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Technical Note

Simplified P-Delta and buckling analysis for inelastic flexibility-based beam-column elements

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1. Introduction

To account for material nonlinearity in frame analysis, a flexibility-based inelastic beam-column element utilizing a lumped plasticity model can be employed. Various researchers have demonstrated the usefulness and efficiency of using the flexibility-based approach in determining the inelastic response of frame members (Spacone *et al.* 1996, Neuenhofer and Filippou 1997). To handle geometric nonlinearities, a solution method can incorporate a simplified geometric stiffness matrix along with an iterative technique to capture the *P-Delta* effect (including of $P-\Delta$ and $P-\delta$ effects). For elastic systems, the geometric stiffness matrix is included in the elastic stiffness matrix of an element in a straightforward way to form the final element stiffness matrix, which is not true for the inelastic beam-column element being considered. Therefore, an approach to account for the $P-\delta$ effect must be considered. The current paper will address this issue and a special procedure to account for the $P-\delta$ effect will be proposed.

2. Element stiffness

In the flexibility-based element formulation, the element stiffness matrix $\boldsymbol{k}$ (a 3×3 matrix) for the essential set is obtained through inversion of the element flexibility matrix $\boldsymbol{f}$, which consists of the elastic flexibility matrix $\boldsymbol{f}_{\mathrm{e}}$ and the hinge flexibility matrix $\boldsymbol{f}_{\mathrm{p}}$ shown in Eq. (1).

$$\boldsymbol{k} = \boldsymbol{f}^{-1} = [\boldsymbol{f}_{\mathrm{e}} + \boldsymbol{f}_{\mathrm{p}}]^{-1} \tag{1}$$

$$\boldsymbol{f}_{\mathrm{e}} = \begin{bmatrix} \frac{L}{EA} & 0 & 0 \\ 0 & \frac{L}{3EI} & \frac{-L}{6EI} \\ 0 & \frac{-L}{6EI} & \frac{L}{3EI} \end{bmatrix} \tag{2}$$

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For two-dimensional force space, each hinge has a 2×2 flexibility matrix, which is in terms of its axial force and bending moment. The hinge at node I affects degrees of freedom r_1 and r_2 while the hinge at node J affects degrees of freedom r_1 and r_3. Hence, the hinge flexibility matrix $\boldsymbol{f}_{\mathrm{p}}$ can be obtained by

$$\boldsymbol{f}_{\mathrm{p}} = \begin{bmatrix} f_{11}^I + f_{11}^J & f_{12}^I & f_{13}^J \\ f_{12}^I & f_{22}^I & 0 \\ f_{13}^J & 0 & f_{33}^J \end{bmatrix} \tag{3}$$

where $\boldsymbol{f}_p^I$ is the flexibility matrix of hinge $I = \begin{bmatrix} f_{11}^I & f_{12}^I \\ f_{12}^I & f_{22}^I \end{bmatrix}$

$\boldsymbol{f}_p^J$ is the flexibility matrix of hinge $J = \begin{bmatrix} f_{11}^J & f_{13}^J \\ f_{13}^J & f_{33}^J \end{bmatrix}$

The definitions and more explanations of the hinge flexibility matrix can be found in literatures (Kim and Engelhardt 2000, Taucer *et al.* 1992). After obtaining the element stiffness matrix $\boldsymbol{k}$ for the essential set, one can calculate the 6×6 element stiffness matrix $\boldsymbol{K}$ for the complete set as follows

$$\boldsymbol{K} = \boldsymbol{A}^T \times \boldsymbol{k} \times \boldsymbol{A} \tag{4}$$

3. Geometric nonlinearity

3.1 Introduction

The term "*P-Delta* effect" is commonly used to describe a type of geometric nonlinearity in which axial compressive forces acting through the displacement of one end of a member relative to the other amplify the lateral bending response of a beam-column. Thus, the *P-Delta* effect influences the transverse bending stiffness of an element. As discussed earlier, the developed solution methodology incorporates a simplified geometric stiffness matrix $\boldsymbol{K}_{\mathrm{G}}$ along with an iterative technique to model the *P-Delta* effect in an approximate manner. This technique employed to simulate the moment increase at the member ends due to the $P - \Delta$ effect is referred to as the equivalent lateral load method or the $P - \Delta$ iterative method. Details of this method can be found in the literature (Chen and Lui 1991). The derivation of $\boldsymbol{K}_{\mathrm{G}}$, which is obtained by assuming cubic shape functions of a beam element, is widely available (McGuire 2000). The geometric stiffness $\boldsymbol{K}_{\mathrm{G}}$ for the complete set in a local coordinate system can be decomposed into two matrices $\boldsymbol{K}_G'$ and $\boldsymbol{K}_G''$

$$\boldsymbol{K}_G = \boldsymbol{K}_G' + \boldsymbol{K}_G'' \tag{5}$$

where $\boldsymbol{K}_G'$ is a stiffness matrix that accounts for the chord rotation ($P - \Delta$) effect, $\boldsymbol{K}_G''$ is a stiffness matrix that accounts for the member curvature ($P - \delta$) effect.

