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Technical Note

Dynamic eigenvalue analysis of structures with interval parameters based on affine arithmetic

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1. Introduction

In the analysis and design of structures, some unavoidable uncertainties should be reasonably taken into account. Probability theory, fuzzy mathematics and interval analysis are the three main tools to handle these uncertainties (Elishakoff 1998, Rao and Berke 1997, Theodorou *et al.* 2007, Galal *et al.* 2008). When data are insufficient to validate the correctness of probability density function and membership function of uncertainties or its functions, which are necessary for probability and fuzzy method respectively, and yet the interval ranges of uncertainties can be given exactly, interval analysis is a natural method to deal with this case.

In recent years, researches on structural analysis using interval method focus mainly on uncertain structural static analysis, static inversion and reliability and so on, also dynamic eigenvalue is referred to Deif (1986) gave the standard eigenvalue solution theorem of symmetrical interval matrix, which was then generalized to the standard eigenvalue of interval matrix by Chen and Qiu and the matrix perturbation method was given in (Chen and Qiu 1994, Qiu and Wang 2005). A direct optimization method was presented for solving eigenvalue problems of interval matrices in Chen and Chen (2000). The available methods mostly deal with the formed interval matrix, not considering the correlations between matrix elements, which may enlarge the range of eigenvalue. Recently, Wang and Li (2004) put forward a global optimization method, moreover, a real-code genetic algorithm is used.

To minish the expansion of computed result by using interval arithmetic directly, an affine arithmetic is given here to study dynamic eigenvalue of structure with uncertain parameters based on interval division, in which the affine forms of interval functions are adopted to describe uncertainties of problems. Through mathematical example and an engineering application, feasibility and validity of this method are illustrated and some important conclusions are obtained.

2. Description of problems

For determinate structures with n-degree-of-freedom, its generalized eigenvalue problem can be

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expressed as follows

$$\boldsymbol{KX} = \lambda \boldsymbol{MX} \tag{1}$$

where $\boldsymbol{K} = [k_{ij}]$ $(i,j = 1,2,\ldots,n)$ is stiffness matrix of n steps; $\boldsymbol{M} = [m_{ij}]$ is mass matrix of n steps; $\boldsymbol{X}$ is eigenvector and λ is eigenvalue.

For uncertain problems, when the elements in stiffness matrix and mass matrix are interval variables, $\boldsymbol{K}$ and $\boldsymbol{M}$ are interval matrixes with $k_{ij} = [\underline{k}_{ij}, \overline{k}_{ij}]$ and $m_{ij} = [\underline{m}_{ij}, \overline{m}_{ij}]$.

Here Eq. (1) becomes a generalized interval eigenvalue question, and the eigenvalue and corresponding eigenvector are turned into interval numbers and vectors respectively.

For mechanical engineering, $\boldsymbol{K}$ and $\boldsymbol{M}$ are functions of uncertain parameter a, such as elastic modulus, mass density of material and structural geometric dimensions. The relativity of some elements in $\boldsymbol{K}$ and $\boldsymbol{M}$ should be taken into account adequately in dynamic analysis.

Suppose that $\boldsymbol{a} = (a_i)_m = (a_1, a_2, \ldots, a_m)^T$ is interval vector composed of structural uncertain parameters, its bounds are expressed as $\boldsymbol{a} \in \boldsymbol{a}^I = [\underline{\boldsymbol{a}}, \overline{\boldsymbol{a}}] = ([\underline{a}_1, \overline{a}_1], [\underline{a}_2, \overline{a}_2], \ldots, [\underline{a}_m, \overline{a}_m])^T$, where $\underline{\boldsymbol{a}}$ and $\overline{\boldsymbol{a}}$ are lower and upper bounds respectively, m is the dimension. And $\boldsymbol{K}$ and $\boldsymbol{M}$ are

$$\boldsymbol{K}(\boldsymbol{a}) = \{\boldsymbol{K}:\boldsymbol{K} = \boldsymbol{K}(\boldsymbol{a}), \underline{\boldsymbol{a}} \le \boldsymbol{a} \le \overline{\boldsymbol{a}}\}, \quad \boldsymbol{M}(\boldsymbol{a}) = \{\boldsymbol{M}:\boldsymbol{M} = \boldsymbol{M}(\boldsymbol{a}), \underline{\boldsymbol{a}} \le \boldsymbol{a} \le \overline{\boldsymbol{a}}\}$$

