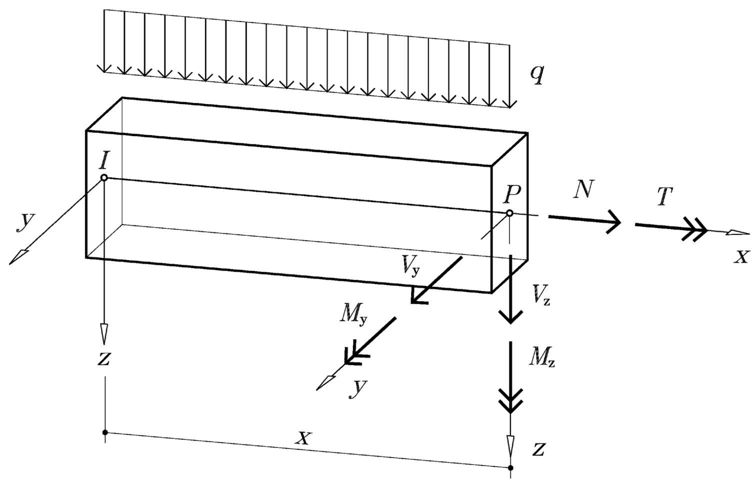

Fig. 1 Force and moment components on a beam section

is established by the following equations.

$$\frac{dN}{dx} + q_x = 0; \quad \frac{dV_y}{dx} + q_y = 0; \quad \frac{dV_z}{dx} + q_z = 0$$

$$\frac{dT}{dx} + m_x = 0; \quad -V_z + \frac{dM_y}{dx} + m_y = 0; \quad V_y + \frac{dM_z}{dx} + m_z = 0 \tag{2}$$

Constitutive relations

According to Hooke's law, stresses and strains are related as follows

$$\varepsilon_x = \frac{\sigma}{E}; \quad \varepsilon_y = -\frac{\nu\sigma}{E}; \quad \varepsilon_z = -\frac{\nu\sigma}{E}$$

$$\gamma_{yz} = 0; \quad \gamma_{xz} = \frac{\tau_z}{G}; \quad \gamma_{xy} = \frac{\tau_y}{G} \tag{3}$$

where $\nu = \frac{E}{2G} - 1$ is the Poison's coefficient.

Compatibility equations

The equations of compatibility of rotations and displacements can be given as

$$-\frac{T}{GI_t} + \frac{d\phi}{dx} - \Theta_x = 0$$

$$-\frac{M_y I_z}{E[I_y I_z - I_{yz}^2]} - \frac{M_z I_{yz}}{E[I_y I_z - I_{yz}^2]} + \frac{d\theta_y}{dx} - \Theta_y = 0$$

$$-\frac{M_y I_{yz}}{E[I_y I_z - I_{yz}^2]} - \frac{M_z I_y}{E[I_y I_z - I_{yz}^2]} + \frac{d\theta_z}{dx} - \Theta_z = 0$$

$$-\frac{N}{EA} + \frac{du}{dx} - \Delta_x = 0$$

$$-\frac{\alpha_y V_y}{GA} - \frac{\alpha_{yz} V_z}{GA} - \theta_z + \frac{dv}{dx} - \Delta_y = 0$$

$$-\frac{\alpha_{yz} V_y}{GA} - \frac{\alpha_z V_z}{GA} + \theta_y + \frac{dw}{dx} - \Delta_z = 0 \tag{4}$$

Eqs. (2) and (4) represent the general mechanical behaviour of a straight beam.

2.5 General equations

Equilibrium and compatibility conditions as well as the constitutive relationships (Eqs. (2), (3) and (4)) yielded the system of twelve differential equations (Gimena 2003).

$$
\begin{aligned}
&\frac{dN}{dx} + q_x = 0\\
&+\frac{dV_y}{dx} + q_y = 0\\
&+\frac{dV_z}{dx} + q_z = 0\\
&\frac{dT}{dx} + m_x = 0\\
&-V_z + \frac{dM_y}{dx} + m_y = 0\\
&V_y + \frac{dM_z}{dx} + m_z = 0\\
&-\frac{T}{GI_t} + \frac{d\phi}{dx} - \Theta_x = 0\\
&-\frac{M_y I_z}{E\left[I_y I_z - I_{yz}^2\right]} - \frac{M_z I_{yz}}{E\left[I_y I_z - I_{yz}^2\right]} + \frac{d\theta_y}{dx} - \Theta_y = 0\\
&-\frac{M_n I_{nb}}{E\left[I_y I_z - I_{yz}^2\right]} - \frac{M_b I_n}{E\left[I_y I_z - I_{yz}^2\right]} + \frac{d\theta_z}{dx} - \Theta_z = 0\\
&-\frac{N}{EA} + \frac{du}{dx} - \Delta_x = 0\\
&-\frac{\alpha_y V_y}{GA} - \frac{\alpha_{yz} V_z}{GA} - \theta_z + \frac{dv}{dx} - \Delta_y = 0\\
&-\frac{\alpha_{yz} V_y}{GA} - \frac{\alpha_z V_z}{GA} + \theta_y + \frac{dw}{dx} - \Delta_z = 0
\end{aligned}
\tag{5}
$$

These can be written in vectorial form as shown below.

$$
\begin{bmatrix}
\nabla & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & \nabla & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & \nabla & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & \nabla & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & -1 & 0 & \nabla & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 1 & 0 & 0 & 0 & \nabla & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -\gamma_1 & 0 & 0 & \nabla & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\gamma_2 & -\gamma_3 & 0 & \nabla & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & -\gamma_3 & -\gamma_4 & 0 & 0 & \nabla & 0 & 0 & 0\\
-\varepsilon_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \nabla & 0 & 0\\
0 & -\varepsilon_2 & -\varepsilon_3 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & \nabla & 0\\
0 & -\varepsilon_3 & -\varepsilon_4 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \nabla
\end{bmatrix}
\begin{Bmatrix} N\\ V_y\\ V_z\\ T\\ M_y\\ M_z\\ \phi\\ \theta_y\\ \theta_z\\ u\\ v\\ w \end{Bmatrix}
+
\begin{Bmatrix} q_x\\ q_y\\ q_z\\ m_x\\ m_y\\ m_z\\ -\Theta_x\\ -\Theta_y\\ -\Theta_z\\ -\Delta_x\\ -\Delta_y\\ -\Delta_z \end{Bmatrix}
=
\begin{Bmatrix} 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0\\ 0 \end{Bmatrix}
\tag{6}
$$

where

$\nabla = \dfrac{d}{dx}$;

