Structural Engineering and Mechanics, Vol. 33, No. 3 (2009) 307-323 307
DOI: http://dx.doi.org/10.12989/sem.2009.33.3.307

Structural matrices of a curved-beam element

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(Received October 16, 2007, Accepted September 2, 2009)

Abstract. This article presents the differential system that governs the mechanical behaviour of a curved-beam element, with varying cross-section area, subjected to generalized load. This system is solved by an exact procedure or by the application of a new numerical recurrence scheme relating the internal forces and displacements at the two end-points of an increase in its centroid-line. This solution has a transfer matrix structure. Both the stiffness matrix and the equivalent load vector are obtained arranging the transfer matrix. New structural matrices have been defined, which permit to determine directly the unknown values of internal forces and displacements at the two supported ends of the curved-beam element. Examples are included for verification.

Keywords: curved beam; differential system; transfer matrix; stiffness matrix; numerical method.

1. Introduction

Traditionally, the laws governing the mechanical behaviour (applying the Euler-Bernuolli and Timoshenko theories) of a naturally curved beam element are expressed in equilibrium and compatibility equations (Love 1944, Timoshenko 1957). Some authors develop these equations by means of energy theories (Moris 1968, Leontovich 1959, Kardestuncer 1974). These two interpretations of the equations have permitted to reach accurate results, either analytical or numerical, only for some types of curved elements; for instance, in the circular arch loaded on its plane (Yamada and Ezawa 1977, Just 1982, Saleeb and Chang 1987, Shi and Voyiadjis 1991, Molari and Ubertini 2006, Tufekci and Arpaci 2006), and loaded out of its plane (Lee 1969), in the parabolic element (Marquis and Wang 1989) or in the uniformly loaded helix (Scordelis 1960).

Other authors present their curved-beam studies, in terms of a set of twelve linear ordinary differential equations, dependent on the arch length (Banan *et al*. 1989, Akoz *et al*. 1991, Yu *et al*. 2002). This set of equilibrium and compatibility equations has facilitated the implementation of new numerical procedures, thus broadening the casuistry of the models and elements to be studied (Haktanir 1995). The analytical methods are limited by: the complexity of the shape of the centroid-line, the cross-section of the element, the characteristics of the material, the load system applied and the type of support fixed (Pestel and Leckie 1963).

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Being not possible to use an exact method in every structure case, approximate numerical procedures have been resorted to. The simplest way to solve the structure problem of a curved-beam element is to substitute the shape of its centroid-line by a polygonal line. In order not to commit any geometrical approximation errors and to obtain accurate results, it is necessary to employ a large number of straight elements, which configure the polygonal line, thus requiring a considerable amount of input data. The numerical procedures to solve the mechanical behaviour of a curved beam are limited to the resolution methods of linear ordinary differential systems with boundary conditions. Among these can be found the so-called shooting method (Lance 1960), finite differences method (Rahman 1991) and finite elements method (Bathe 1996), the latter being the method most used. The finite elements proposed in the literature are generally generated from polynomial or trigonometric approximations. In certain curved beams cases, these elements are not as approximate as is considered to be necessary (Litewka and Rakowski 1996).

This article presents the differential system governing the structural behaviour of a curved beam (Gimena *et al.* 2008) and gives the steps to be followed for using either an exact solution or a numerical approximation. This numerical procedure does not increase the number of unknown quantities and allows the finding of an accurate solution under the Euler-Bernuolli and Timoshenko theories.

The solution obtained by these resolution procedures has a transfer matrix structure. With this linear solution as a starting point, it is possible, by a rearrangement of terms using simple algebraic operations (Gimena *et al.* 2003), to obtain both the stiffness matrix and the equivalent load vector of a generic curved-beam element. By entering the values contributed by each support in the transfer matrix expression, new matrices (boundary matrices) can be defined, which give the unknown stress and deformation values in the two end points.

The stiffness and boundary matrices of a bar subjected to flexure load and the semi-elliptic arch loaded into its plane have been included for verification.

2. Basic equations

A curved-beam element has been defined as that generated by a plane section which centroid runs orthogonally through all the points of an axis line. The curve is expressed by the vector radius $\mathbf{r} = \mathbf{r}(s)$, s (arc length of the axis line) being the only independent variable of the linear structure problem.