3.2 Adding a geometric stiffness matrix

For elastic systems, the geometric stiffness matrix $\boldsymbol{K}_{\mathrm{G}}$ can be included to the elastic stiffness matrix $\boldsymbol{K}_{\mathrm{e}}$ of an element in a straightforward way to form the final element stiffness matrix $\boldsymbol{K} = \boldsymbol{K}_{\mathrm{e}} + \boldsymbol{K}_G$ However, for the inelastic beam-column element being considered, the geometric stiffness matrix $\boldsymbol{K}_G''$ accounting for the member curvature ($P - \delta$) effect cannot be added directly to the element tangent stiffness matrix $\boldsymbol{K}_{\mathrm{t}}$ because the tangent stiffness matrix $\boldsymbol{K}_{\mathrm{t}}$ is derived from the tangent element stiffness matrix $\boldsymbol{k}_{\mathrm{t}}$ for the essential set using Eq. (4) in which it is obtained through the inversion of the tangent flexibility matrix $\boldsymbol{f}_{\mathrm{t}}$ as expressed by Eq. (1). In the way it is formulated, the tangent stiffness matrix $\boldsymbol{K}_{\mathrm{t}}$ already accounts for inelastic material behavior in which axial and/or flexural yielding may already occur. Therefore, in this case, a special procedure is needed for computing the element tangent stiffness matrix.

Because the tangent flexibility matrix $\boldsymbol{f}_{\mathrm{t}}$ is determined using the essential set (three deformations without rigid body motion), $\boldsymbol{K}_G$ corresponding to the essential set will be considered. The geometric stiffness matrix ($P - \delta$ effect) $\boldsymbol{k}_{\mathrm{g}}''$ for the essential set can be written as

$$\boldsymbol{k}_{\mathrm{g}}'' = \frac{P}{L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{2L^2}{15} & \dfrac{-L^2}{30} \\ 0 & \dfrac{-L^2}{30} & \dfrac{2L^2}{15} \end{bmatrix} \tag{6}$$

where $\boldsymbol{K}_G'' = \boldsymbol{A}^T \times \boldsymbol{k}_{\mathrm{g}}'' \times \boldsymbol{A}$

3.3 A proposed procedure

To solve the problem addressed in Section 3.2, a special procedure to account for the $P - \delta$ effect is proposed. This process is simple and does not affect the main solution procedure of the nonlinear algorithm being used.

Let's consider the adding of $\boldsymbol{k}_{\mathrm{g}}''$ to $\boldsymbol{k}_{\mathrm{t}}$ which can create inconsistency in violating the yield limit. Instead of doing so, the geometric stiffness matrix $\boldsymbol{k}_{\mathrm{g}}''$ may be included in the elastic stiffness matrix $\boldsymbol{k}_{\mathrm{e}}$ to form the elastic-geometric stiffness matrix $\boldsymbol{k}_{\mathrm{eg}}$ as following

$$\boldsymbol{k}_{\mathrm{eg}} = \boldsymbol{k}_{\mathrm{g}} + \boldsymbol{k}_{\mathrm{g}}''$$

$$= \begin{bmatrix} \dfrac{AE}{L} & 0 & 0 \\ 0 & \dfrac{4EI}{L} & \dfrac{2EI}{L} \\ 0 & \dfrac{2EI}{L} & \dfrac{4EI}{L} \end{bmatrix} + \frac{P}{L}\begin{bmatrix} 1 & 0 & 0 \\ 0 & \dfrac{2L^2}{15} & \dfrac{-L^2}{30} \\ 0 & \dfrac{-L^2}{30} & \dfrac{2L^2}{15} \end{bmatrix} \tag{7}$$

Then, the elastic flexibility matrix accounting for geometric nonlinearity $\boldsymbol{f}_{\mathrm{eg}}$ can be obtained through the inversion of $\boldsymbol{k}_{\mathrm{eg}}$

$$\boldsymbol{f}_{\mathrm{eg}} = \boldsymbol{k}_{\mathrm{eg}}^{-1} \tag{8}$$

To account for geometric nonlinearity, the elastic flexibility matrix $\boldsymbol{f}_{\mathrm{eg}}$ appearing in Eq. (8) is used to replace the one in Eq. (2). With this replacement, the procedure to obtain the tangent stiffness matrix remains unchanged, and the violation of the yield limit is avoided. After the tangent stiffness matrix for the complete set $\boldsymbol{K}_{\mathrm{t}}$ is obtained, $\boldsymbol{K}_G'$ is later added to account for the $P-\Delta$ effect.

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