$\varepsilon_1 = \dfrac{1}{EA}$; $\varepsilon_2 = \dfrac{\alpha_y}{GA}$; $\varepsilon_3 = \dfrac{\alpha_{yz}}{GA}$; $\varepsilon_4 = \dfrac{\alpha_z}{GA}$

$$\gamma_1 = \frac{1}{GI_t};\ \gamma_2 = \frac{I_z}{E[I_y I_z - I_{yz}^2]};\ \gamma_3 = \frac{I_{yz}}{E[I_y I_z - I_{yz}^2]};\ \gamma_4 = \frac{I_y}{E[I_y I_z - I_{yz}^2]}$$

where $N, V_y, V_z, T, M_y, M_z, \phi, \theta_y, \theta_z, u, v, w$ are the twelve components of forces, moments, rotations and displacements of the state and $q_x, q_y, q_z, m_x, m_y, m_z$ and $\Theta_x, \Theta_y, \Theta_z, \Delta_x, \Delta_y, \Delta_z$ are the twelve forces and moments and rotations and displacements of the actions distributed along the axis of the beam.

Note the comprehensive order in which the functions are arranged in the system; forces, moments, rotations and displacements and the various effects are coupled.

2.6 Special cases

The above equation represents the mechanical behaviour of a general straight beam under different types of effects, i.e., traction, compression, torsional and bending which are combined and their effects interact because of the cross section relationships.

In a particular case when principal axes coincide with the section ones $I_{yz} = \alpha_{yz} = 0$ and neglecting shearing deformation $\alpha_y = \alpha_z = 0$ the whole system of equations can be uncoupled into different subsystems to approach to the special cases.

If the displacements and rotations of the actions are zero

Traction and compression effect

$$\begin{bmatrix} \nabla & 0 \\ -\frac{1}{EA} & \nabla \end{bmatrix} \begin{Bmatrix} N \\ u \end{Bmatrix} + \begin{Bmatrix} q_x \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{7}$$

Torsion effect

$$\begin{bmatrix} \nabla & 0 \\ -\frac{1}{GI_t} & \nabla \end{bmatrix} \begin{Bmatrix} T \\ \phi \end{Bmatrix} + \begin{Bmatrix} m_x \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{8}$$

Bending effect – Pure flexion about z axis

$$\begin{bmatrix} \nabla & 0 & 0 & 0 \\ 1 & \nabla & 0 & 0 \\ 0 & -\frac{1}{EI_z} & \nabla & 0 \\ 0 & 0 & -1 & \nabla \end{bmatrix} \begin{Bmatrix} V_y \\ M_z \\ \theta_z \\ v \end{Bmatrix} + \begin{Bmatrix} q_y \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{9}$$

Bending effect – Pure flexion about y axis

$$\begin{bmatrix} \nabla & 0 & 0 & 0 \\ -1 & \nabla & 0 & 0 \\ 0 & -\frac{1}{EI_y} & \nabla & 0 \\ 0 & 0 & 1 & \nabla \end{bmatrix} \begin{Bmatrix} V_z \\ M_y \\ \theta_y \\ w \end{Bmatrix} + \begin{Bmatrix} q_z \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{10}$$

A unique set of twelve linear first order differential equations which represent the general mechanical behaviour of straight beam is established. It is also shown that this system of equations could be, for special cases, uncoupled into traction, torsion and bending systems. The method does not distinguish between different types of beams as statically determinate or indeterminate.

3. Differential quadrature method

This problem of structural behavior of the classical beam theory, either prismatic or non-prismatic, with different loads and supports, could easily be solved using Differential Quadrature Method (DQM) which was introduced by Bellman and Casti (1971). With the application of boundary conditions as per Wilson's method (Wilson 2002) DQM will also be straight forward and easy to use by the engineers. Since the introduction of this method, applications of the differential quadrature method to various engineering problems have been investigated and their success has shown the potential of the method as an attractive numerical analysis tool. The basic idea of DQM is to quickly compute the derivatives of a function at any grid point within its bounded domain by estimating the weighted sum of the values of the functions at a small set of points related to the domain. In the originally derived DQM, Lagrangian interpolation polynomial was used by Bert and Malik (1996) and (Bert *et al.* 1993, 1994). A subsequent approach of the original differential quadrature approximation is called Harmonic Differential Quadrature (HDQ) originally proposed by (Striz *et al* 1995). Ulker and Civalek (2004) applied HDQ to the bending analysis of beams and plates. Wu and Liu (2000) applied the generalized differential quadrature to axi-symmetric bending solution of shells of revolution which includes both long and short cylinders and storage tanks with stepped wall thickness. Lam and Li (2000) applied the differential quadrature for finding the frequency of rotating multi-layered conical shells. Unlike DQM, HDQ uses harmonic or trigonometric functions as test functions. As the name of the test function suggested, this method is called the HDQ method. All the problems in this paper have demonstrated that the application of the DQM and HDQ will lead to accurate results with less computational effort and that there is a potential that the method may become alternative to conventional methods such as Finite Difference, Finite Element and Boundary Element methods.

For a function $f(\xi)$, DQM approximation of the m^{th} order derivative at the i^{th} sampling point is given by

$$\frac{d^m}{d\xi^m}\begin{Bmatrix} f(\xi_1) \\ f(\xi_2) \\ \vdots \\ f(\xi_n) \end{Bmatrix} = C_{ij}^{(m)}\begin{Bmatrix} f(\xi_1) \\ f(\xi_2) \\ \vdots \\ f(\xi_n) \end{Bmatrix} \quad \text{for } i,j = 1,2,\ldots,n \tag{11}$$

where n is the number of the sampling points.

Assuming Lagrangian interpolation polynomial

$$f(\xi) = \frac{M(\xi)}{(\xi-\xi_i)M_1(\xi_i)} \quad \text{for } i = 1,2,\ldots,n \tag{12}$$

where

$$M(\xi) = \prod_{j=1}^{n}(\xi - \xi_j); \quad M_1(\xi_i) = \prod_{j=1, j\neq i}^{n}(\xi_i - \xi_j) \quad \text{for } i,j = 1,2,\ldots,n$$

Substituting Eq. (12) in Eq. (11) leads to

$$C_{ij}^{(1)} = \frac{M_1(\xi_i)}{(\xi_i - \xi_j)M_1(\xi_j)}; \quad \text{for} \quad i,j = 1,2,\ldots,n;\ i\neq j$$

$$C_{ii}^{(1)} = -\sum_{\substack{j=1\\ j\neq i}}^{n} C_{ij}^{(1)}; \qquad \text{for} \quad i = 1,2,\ldots,n \tag{13}$$

The second and third and higher derivative can be calculated as

$$C_{ij}^{(2)} = \sum_{k=1}^{n} C_{ik}^{(1)} C_{kj}^{(1)}; \qquad \text{for} \quad i = j = 1,2,\ldots,n$$

$$C_{ij}^{(m)} = \sum_{k=1}^{n} C_{ik}^{(1)} C_{kj}^{(m-1)}; \quad \text{for} \quad i = j = 1,2,\ldots,n \tag{14}$$

and the number of the sampling points $n > m$.