The reference system used for annotating the known and unknown functions is the Frenet-Serret trihedron. Its unit vectors tangent $\mathbf{t}$, normal $\mathbf{n}$ and binormal $\mathbf{b}$ are

$$\mathbf{t} = D\mathbf{r};\ \mathbf{n} = \frac{D\mathbf{t}}{|D\mathbf{t}|};\ \ \mathbf{b} = \mathbf{t} \times \mathbf{n} \tag{1}$$

Where, $D = d/ds$ is the derivative respect the parameter s.

The natural equations of the centroid line are expressed by the flexion curvature $\chi = \chi(s)$ and the torsion curvature $\tau = \tau(s)$ (Sokolnikoff and Redeffer 1958).

Accepting the habitual principles and hypotheses of the strength of materials and only considering the stresses associated with the normal section to the curve (σ, τ_n, τ_b), the geometric characteristics of the section are: area $A(s)$, shearing coefficients $\alpha_n(s)$, $\alpha_{nb}(s)$, $\alpha_{bn}(s)$, $\alpha_b(s)$, and moments of inertia $I_t(s)$, $I_n(s)$, $I_b(s)$, $I_{nb}(s)$. The longitudinal $E(s)$ and transversal $G(s)$ moduli give the elasticity condition of the material.

Applying the laws of equilibrium and kinematics on a differential element of the curve, equations expressing the mechanical behaviour of a curved-beam element can be obtained (Gimena *et al.* 2008)

$$\begin{aligned} &D\mathbf{V} && +\mathbf{q} = 0 \\ &\mathbf{t}\times\mathbf{V}+D\mathbf{M} && +\mathbf{k} = 0 \\ &\quad -\mathbf{M}[\theta_{\mathbf{M}}]+D\theta && -\Theta = 0 \\ &-\mathbf{V}[\mathbf{u_V}] \quad +\mathbf{t}\times\theta+D\mathbf{u} && -\Delta = 0 \end{aligned} \tag{2}$$

The first two rows of system Eq. (2) represent the equilibrium equations.
The vectors intervening in the equilibrium are:

Internal force $\mathbf{V} = N\mathbf{t}+V_n\mathbf{n}+V_b\mathbf{b} = \int_A \sigma dA\,\mathbf{t}+\int_A \tau_n dA\,\mathbf{n}+\int_A \tau_b dA\,\mathbf{b}$

Internal moment $\mathbf{M} = T\mathbf{t}+M_n\mathbf{n}+M_b\mathbf{b} = \int_A (\tau_b n-\tau_n b)dA\,\mathbf{t}+\int_A \sigma bA\,\mathbf{n}-\int_A \sigma ndA\,\mathbf{b}$

Load force $\mathbf{q} = q_t\mathbf{t}+q_n\mathbf{n}+q_b\mathbf{b}$

Load moment $\mathbf{k} = k_t\mathbf{t}+k_n\mathbf{n}+k_b\mathbf{b}$

The last two rows of system Eq. (2) represent the kinematics equations:

Rotation $\theta = \phi\mathbf{t}+\theta_n\mathbf{n}+\theta_b\mathbf{b}$

Displacement $\mathbf{u} = u\mathbf{t}+v\mathbf{n}+w\mathbf{b}$

Load rotation $\Theta = \Theta_t\mathbf{t}+\Theta_n\mathbf{n}+\Theta_b\mathbf{b}$

Load displacement $\Delta = \Delta_t\mathbf{t}+\Delta_n\mathbf{n}+\Delta_b\mathbf{b}$

In the kinematics equations the following matrices are involved:

Rotation matrix produced by the moments $[\theta_{\mathbf{M}}] = \begin{bmatrix} \frac{1}{GI_t} & 0 & 0 \\ 0 & \frac{I_b}{E[I_nI_b-I_{nb}^2]} & \frac{I_{nb}}{E[I_nI_b-I_{nb}^2]} \\ 0 & \frac{I_{nb}}{E[I_nI_b-I_{nb}^2]} & \frac{I_n}{E[I_nI_b-I_{nb}^2]} \end{bmatrix}$

Displacement matrix produced by the forces $[\mathbf{u_V}] = \begin{bmatrix} \frac{1}{EA} & 0 & 0 \\ 0 & \frac{\alpha_n}{GA} & \frac{\alpha_{nb}}{GA} \\ 0 & \frac{\alpha_{bn}}{GA} & \frac{\alpha_b}{GA} \end{bmatrix}$