Then the set of structural generalized eigenvalue is

$$\boldsymbol{\Gamma} = \{\lambda:\boldsymbol{K}(\boldsymbol{a})\boldsymbol{X} = \lambda \boldsymbol{MX}, \underline{\boldsymbol{a}} \le \boldsymbol{a} \le \overline{\boldsymbol{a}}\} \tag{2}$$

Seen form Eq. (2), structural eigenvalue is also interval variables denoted as $\lambda_j = \lambda_j^I = [\underline{\lambda}_j, \overline{\lambda}_j]$ with $\underline{\lambda}_j = \min\limits_{\underline{\boldsymbol{a}} \le \boldsymbol{a} \le \overline{\boldsymbol{a}}} \{\boldsymbol{K}(\boldsymbol{a})\boldsymbol{X} = \lambda \boldsymbol{M}(\boldsymbol{a})\boldsymbol{X}\}$ and $\overline{\lambda}_j = \max\limits_{\underline{\boldsymbol{a}} \le \boldsymbol{a} \le \overline{\boldsymbol{a}}} \{\boldsymbol{K}(\boldsymbol{a})\boldsymbol{X} = \lambda \boldsymbol{M}(\boldsymbol{a})\boldsymbol{X}\}$ $(j = 1,2,\ldots,n)$.

3. Affine arithmetic

3.1 Affine form and its interval transform

Specifically, in affine arithmetic (Stolfi and de Figueiredo 2003) a partially unknown quantity x is represented by an affine form $\hat{x}$, which is a first-degree polynomial

$$\hat{x} = x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 + \ldots + x_t\varepsilon_t \tag{3}$$

Here we call x_0 the central value of the affine form $\hat{x}$; the coefficients $x_j \in R(j = 1,2,\ldots,t)$ are the jth partial deviations of $\hat{x}$. Each $\varepsilon_j \in [-1,1](j = 1,2,\ldots,t)$ is the jth noise symbol which stands for an independent source of error or uncertainty that contributions to the total uncertainty of the quantity x. The corresponding x_j gives the magnitude of that contribution.

Conversions between affine forms and intervals are defined: given an ordinary interval $[\underline{x}, \overline{x}]$ representing a quantity x, the corresponding affine form can be written as

$$\hat{x} = x_0 + x_1\varepsilon_x, \quad \text{where} \quad x_0 = (\underline{x} + \overline{x})/2, \quad x_1 = (\overline{x} - \underline{x})/2$$

Conversely, given an affine form $\hat{x} = x_0 + x_1\varepsilon_1 + x_2\varepsilon_2 + \ldots + x_t\varepsilon_t$, the corresponding interval is

$$[\underline{x}, \overline{x}] = [x_0 - \xi, x_0 + \xi], \text{ where } \quad \xi = \sum_{j=1}^{t} |x_j|.$$

3.2 Modified Affine arithmetic

The regular affine arithmetic is improved in this paper, whose main idea is to divide the independent variables $[\underline{a}_i, \overline{a}_i](i = 1, \ldots, m)$ step by step, then to do affine operation in every sub-interval and solve corresponding structural generalized eigenvalue.

A more formal description of the procedure of this method goes like this:

(1) Confirm structural independent interval variables $a_i(i = 1, \ldots, m)$. Given $k = 0$, the partition number of interval $n_k = 1$, the allowable error $\delta > 0$.

(2) Divide the intervals $[\underline{a}_i, \overline{a}_i]$ into n_k parts according to desired accuracy.

$$[\underline{a}_i, \overline{a}_i] \Rightarrow [\underline{a}_i, \underline{a}_i + 1/n_k*(\overline{a}_i - \underline{a}_i)], [\underline{a}_i + 1/n_k*(\overline{a}_i - \underline{a}_i), \underline{a}_i + 2/n_k*(\overline{a}_i - \underline{a}_i)], \ldots, [\overline{a}_i - 1/n_k*(\overline{a}_i - \underline{a}_i), \overline{a}_i]$$

Then we transform the split intervals into several affine representations where the coefficients of uncertainty variables are $1/N$ of the original coefficients.

(3) No longer introduce any new uncertainty symbol as a result of each basic non-affine arithmetic operation, keep the affine form in $\boldsymbol{K}$ and $\boldsymbol{M}$ as certain representations. Then solve the structural generalized eigenvalue in Eq. (5).

(4) Calculate the expressions of $\lambda_j(j = 1, \ldots, n)$ by affine arithmetic.

(5) Search the $\lambda_{j\max}^{(k)}$ and $\lambda_{j\min}^{(k)}$ $(j = 1, \ldots, n)$ from the jth generalized eigenvalue λ_j.