A natural and often convenient choice for the sampling point is that of equally spaced points or Chebyshev-Gauss-Lobatto (CGL) mesh distribution as given by Eq. (15). For the sampling points, we adopt the well accepted CGL mesh distribution and its normalized form is given by (Shu 2000) as

$$\xi_i = \frac{1}{2}\left[1 - \cos\frac{(i-1)}{(n-1)}\pi\right] \tag{15}$$

where $\xi_i = \dfrac{x_i}{L}$

being L the length of the beam that is divided into $n-1$ divisions or n sampling points in case of DQ method and x_i is the distance from the left end of the beam.

4. Harmonic Differential Quadrature method (HDQ)

The Harmonic test function $h_i(\xi)$ used in HDQ method is defined as

$$h_i(\xi) = \frac{\prod_{k=0,\ k\neq i}^{n} \sin[\pi(\xi - \xi_k)/2]}{\prod_{k=0,\ k\neq i}^{n} \sin[\pi(\xi_i - \xi_k)/2]} \tag{16}$$

According to the HDQ, the weighting coefficients of the first order derivative C_{ij}^{1} denoted as $C(i,j,1)$, for $i\neq j$ is obtained using the form (Ulker and Civalek 2004)

$$C(i,j,1) = \frac{\pi P(\xi_i)/2}{P(\xi_j)\sin[\pi(\xi_i - \xi_j)/2]} \qquad \text{for} \quad i = j = 1, 2, \ldots, n \tag{17}$$

$$C(i,i,1) = -\sum_{\substack{j=1 \\ j \neq i}}^{n} C(i,j,1) \qquad \text{for} \quad i = 1, 2, \ldots, n \tag{18}$$

where

$$P(\xi_i) = \prod_{j=1,\, j \neq i}^{n} \sin[\pi(\xi_i - \xi_j)/2] \qquad \text{for} \quad i = 1, 2, \ldots, n$$

The weighting coefficients of the second order derivative are given by

$$C(i,j,2) = C(i,j,1)\{2C(i,j,1) - \pi\cot[(\xi_i - \xi_j)/2]\} \qquad \text{for} \quad i = j = 1, 2, \ldots, n$$

$$C(i,i,2) = -\sum_{\substack{j=1 \\ j \neq i}}^{n} C(i,j,2) \qquad \text{for} \quad i = j = 1, 2, \ldots, n \tag{19}$$

Higher order derivatives can be obtained using Eq. (14).

5. Solution of pure flexion problems

5.1 Applying DQM to the system of differential equations

The governing equations of pure flexion problems are given by Eq. (9)

$$\begin{bmatrix} \nabla & 0 & 0 & 0 \\ 1 & \nabla & 0 & 0 \\ 0 & -\dfrac{1}{EI} & \nabla & 0 \\ 0 & 0 & -1 & \nabla \end{bmatrix} \begin{Bmatrix} V \\ M \\ \theta \\ v \end{Bmatrix} + \begin{Bmatrix} q \\ 0 \\ 0 \\ 0 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix}$$

Assume the beam is idealized into n sampling points as given by CGL mesh distribution or uniform distribution. ∇V at any j^{th} point is given by

$$\nabla V_j = C(j, 1{:}n, 1)\{V\}/L \tag{20}$$

Hence $\nabla\{V\}$ at all the sampling points can be calculated as

$$\nabla\{V\} = C(:,:,1)\{V\}/L \tag{21}$$

Hence $\{V\}, \{M\}, \{\theta\}, \{v\}$ at n sampling points are given by

$$\begin{bmatrix} [C(:,:,1)/L] & [0] & [0] & [0] \\ [\mathbf{I}] & [C(:,:,1)/L] & [0] & [0] \\ [0] & -[1/EI] & [C(:,:,1)/L] & [0] \\ [0] & [0] & -[\mathbf{I}] & [C(:,:,1)/L] \end{bmatrix} \begin{Bmatrix} \{V\} \\ \{M\} \\ \{\theta\} \\ \{v\} \end{Bmatrix} = \begin{Bmatrix} \{q\} \\ \{0\} \\ \{0\} \\ \{0\} \end{Bmatrix} \tag{22}$$

where $[1/EI]$ is a diagonal matrix consisting of $1/EI$ at sampling points.
Eq. (22) is rewritten as

$$\underset{4n\times 4n}{[D]}\underset{4n\times 1}{\{r\}} = \underset{4n\times 1}{\{R\}} \tag{23}$$

where $[D]$ is an un-symmetric matrix.
Multiplying both sides of the Eq. (23) by $[D]^T$ we get

$$[D]^T[D]\{r\} = [D]^T\{R\} \text{ or } [G]\{r\} = \{E\} \tag{24}$$

where $\{r\}$, from 1 to n denotes the shear at sampling points, n+1 to $2n$ denotes moments at the sampling points, $2n$+1 to $3n$ denotes slopes at the sampling points and $3n$+1 to $4n$ denotes deflections at the sampling points.