Differential system Eq. (2) can be written in vector mode

$$D\mathbf{e}(s) = [\mathbf{T_D}(s)]\mathbf{e}(s)+\mathbf{q}(s) \tag{3}$$

where:

$\mathbf{e}(s) = \{\mathbf{V}, \mathbf{M}, \theta, \mathbf{u}\}^T = \{N, V_n, V_b, T, M_n, M_b, \phi, \theta_n, \theta_b, u, v, w\}^T$ is the effect of internal forces, moments, rotations and displacements;

$[\mathbf{T_D}(s)] = \begin{bmatrix} [\mathbf{F}] & [\mathbf{0}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{D}] & [\mathbf{F}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{0}] & -[\theta_\mathbf{M}] & [\mathbf{F}] & [\mathbf{0}] \\ -[\mathbf{u_V}] & [\mathbf{0}] & [\mathbf{D}] & [\mathbf{F}] \end{bmatrix}$ with $[\mathbf{F}] = \begin{bmatrix} 0 & \chi & 0 \\ -\chi & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix}$ and $[\mathbf{D}] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$ represent the geometry of the curve (infinitesimal transfer matrix), and

$\mathbf{q}(s) = \{\mathbf{q}, \mathbf{k}, \Theta, \Delta\}^T = \{-q_t, -q_n, -q_b, -k_t, -k_n, -k_b, \Theta_t, \Theta_n, \Theta_b, \Delta_t, \Delta_n, \Delta_b\}^T$ is the generalized load applied.

This general expression of the mechanical behaviour of the beam, which expresses the relationship between the unknown effect $\mathbf{e}(s)$ (internal forces and displacements) produced by the load $\mathbf{q}(s)$ adopted, is a unique system of linear ordinary differential equations.

3. Resolution procedures

3.1 Analytical exact

The exact solution of the mechanical behaviour of the curved element only exists if Eq. (3) is solved analytically. In that case, the structure of the solution, linear and depending on twelve integration constants **c**, can be written as follows (Zheng *et al.* 2000)

$$\mathbf{e}(s) = [\mathbf{T}(s)]\mathbf{c} + \mathbf{q_T}(s) \tag{4}$$

The first term of the addend represents the solution that depicts the system as a homogeneous one. The second is a particular solution of the differential system.

The values of the integration constants can be expressed in terms of the initial end-point on the axis of the curved element **I**

$$\mathbf{c} = [\mathbf{T}(s_\mathrm{I})]^{-1}\mathbf{e}(s_\mathrm{I}) - [\mathbf{T}(s_\mathrm{I})]^{-1}\mathbf{q_T}(s_\mathrm{I}) \tag{5}$$

Substituting this value of the integration constants in Eq. (4) and particularizing at the end-point of the curved element **II**, the transfer matrix expression is obtained.

$$\mathbf{e}(s_\mathrm{II}) = [\mathbf{T}(s_\mathrm{I}, s_\mathrm{II})]\mathbf{e}(s_\mathrm{I}) + \mathbf{q_T}(s_\mathrm{I}, s_\mathrm{II}) \tag{6}$$

Where,

$\mathbf{T}(s_\mathrm{I}, s_\mathrm{II}) = [\mathbf{T}(s_\mathrm{II})][\mathbf{T}(s_\mathrm{I})]^{-1}$ is the transfer matrix, and

$\mathbf{q_T}(s_\mathrm{I}, s_\mathrm{II}) = \mathbf{q_T}(s_\mathrm{II}) - [\mathbf{T}(s_\mathrm{II})][\mathbf{T}(s_\mathrm{I})]^{-1}\mathbf{q_T}(s_\mathrm{I})$ is the load transmission vector.

The algebraic system Eq. (6) is composed of twelve equations. Each support condition supplies six null internal force and displacement values. With these values being known, the number of unknown values to be determined is reduced to twelve, coinciding with the number of equations to be solved.