(6) Let iteration number be $k = k + 1$. When $k = 1$, $n_k = 2$; when $k = 2$, $n_k = 4$; when $k \geq 3$, then $n_k = 2^{k-1} \times 5$.

(7) Repeat the step (2) to step (5), then we obtain a new round of $\lambda_{j\max}^{(k-1)}$ and $\lambda_{j\min}^{(k-1)}$.

(8) Calculate $\left|\lambda_{j\max}^{(k+1)} - \lambda_{j\max}^{(k)}\right|$ and $\left|\lambda_{j\min}^{(k+1)} - \lambda_{j\min}^{(k)}\right|$. If $\max\left\{\dfrac{\left|\lambda_{j\max}^{(k+1)} - \lambda_{j\max}^{(k)}\right|}{\lambda_{j\max}^{(k)}}, \dfrac{\left|\lambda_{j\min}^{(k+1)} - \lambda_{j\min}^{(k)}\right|}{\lambda_{j\min}^{(k)}}\right\} \leq \delta$ $(j = 1, 2, \cdots, n)$,

return to step (9), otherwise to step (6).

(9) Output generalized eigenvalue $[\underline{\lambda}_j, \overline{\lambda}_j]$, where $\overline{\lambda}_j = \lambda_{j\max}^{(k+1)}$ and $\underline{\lambda}_j = \lambda_{j\min}^{(k+1)}$.

4. Examples

Example 1: Solve the range of $f = x_1 \dfrac{x_3 + x_2}{x_3 - x_2} + (x_4 - 3)^2$, where $x_1 \in [2, 3], x_2 \in [6, 18], x_3 \in [1, 3]$, $x_4 \in [-3, 7]$.

By using interval arithmetic, the result of original function is [–45, 5.1765], whereas the result of changed form $f = x_1\left(1 - \dfrac{2}{1 - x_3/x_2}\right) + (x_4 - 3)^2$ is [–33, 33.7647]. If the affine arithmetic based on interval division is adopted, let x_1, x_2, x_3, x_4 be independent variable, allowable error be $\delta = 0.001$, when the partition number is $n_k = 5$ we obtain the result [–9, 33.7647] which is consistent with the factual solution domain.

Example 2: Fig. 1 shows a multi-story frame structure. Structural stiffness parameters are $[k_1] = [2000, 2020]$ N/m, $[k_2] = [1800, 1850]$ N/m, $[k_3] = [1600, 1630]$ N/m, $[k_4] = [1400, 1420]$ N/m and $[k_5] = [1200, 1210]$ N/m. The mass parameters are $[m_1] = [29, 31]$ kg, $[m_2] = [26, 28]$

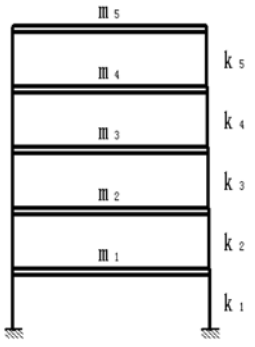

Fig. 1 A multi-story frame

Table 1 Eigenvalue of the frame structure

	Global optimization method (Wang and Li 2004)	The proposed method	Matrix perturbation method
λ_1	[5.858, 6.502]	[5.8581, 6.5020]	[4.617, 7.830]
λ_2	[42.029, 46.309]	[42.0293, 46.3088]	[40.643, 47.820]
λ_3	[98.856, 108.689]	[98.8564, 108.6890]	[99.180, 109.399]
λ_4	[158.051, 173.778]	[158.0514, 173.7776]	[157.848, 174.002]

kg, $[m_3] = [26, 28]$ kg, $[m_4] = [24, 26]$ kg and $[m_5] = [17, 19]$ kg. To solve the structural eigenvalue. By using affine arithmetic based interval division, let allowable error be $\delta = 0.001$, when $n_k = 2$, the iteration stops. Table 1 shows the eigenvalue of the frame structure by using the improved affine arithmetic, the global optimization method and the matrix perturbation method respectively.

Affine arithmetic takes structural interval parameters as independent variables and considers the correlatives between elements of mass matrix and stiffness matrix adequately. However, the matrix perturbation method takes into account no effects of parameters on matrix elements, and solves the corresponding interval generalized eigenvalue after the stiffness matrix and mass matrix formed perfectly. Table 1 shows the results solved by the proposed method in this paper are almost consistent with that solved by global optimization method, and its solution domain of eigenvalue is subset of that by using matrix perturbation method.

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