Boundary conditions
Since it is a system of four differential equations of first order, four boundary conditions should be given.
The boundary conditions will be applied as follows.
Clamped - Pinned

$$\begin{aligned} v(x=0) &= 0;\ G[4n+1, 3n+1] = 1 \\ \theta(x=0) &= 0;\ G[4n+2, 2n+1] = 1 \\ v(x=L) &= 0;\ G[4n+3, 4n] = 1 \\ M(x=L) &= 0;\ G[4n+4, 2n] = 1 \end{aligned} \tag{25}$$

Clamped – Clamped

$$\begin{aligned} v(x=0) &= 0;\ G[4n+1, 3n+1] = 1 \\ \theta(x=0) &= 0;\ G[4n+2, 2n+1] = 1 \\ v(x=L) &= 0;\ G[4n+3, 4n] = 1 \\ \theta(x=L) &= 0;\ G[4n+4, 3n] = 1 \end{aligned} \tag{26}$$

Pinned – Pinned

$$\begin{aligned} v(x=0) &= 0;\ G[4n+1, 3n+1] = 1 \\ M(x=0) &= 0;\ G[4n+2, 2n+1] = 1 \\ v(x=L) &= 0;\ G[4n+3, 4n] = 1 \\ M(x=L) &= 0;\ G[4n+4, 2n] = 1 \end{aligned} \tag{27}$$

Clamped – Free

$$\begin{aligned} &v(x=0)=0;\ G[4n+1,3n+1]=1 \\ &\theta(x=0)=0;\ G[4n+2,2n+1]=1 \\ &M(x=L)=0;\ G[4n+3,2n]=1 \\ &V(x=L)=0;\ G[4n+4,n]=1 \end{aligned} \tag{28}$$

Wilson's method of applying boundary conditions (Wilson 2002)
In general, the boundary conditions are given by

$$\underset{4\times 4n}{[G]_1}\underset{4n\times 1}{\{r\}}=\underset{4\times 1}{\{E\}_1} \tag{29}$$

Combining governing equations and boundary conditions we get

$$\begin{bmatrix} \underset{4n\times 4n}{[G]} \\ \underset{4\times 4n}{[G]_1} \end{bmatrix}\{r\}=\begin{Bmatrix} \underset{4n\times 1}{\{E\}} \\ \underset{4\times 1}{\{E\}_1} \end{Bmatrix} \tag{30}$$

Using Lagrange multiplier approach as recommended by (Wilson 2002), Eq. (30) can be modified to square matrix as

$$\begin{bmatrix} [G] & [G]_1^T \\ [G]_1 & [0] \end{bmatrix}\begin{Bmatrix} \{r\} \\ \{\lambda\} \end{Bmatrix}=\begin{Bmatrix} \{E\} \\ \{E\}_1 \end{Bmatrix} \tag{31}$$

The above equation has both equilibrium and equation of geometry. Solving Eq. (31) as an equilibrium problem, one will be able to get the shear, moment, rotations and deflections all along the sampling points of the beam.

5.2 Example 1. Beam fixed at both ends subjected to uniform distributed load (udl)

Fig. 2 shows the bending moment diagram of a fixed - fixed beam of span 10 m subjected to uniformly distributed load of 1 kN/m. Even though the above method is simple and can handle discontinuity in geometry and loading, the method involves the solution of large number of equations. To handle discontinuity in geometry and loading we will present the following method.

Fig. 2 Bending moment diagram for a fixed beam under constant loading

5.3 Applying DQM to a high order differential equation

The governing equation for flexion problem (assume $D = EI$ varies along the length) expresses equilibrium in terms of displacement for the beam flexion problem. The elastic line equation, combines equilibrium, constitutive and compatibility equations

$$D\frac{d^4v}{dx^4}+2\frac{dD}{dx}\frac{d^3v}{dx^3}+\frac{d^2D}{dx^2}\frac{d^2v}{dx^2} = q \tag{32}$$

Eq. (32) is written in differential quadrature form as

$$[[k]C(:,:,4)/L^4+2[\alpha]C(:,:,3)/L^3+[\beta]C(:,:,2)/L^2]\{v\} = \{q\} \tag{33}$$

where

$$[k]=\begin{bmatrix} D_1 & 0 & \cdots & 0 \\ 0 & D_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & D_n \end{bmatrix};\ [\alpha]=\begin{bmatrix} \alpha_1 & 0 & \cdots & 0 \\ 0 & \alpha_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \alpha_n \end{bmatrix};\ [\beta]=\begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \beta_n \end{bmatrix}$$

being $\alpha_i = C(i,1:n,1)\{D\}/L$ and $\beta_i = C(i,1:n,2)\{D\}/L^2$

Eq. (33) is rewritten as

$$[G]\{v\} = \{q\} \tag{34}$$

Unlike the method exposed in section 5.1, now $[G]$ matrix is un-symmetric.

Boundary conditions must be applied similar to method in 5.1 section but in term of displacements as shown below. Eq. (34) can be solved as an equilibrium problem.

Boundary conditions

Since it is a fourth order differential equation, four boundary conditions should be given. The boundary conditions will be applied as follows.

Clamped - Pinned

$$\begin{aligned} &v(x=0) = 0;\ G[n+1,1]=1 \\ &v'(x=0) = 0;\ G[n+2,1:n]=C(1,1:n,1)/L \\ &v(x=L) = 0;\ G[n+3,n]=1 \\ &v''(x=L) = 0;\ G[n+4,1:n]=C(n,1:n,2)/L^2 \end{aligned} \tag{35}$$

Clamped – Clamped

$$\begin{aligned} &v(x=0) = 0;\ G[n+1,1]=1 \\ &v'(x=0) = 0;\ G[n+2,1:n]=C(1,1:n,1)/L \\ &v(x=L) = 0;\ G[n+3,n]=1 \\ &v'(x=L) = 0;\ G[n+4,1:n]=C(n,1:n,1)/L \end{aligned} \tag{36}$$

Pinned – Pinned

$$\begin{aligned} &v(x=0) = 0;\ G[n+1,1] = 1 \\ &v''(x=0) = 0;\ G[n+2,1{:}n] = C(1,1{:}n,2)/L^2 \\ &v(x=L) = 0;\ G[n+3,n] = 1 \\ &v''(x=L) = 0;\ G[n+4,1{:}n] = C(n,1{:}n,2)/L^2 \end{aligned} \tag{37}$$

Clamped - Free

$$\begin{aligned} &v(x=0) = 0;\ G[n+1,1] = 1 \\ &v'(x=0) = 0;\ G[n+2,1{:}n] = C(1,1{:}n,1)/L \\ &v''(x=L) = 0;\ G[n+3,1{:}n] = C(n,1{:}n,2)/L^2 \\ &v'''(x=L) = 0;\ G[n+4,1{:}n] = C(n,1{:}n,3)/L^3 \end{aligned} \tag{38}$$

Wilson's method of applying boundary conditions (Wilson 2002)

In general, the boundary conditions are given by (in case of buckling and free vibration problems)

$$\underset{4\times n}{[G]_1}\ \underset{n\times 1}{\{v\}} = \underset{4\times 1}{\{E\}_1} \tag{39}$$

Combining governing equations and boundary conditions we get

$$\begin{bmatrix} \underset{n\times n}{[G]} \\ \underset{4\times n}{[G]_1} \end{bmatrix} \{v\} = \begin{Bmatrix} \underset{n\times 1}{\{E\}} \\ \underset{4\times 1}{\{E\}_1} \end{Bmatrix} \tag{40}$$

Using Lagrange multiplier approach as recommended by (Wilson 2002), Eq. (40) can be modified to square matrix as

$$\begin{bmatrix} [G] & [G]_1^T \\ [G]_1 & [0] \end{bmatrix} \begin{Bmatrix} \{v\} \\ \{\lambda\} \end{Bmatrix} = \begin{Bmatrix} \{E\} \\ \{E\}_1 \end{Bmatrix} \tag{41}$$

The above equation has both equilibrium and equation of geometry. Writing the right hand side of the Eq. (41) as $P[E]\begin{Bmatrix} v \\ \lambda \end{Bmatrix}$ and solving it as an eigenvalue problem, one will be able to get either the buckling load or natural frequency.