Having found all the internal forces and displacements values at the ends of the curved-beam element, the following equation can be employed to obtain the solution to the problem at any point on the element.

$$\mathbf{e}(s) = [\mathbf{T}(s_\mathrm{I}, s)]\mathbf{e}(s_\mathrm{I}) + \mathbf{q}_\mathbf{T}(s_\mathrm{I}, s) \tag{7}$$

Where,

$\mathbf{T}(s_\mathrm{I}, s) = [\mathbf{T}(s)][\mathbf{T}(s_\mathrm{I})]^{-1}$ is the transfer matrix at the s point, and

$\mathbf{q}_\mathbf{T}(s_\mathrm{I}, s) = \mathbf{q}_\mathbf{T}(s) - [\mathbf{T}(s)][\mathbf{T}(s_\mathrm{I})]^{-1}\mathbf{q}_\mathbf{T}(s_\mathrm{I})$ is the load transmission vector at the s point.

In view of the possible difficulty of obtaining an analytical solution, it is usual to resort to approximate solutions, based on numerical calculus.

3.2 Numerical approximate

Starting from the differential system Eq. (3), the relationship between two points i and $i+1$ at a distance of length Δs using the fourth order Runge-Kutta approximation (Pestel and Leckie 1963) is given

$$\mathbf{e}(s_{i+1}) = \mathbf{e}(s_i) + \frac{\Delta s}{6}(\mathbf{K}_1 + 2\mathbf{K}_2 + 2\mathbf{K}_3 + \mathbf{K}_4) \tag{8}$$

Where,

$$\mathbf{K}_1 = [\mathbf{T}_\mathbf{D}(s_i)]\mathbf{e}(s_i) + \mathbf{q}(s_i)$$

$$\mathbf{K}_2 = [\mathbf{T}_\mathbf{D}(s_{i+1/2})]\left[\mathbf{e}(s_i) + \frac{\Delta s}{2}\mathbf{K}_1\right] + \mathbf{q}(s_{i+1/2})$$

$$\mathbf{K}_3 = [\mathbf{T}_\mathbf{D}(s_{i+1/2})]\left[\mathbf{e}(s_i) + \frac{\Delta s}{2}\mathbf{K}_2\right] + \mathbf{q}(s_{i+1/2})$$

$$\mathbf{K}_4 = [\mathbf{T}_\mathbf{D}(s_{i+1})][\mathbf{e}(s_i) + \Delta s\mathbf{K}_3] + \mathbf{q}(s_{i+1})$$

The resulting finite system has the following expression

$$\mathbf{e}(s_{i+1}) = [\mathbf{T}(s_i)]\mathbf{e}(s_i) + \mathbf{q}_\mathbf{T}(s_i) \tag{9}$$

Where,

$$\begin{aligned}[\mathbf{T}(s_i)] = {} & [\mathbf{I}] + \frac{\Delta s}{6}[[\mathbf{T}_\mathbf{D}(s_{i+1})] + 4[\mathbf{T}_\mathbf{D}(s_{i+1/2})] + [\mathbf{T}_\mathbf{D}(s_i)]] + \\ & + \frac{\Delta s^2}{6}[[\mathbf{T}_\mathbf{D}(s_{i+1})][\mathbf{T}_\mathbf{D}(s_{i+1/2})] + [\mathbf{T}_\mathbf{D}(s_{i+1/2})]^2 + [\mathbf{T}_\mathbf{D}(s_{i+1/2})][\mathbf{T}_\mathbf{D}(s_i)]] + \\ & + \frac{\Delta s^3}{12}[[\mathbf{T}_\mathbf{D}(s_{i+1})][\mathbf{T}_\mathbf{D}(s_{i+1/2})]^2 + [\mathbf{T}_\mathbf{D}(s_{i+1/2})]^2[\mathbf{T}_\mathbf{D}(s_i)]] + \\ & + \frac{\Delta s^4}{24}[\mathbf{T}_\mathbf{D}(s_{i+1})][\mathbf{T}_\mathbf{D}(s_{i+1/2})]^2[\mathbf{T}_\mathbf{D}(s_i)]\end{aligned}$$

[**I**] is the identity matrix of order 12.