6. Analysis of discontinuity in load and geometry. Influence lines

Consider a non-prismatic beam idealized into 12 elements say, shown in Fig. 3, with geometric discontinuities and load discontinuities. Assume each element has n (say n=11) sampling points and hence total number of sampling points for ne=12 elements is 11×12=132. Assume for each element the flexural rigidities are given at 11 sampling points as

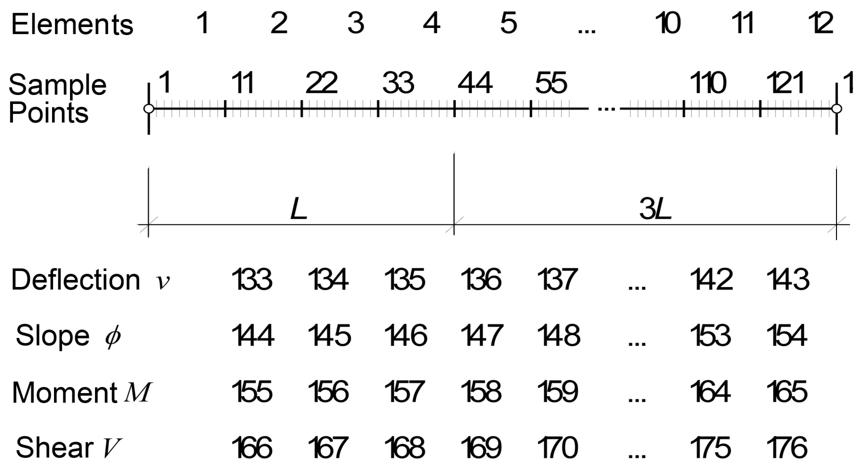

Fig. 3 Sample points numbering

$$[D] = \begin{bmatrix} D(1,1) & D(1,2) & \dots & D(1,11) \\ D(2,1) & \ddots & & \\ \vdots & & & \vdots \\ D(ne,1) & & \dots & D(ne,11) \end{bmatrix}_{12\times 11} \tag{42}$$

The beam is subjected to concentrated loads and concentrated moment and udl and also there is a discontinuity in geometry. Concentrated load and concentrated moment mean that there is a discontinuity in loading. It should be specified where the various discontinuities are considered applied. In fact, such discontinuities can be located at points on the boundary of the element. Appropriately the singularities or pathological functions are closely stated in subsequent rows at the end of section 6.2.

For example the governing equation for element 1 (applying Eq. (33)) is written as

$$[G]_1\{v\}_1 = [[k]C(:,:,4)/L_1^4 + 2[\alpha]C(:,:,3)/L_1^3 + [\beta]C(:,:,2)/L_1^2]\{v\}_1 = \{q\}_1 \tag{43}$$

Such equations for each element are written as

$$\begin{bmatrix} [G]_1 & [0] & & \dots & & [\ ] \\ [0] & [G]_2 & & & & \\ \vdots & & [G]_3 & & & \vdots \\ & & & \ddots & & \\ & & & & [G]_{11} & [0] \\ [0] & & & \dots & [0] & [G]_{12} \end{bmatrix} \begin{Bmatrix} \{v\}_1 \\ \{v\}_2 \\ \{v\}_3 \\ \vdots \\ \{v\}_{12} \end{Bmatrix} = \begin{Bmatrix} \{q\}_1 \\ \{q\}_2 \\ \{q\}_3 \\ \vdots \\ \{q\}_{12} \end{Bmatrix} \tag{44}$$

where $\{v\}_i$ denotes the displacements at the sampling points of element *i*.

6.1 Compatibility at internal node 2

1) The displacement of the 11th sampling point of the element 1 is equal to the displacement of the first sampling point of the element 2 and this can be written as

$$v_{11} - v_{12} = \Lambda_Q = 0 \quad \text{or} \quad v_{11} = v_{12} \tag{45}$$

which can be written in matrix form as

$G(133, 11) = 1, \qquad G(133, 12) = -1$

By applying Wilson's method

$G(11, 133) = 1, \qquad G(12, 133) = -1$

2) The rotation to the left of the 2nd node is equal to the rotation to the right of the node 2 and this can be written as

$$(\theta_2)_L - (\theta_2)_R = \Theta_Q = 0 \quad \text{or} \quad (\theta_2)_L = (\theta_2)_R \tag{46}$$

where

$(\theta_2)_L = G(144, 1{:}11)\{v\}_1 = [C(n, 1{:}n, 1)/L_1]\{v\}_1$ or $G(144, 1{:}11) = C(n, 1{:}n, 1)/L_1$

$(\theta_2)_R = G(144, 12{:}22)\{v\}_2 = -[C(1, 1{:}n, 1)/L_2]\{v\}_2$

or $G(144, 12{:}22) = -C(1, 1{:}n, 1)/L_2$

By applying Wilson's method

$G(1{:}11, 144) = G(144, 1{:}11)^T$ and $G(12{:}22, 144) = G(144, 12{:}22)^T$

6.2 Equilibrium at internal node 2

1) Sum of the moments at the internal node 2 must be equal to zero i.e., moment to the left of the node must be equal to the moment at the right of node 2 and this is written as

$$(M_2)_L - (M_2)_R = M_Q = 0 \quad \text{or} \quad (M_2)_L = (M_2)_R \tag{47}$$

$(M_2)_L = G(155, 1{:}11)\{v\}_1 = (k_{11})^{(1)}[C(n, 1{:}n, 2)/L_1^2]\{v\}_1$

or $G(155, 1{:}11) = (k_{11})^{(1)}C(n, 1{:}n, 2)/L_1^2$

$(M_2)_R = G(155, 12{:}22)\{v\}_2 = -(k_1)^{(2)}[C(1, 1{:}n, 2)/L_2^2]\{v\}_2$

or $G(155, 12{:}22) = -(k_1)^{(2)}C(1, 1{:}n, 2)/L_2^2$

By applying Wilson's method

$G(1{:}11, 155) = G(155, 1{:}11)^T$

and $G(12{:}22, 155) = G(155, 12{:}22)^T$

2) Sum of the shears at the internal node 2 must be equal to zero i.e., shear to the left of the node must be equal to the shear at the right of node 2 and this is written as

$$(V_2)_L-(V_2)_R = V_Q = 0 \quad \text{or} \quad (V_2)_L=(V_2)_R \tag{48}$$

$$(V_2)_L = G(166,1{:}11)\{v\}_1 = [(k_{11})^{(1)}C(n,1{:}n,3)/L_1^3+(\alpha_{11})^{(1)}C(n,1{:}n,2)/L_1^2]\{v\}_1$$

or $G(166,1{:}11) = [(k_{11})^{(1)}C(n,1{:}n,3)/L_1^3+(\alpha_{11})^{(1)}C(n,1{:}n,2)/L_1^2]$

$$(V_2)_R = G(166,12;22)\{v\}_2 = -[(k_1)^2C(1,1{:}n,3)/L_2^3+(\alpha_1)^{(2)}C(n,1{:}n,2)/L_2^2]\{v\}_2$$

or $G(166,12;22) = -[(k_1)^2C(1,1{:}n,3)/L_2^3+(\alpha_1)^{(2)}C(n,1{:}n,2)/L_2^2]$

By applying Wilson's method

$G(1{:}11,166) = G(166,1{:}11)^T$

and $G(12{:}22,166) = G(166,12{:}22)^T$

Similarly compatibility equations must be established at all the interior joints. In addition one has to give support boundary conditions as discussed in method presented in section 5.3.

If $V_Q \neq 0$ at any interior joint as concentrated load acting at the joint.

If $M_Q \neq 0$ at any interior joint as concentrated moment acting at the joint.

If $\Theta_Q \neq 0$ at any interior joint one can get the influence line diagram for moment at that joint.

If $\Lambda_Q \neq 0$ at any interior joint, then one can get the influence line diagram for shear at that point.

6.3 Example 2. Continuous beam. Influence lines

Consider a two span continuous beam of each span of 12 m and uniform section I-shaped. Fig. 4 and 5 show the influence lines for shear and bending moment at midpoint of first span. Figs. 6 and 7 show bending moment diagram and deformed shape due to concentrated moment acting at mid point of first span. Figs. 8 and 9 show the bending moment diagram and deformed shape for concentrated load acting at mid point of first span. Figs. 10 and 11 show the bending moment and deflected shape when the beam is subjected to uniformly distributed load.

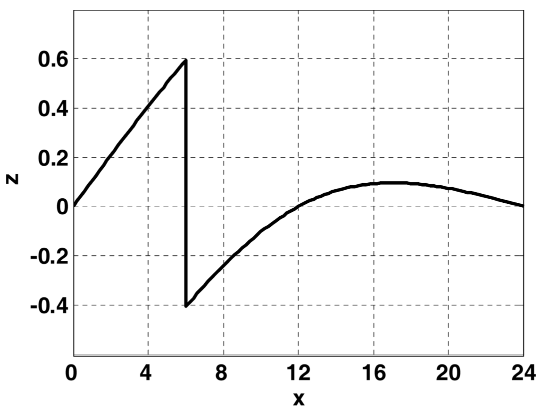

Fig. 4 Influence line for shear at midpoint of the first span

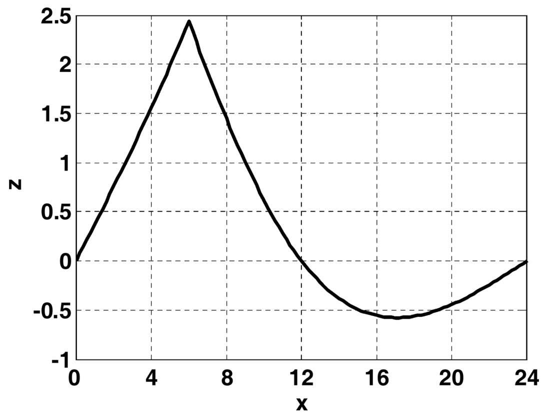

Fig. 5 Influence line for moment at midpoint of the first span

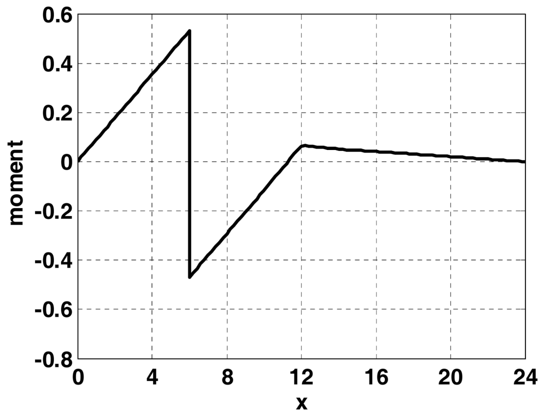

Fig. 6 Moment diagram (concentrated moment at midpoint of the first span)