$$\mathbf{q_T}(s_i) = \frac{\Delta s}{6}[\mathbf{q}(s_{i+1}) + 4\mathbf{q}(s_{i+1/2}) + \mathbf{q}(s_i)] +$$

$$+ \frac{\Delta s^2}{6}[[\mathbf{T_D}(s_{i+1})]\mathbf{q}(s_{i+1/2}) + [\mathbf{T_D}(s_{i+1/2})]\mathbf{q}(s_{i+1/2}) + [\mathbf{T_D}(s_{i+1/2})]\mathbf{q}(s_i)] +$$

$$+ \frac{\Delta s^3}{12}[[\mathbf{T_D}(s_{i+1})][\mathbf{T_D}(s_{i+1/2})]\mathbf{q}(s_{i+1/2}) + [\mathbf{T_D}(s_{i+1/2})]^2\mathbf{q}(s_i)] +$$

$$+ \frac{\Delta s^4}{24}[[\mathbf{T_D}(s_{i+1})][\mathbf{T_D}(s_{i+1/2})]^2\mathbf{q}(s_i)]$$

Using the previous Eq. (9), it is possible to relate the internal forces and displacements at the initial end **I** of the curve to those of the generic point $i + 1$, obtaining

$$\mathbf{e}(s_{i+1}) = \left[\prod_{j=0}^{j=i}[\mathbf{T}(s_j)]\right]\mathbf{e}(s_\mathbf{I}) + \sum_{j=0}^{j=i}\left[\prod_{k=j+1}^{k=i}[\mathbf{T}(s_k)]\right]\mathbf{q_T}(s_j) \tag{10}$$

Both extreme ends can be related in the same way. For a number n of intervals, **II** being the final-end, the Eq. (10) gives

$$\mathbf{e}(s_\mathbf{II}) = [\mathbf{T}(s_\mathbf{I}, s_\mathbf{II})]\mathbf{e}(s_\mathbf{I}) + \mathbf{q}(s_\mathbf{I}, s_\mathbf{II}) \tag{11}$$

Where,

$[\mathbf{T}(s_\mathbf{I}, s_\mathbf{II})] = \prod_{j=0}^{j=n-1}[\mathbf{T}(s_j)]$ is the transfer matrix, and

$\mathbf{q_T}(s_\mathbf{I}, s_\mathbf{II}) = \prod_{j=0}^{j=n-1}\left[\prod_{k=j+1}^{k=n-1}[\mathbf{T}(s_k)]\right]\mathbf{q_T}(s_j)$ is the load transmission vector.

The system Eq. (11) always contains twelve algebraic equations, regardless of the number of intervals adopted. The six null internal forces and displacements values provided by each support are identified with the boundary conditions. Thus, the number of unknown quantities to be determined is reduced to twelve, coinciding with the number of equations to be solved.

Having found all the stress and deformation values at the ends of the curved element, Eq. (10) can be employed to obtain the solution to the problem in any other point on the directrix of the element.

4. Structural matrices

4.1 Transfer matrix

Either from the analytical exact Eq. (6) or numerical approximate Eq. (11) system of equations, the transfer matrix can directly be determined

$$\mathbf{e}(s_{\mathrm{II}}) = \begin{bmatrix} \mathbf{V}_{\mathrm{II}} \\ \mathbf{M}_{\mathrm{II}} \\ \theta_{\mathrm{II}} \\ \mathbf{u}_{\mathrm{II}} \end{bmatrix} = [\mathbf{T}(s_{\mathrm{I}}, s_{\mathrm{II}})]\mathbf{e}(s_{\mathrm{I}}) + \mathbf{q}_{\mathrm{T}}(s_{\mathrm{I}}, s_{\mathrm{II}}) =$$

$$= \begin{bmatrix} [\mathbf{T}_{\mathrm{VV}}] & [\mathbf{T}_{\mathrm{VM}}] & [\mathbf{T}_{\mathrm{V}\theta}] & [\mathbf{T}_{\mathrm{Vu}}] \\ [\mathbf{T}_{\mathrm{MV}}] & [\mathbf{T}_{\mathrm{MM}}] & [\mathbf{T}_{\mathrm{M}\theta}] & [\mathbf{T}_{\mathrm{Mu}}] \\ [\mathbf{T}_{\theta\mathrm{V}}] & [\mathbf{T}_{\theta\mathrm{M}}] & [\mathbf{T}_{\theta\theta}] & [\mathbf{T}_{\theta\mathrm{u}}] \\ [\mathbf{T}_{\mathrm{uV}}] & [\mathbf{T}_{\mathrm{uM}}] & [\mathbf{T}_{\mathrm{u}\theta}] & [\mathbf{T}_{\mathrm{uu}}] \end{bmatrix} \begin{bmatrix} [\mathbf{V}_{\mathrm{I}}] \\ [\mathbf{M}_{\mathrm{I}}] \\ [\theta_{\mathrm{I}}] \\ [\mathbf{u}_{\mathrm{I}}] \end{bmatrix} + \begin{bmatrix} [\mathbf{q}_{\mathrm{V}}] \\ [\mathbf{q}_{\mathrm{M}}] \\ [\mathbf{q}_{\theta}] \\ [\mathbf{q}_{\mathrm{u}}] \end{bmatrix} \tag{12}$$

Expressed in a compact form it can be written as follows

$$\mathbf{e}_{\mathrm{II}} = [\mathbf{T}]\mathbf{e}_{\mathrm{I}} + \mathbf{q}_{\mathrm{T}} \tag{13}$$

The structural *transfer matrix equation* is annotated.

4.2 Stiffness matrix

With the exact Eq. (6) or approximate Eq. (11) transfer matrix formulas, as a starting point, its stiffness equivalent notation can be expressed (Gimena *et al*. 2003).

The former terms of the above Eq. (12) are subsequently arrayed, yielding

$$\begin{bmatrix} [\mathbf{T}_{\mathrm{VV}}] & [\mathbf{T}_{\mathrm{VM}}] & [\mathbf{I}] & [\mathbf{0}] \\ [\mathbf{T}_{\mathrm{MV}}] & [\mathbf{T}_{\mathrm{MM}}] & [\mathbf{0}] & [\mathbf{I}] \\ [\mathbf{T}_{\theta\mathrm{V}}] & [\mathbf{T}_{\theta\mathrm{M}}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{T}_{\mathrm{uV}}] & [\mathbf{T}_{\mathrm{uM}}] & [\mathbf{0}] & [\mathbf{0}] \end{bmatrix} \begin{bmatrix} [-\mathbf{V}_{\mathrm{I}}] \\ [-\mathbf{M}_{\mathrm{I}}] \\ [\mathbf{V}_{\mathrm{II}}] \\ [\mathbf{M}_{\mathrm{II}}] \end{bmatrix} = \begin{bmatrix} [\mathbf{T}_{\mathrm{Vu}}] & [\mathbf{T}_{\mathrm{V}\theta}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{T}_{\mathrm{Mu}}] & [\mathbf{T}_{\mathrm{M}\theta}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{T}_{\theta\mathrm{u}}] & [\mathbf{T}_{\theta\theta}] & [\mathbf{0}] & -[\mathbf{I}] \\ [\mathbf{T}_{\mathrm{uu}}] & [\mathbf{T}_{\mathrm{u}\theta}] & -[\mathbf{I}] & [\mathbf{0}] \end{bmatrix} \begin{bmatrix} [\mathbf{q}_{\mathrm{V}}] \\ [\mathbf{q}_{\mathrm{M}}] \\ [\mathbf{q}_{\theta}] \\ [\mathbf{q}_{\mathrm{u}}] \end{bmatrix} \tag{14}$$

Where [**I**] is the identity matrix of order 3.

Finally, the stiffness matrix expression is determined, extracting the vector of the reactions from Eq. (14)