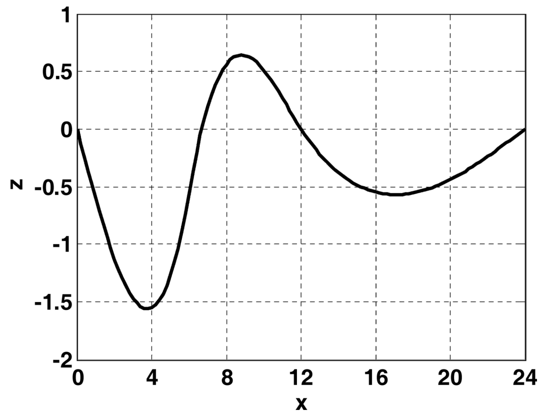

Fig. 7 Deflected shape due to concentrated moment at midpoint of the first span

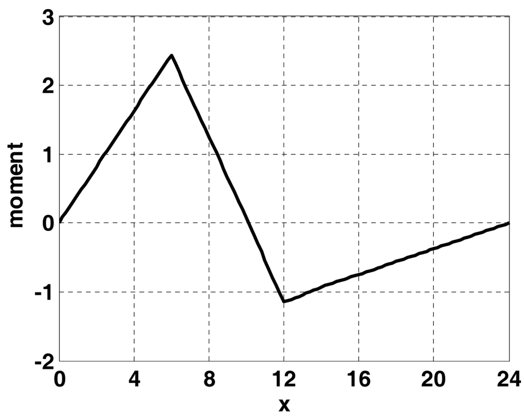

Fig. 8 Moment diagram due to concentrated load at midpoint of the first span

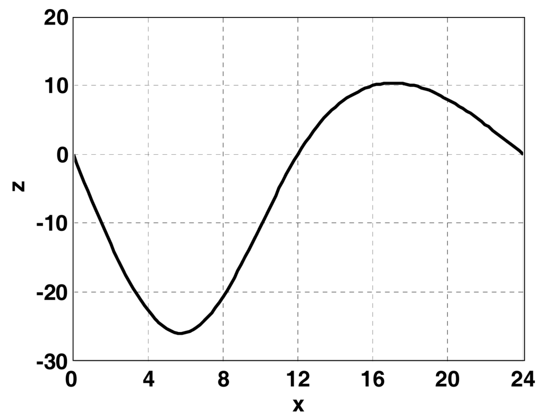

Fig. 9 Deflected shape due to concentrated load at midpoint of the first span

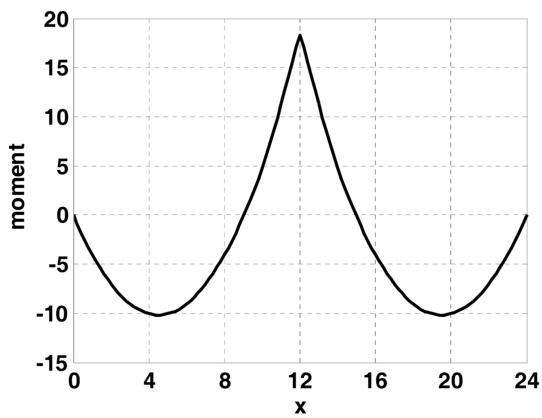

Fig. 10 Bending moment diagram for uniform distributed load in both spans

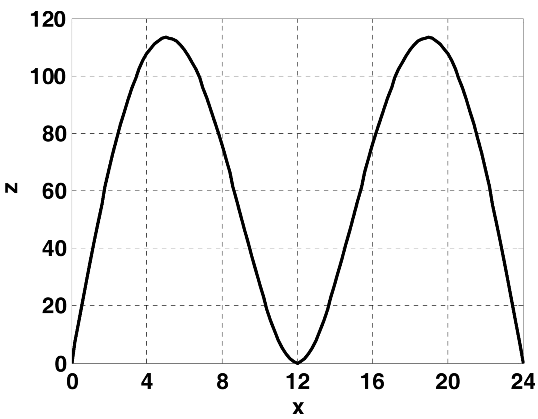

Fig. 11 Deflected shape due to udl in both spans

7. Buckling analysis

The governing equation for buckling of a beam is given by

$$D\frac{d^4v}{dx^4}+2\frac{dD}{dx}\frac{d^3v}{dx^3}+\frac{d^2D}{dx^2}\frac{d^2v}{dx^2} = -P\frac{d^2v}{dx^2} \tag{49}$$

For buckling problem Eq. (49) is modified as

$$\begin{bmatrix} [G]_1 & [0] & \cdots & [0] \\ [0] & [G]_2 & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \cdots & [0] & [G]_n \end{bmatrix} \begin{Bmatrix} \{v_1\} \\ \{v_2\} \\ \vdots \\ \{v_n\} \end{Bmatrix} = -P \begin{bmatrix} [E]_1 & [0] & \cdots & [0] \\ [0] & [E]_2 & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \cdots & [0] & [E]_n \end{bmatrix} \begin{Bmatrix} \{v_1\} \\ \{v_2\} \\ \vdots \\ \{v_n\} \end{Bmatrix} \tag{50}$$

where $[G]_1 = [k]C(:,:,4)/L_1^4+2[\alpha]C(:,:,3)/L_1^3+[\beta]C(:,:,2)/L_1^2$ and $[E]_1 = -C(:,:,2)/L_1^2$ and the boundary conditions are applied as before.

7.1 Example 3. Continuous beam. Buckling load

The same continuous beam (See example 2) is subjected to axial load. The first span has moment of inertia of 2 and second span has moment of inertia of 1. The buckling load by differential quadrature is obtained as 0.09363*EI* whereas the value obtained by Finite Element method is 0.093150*EI*. The buckled shape is also shown in Fig. 12. The beam is idealized into two elements (two spans). It is seen from Fig. 13 that (10 divisions) 11 sampling points are enough to achieve the desired accuracy.

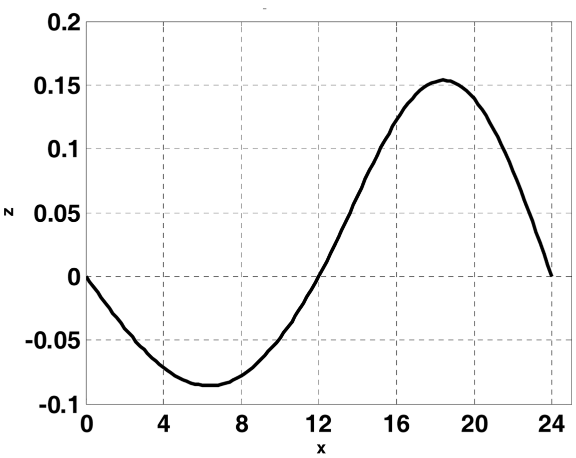

Fig. 12 Buckled shape of a continuous beam

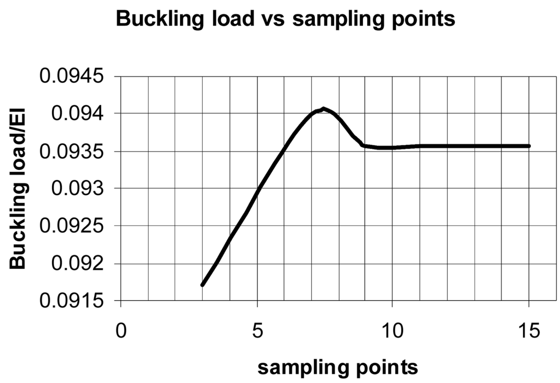

Fig. 13 Variation of the buckling load with respect to sampling points

8. Free vibrations of beams

The governing equation for free vibration of a beam is given by

$$\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 v}{\partial x^2}\right) + P\left(\frac{\partial^2 v}{\partial x^2}\right) = \rho A \frac{\partial^2 v}{\partial t^3} \tag{51}$$

or

$$\begin{bmatrix} [G]_1 & [0] & \dots & [0] \\ [0] & [G]_2 & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & [G]_n \end{bmatrix} \begin{Bmatrix} \{v_1\} \\ \{v_2\} \\ \vdots \\ \{v_n\} \end{Bmatrix} = -P \begin{bmatrix} [E]_1 & [0] & \dots & [0] \\ [0] & [E]_2 & & \vdots \\ \vdots & & \ddots & [0] \\ [0] & \dots & [0] & [E]_n \end{bmatrix} \begin{Bmatrix} \{v_1\} \\ \{v_2\} \\ \vdots \\ \{v_n\} \end{Bmatrix} \tag{52}$$

where $[G]_1 = [k]C(:,:,4)/L_1^4 + 2[\alpha]C(:,:,3)/L_1^3 + ([\beta] + P[I])C(:,:,2)/L_1^2$
being $[E]_1 = diag[EA]$

The boundary conditions are applied as explained above.

8.1 Example 4. Non-prismatic cantilever. Free vibration

The Non-prismatic thin-walled cantilever beam shown in Fig. 14(a) is analysed for free vibration and the first five fundamental frequencies are obtained. The results are in good agreement with Wekezer (Wekezer 1987, 1989) and are tabulated in Table 1. The cross sectional properties are given in Fig. 14(b). The first mode shape is shown in Fig. 15.

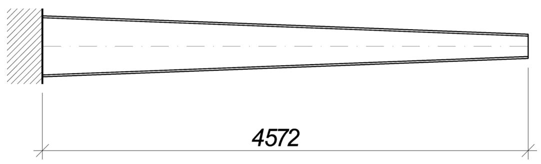

Fig. 14(a) Free vibration of a non-prismatic thin walled beam

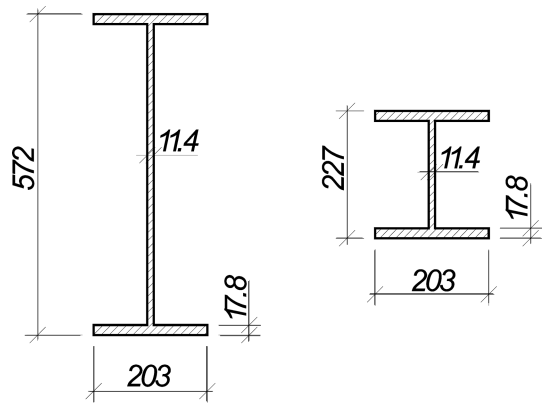

Fig. 14(b) Cross section geometry at initial fixed and final free end

Table 1 Fundamental frequencies of non-prismatic beam

ω rad/sec	DQM	Wekezer
ω_1	42	42
ω_2	117	125
ω_3	197	191
ω_4	250	255
ω_5	400	398

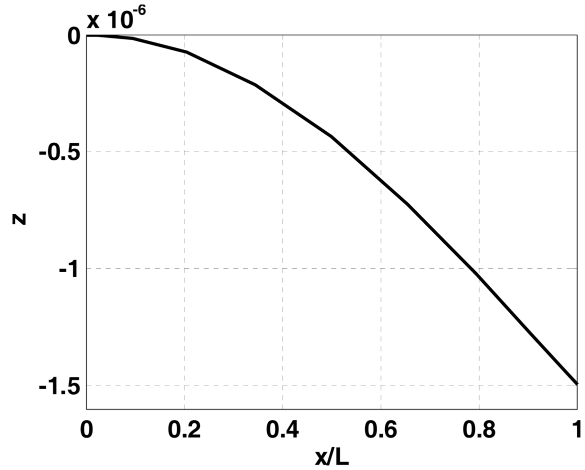

Fig. 15 Fundamental mode shape

The material properties were assumed as, Young's modulus $E = 206.8$ GPa, mass density $\rho = 1250$ kg/m^3 and Poisson's ratio $\nu = 0.375$.

8.2 Example 5. Pinned-pinned column. Frequencies and buckling load

To find the buckling load of a pinned-pinned column given $E = 200$ GPa, I=0.000038 m^4 mass density $\rho = 7800$ kg/m^3, $A = 1/7800$ m^2 and span = 12 m. By trial and error by giving various axial loads natural frequencies are calculated. The load at which the frequency becomes imaginary will give the buckling load. For the column, the buckling load is obtained as 520895 N which agrees with Euler critical load. When the axial load is zero the natural frequency $\omega_n = 188.9$ rad/sec thus obtained agrees with Euler beam with simple support conditions. When the axial load is tensile say $P = -300000$ N, the natural frequency is $\omega_n = 237.19$ rad/sec which agrees with the value

$$\left[\frac{\omega}{\omega_n(P=0)}\right]^2 = 1 - \frac{P}{P_{cr}} = 1.5759 \tag{53}$$

9. Conclusions

A unified formulation expressed in a unique differential system equation is presented to simulate the structural behavior of the classical beam theory. The DQ and HDQ methods are applied to solve equilibrium analysis of beam problems, stability problems and free vibration problems. Unlike Rayleigh-Ritz methods, DQ and HDQ methods do not need the construction of an admissible function that satisfies boundary conditions a priori. Accurate results are obtained for the problem even when a smaller number of discrete points are used to discretize the domain. The application is convenient for solving problem governed by the higher order differential equations and matrix operations could be performed using symbolic mathematic software as MATLAB with ease. It is also easy to write algebraic equations in the place of differential equations and application of boundary conditions is also an easy task. It is also explained how Lagrangian multiplier method is used to convert rectangular matrix to square matrix by incorporating boundary conditions using Wilson's procedure. Results with high accuracy are obtained in all study cases. DQM and HDQ are computationally efficient and DQM is straight forward that the general procedure can be easily employed for handling problems with other boundary conditions. It is also shown in the paper how discontinuity on the boundary of the elements can be handled to obtain shear force diagram and bending moment diagram due to concentrated loads and moments and influence lines for moment and shear due to concentrated rotations and deflections. It is expected that DQM will find a wide range of applications in structural engineering (Rajasekaran 2007b).

Acknowledgements

The authors thank the management, Dr R Rudramoorthy, Principal, PSG College of Technology for providing necessary facilities to authors to carry out the research work reported in this paper.

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