$$\begin{bmatrix} [-\mathbf{V}_{\mathrm{I}}] \\ [-\mathbf{M}_{\mathrm{I}}] \\ [\mathbf{V}_{\mathrm{II}}] \\ [\mathbf{M}_{\mathrm{II}}] \end{bmatrix} = \begin{bmatrix} [\mathbf{T}_{\mathrm{VV}}] & [\mathbf{T}_{\mathrm{VM}}] & [\mathbf{I}] & [\mathbf{0}] \\ [\mathbf{T}_{\mathrm{MV}}] & [\mathbf{T}_{\mathrm{MM}}] & [\mathbf{0}] & [\mathbf{I}] \\ [\mathbf{T}_{\theta\mathrm{V}}] & [\mathbf{T}_{\theta\mathrm{M}}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{T}_{\mathrm{uV}}] & [\mathbf{T}_{\mathrm{uM}}] & [\mathbf{0}] & [\mathbf{0}] \end{bmatrix}^{-1} \left[\begin{bmatrix} [\mathbf{T}_{\mathrm{Vu}}] & [\mathbf{T}_{\mathrm{V}\theta}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{T}_{\mathrm{Mu}}] & [\mathbf{T}_{\mathrm{M}\theta}] & [\mathbf{0}] & [\mathbf{0}] \\ [\mathbf{T}_{\theta\mathrm{u}}] & [\mathbf{T}_{\theta\theta}] & [\mathbf{0}] & -[\mathbf{I}] \\ [\mathbf{T}_{\mathrm{uu}}] & [\mathbf{T}_{\mathrm{u}\theta}] & -[\mathbf{I}] & [\mathbf{0}] \end{bmatrix} \begin{bmatrix} [\mathbf{u}_{\mathrm{I}}] \\ [\theta_{\mathrm{I}}] \\ [\mathbf{u}_{\mathrm{II}}] \\ [\theta_{\mathrm{II}}] \end{bmatrix} + \begin{bmatrix} [\mathbf{q}_{\mathrm{V}}] \\ [\mathbf{q}_{\mathrm{M}}] \\ [\mathbf{q}_{\theta}] \\ [\mathbf{q}_{\mathrm{u}}] \end{bmatrix} \right] \tag{15}$$

Expressed in a compact form it can be written as follows

$$\mathbf{f} = [\mathbf{K}]\mathbf{u} + \mathbf{q}_{\mathrm{K}} \tag{16}$$

Being **f** and **u** the vectors of reactions and of displacements at both ends, [**K**] the stiffness matrix, and $\mathbf{q}_{\mathrm{K}}$ the equivalent load vector, in the *stiffness matrix equation.*

4.3 Boundary matrices

This article denotes as boundary matrices those, which multiplied by the load transmission vector, give the unknown values of the effect vectors associated with the supports. The exact Eq. (6) or approximate Eq. (11) transfer annotation is rearranged in such a way that the effect vectors of the curved element end-points are in the first term of equation

$$[\mathbf{T}(s_{\mathrm{I}}, s_{\mathrm{II}})]\mathbf{e}(s_{\mathrm{I}}) - \mathbf{e}(s_{\mathrm{II}}) = -\mathbf{q}_{\mathrm{T}}(s_{\mathrm{I}}, s_{\mathrm{II}}) \tag{17}$$

The support structure conditions are applied on the previous Eq. (17), i.e. the null values contributed by these conditions are entered and the unknown effect vector values $\mathbf{e}(s_{\mathrm{I}}, s_{\mathrm{II}})$ of the end-points of the curved element are found:

$$\mathbf{e}(s_{\mathrm{I}}, s_{\mathrm{II}}) = [\mathbf{B}]\mathbf{q}_{\mathrm{T}}(s_{\mathrm{I}}, s_{\mathrm{II}}) \tag{18}$$

where [**B**] is the boundary matrix annotated in the *boundary matrix equation.*

5. Examples

5.1 Structural matrices of a bar under flexure

The differential system that governs the mechanical behaviour of a bar under flexion, with isotropic homogeneous material, constant section, principal axes of inertia coinciding with those of the section, and neglecting the shearing deformation, is given by Eq. (19)

$$\begin{aligned}
&\frac{dV_z}{dx} && + q_z = 0 \\
-V_z + &\frac{dM_y}{dx} && + k_y = 0 \\
-\frac{M_y}{EI_y} + &\frac{d\theta_y}{dx} && - \Theta_y = 0 \\
+\theta_y + &\frac{dw}{dx} && - \Delta_z = 0
\end{aligned} \tag{19}$$

A bar under a generalized flexure load is selected Fig. 1 and the sequence of operations used to obtain, transfer, stiffness and boundary matrices is presented:

The analytical solution of differential system Eq. (17) is

$$\begin{aligned}
V_z(x) &= C_1 && + p_1(x) \\
M_y(x) &= C_1 x + C_2 && + p_2(x) \\
\theta_y(x) &= C_1 \frac{x^2}{2EI_y} + C_2 \frac{x}{EI_y} + C_3 && + p_3(x) \\
w(x) &= -C_1 \frac{x^3}{6EI_y} - C_2 \frac{x^2}{2EI_y} - C_3 x + C_4 && + p_4(x)
\end{aligned} \tag{20}$$

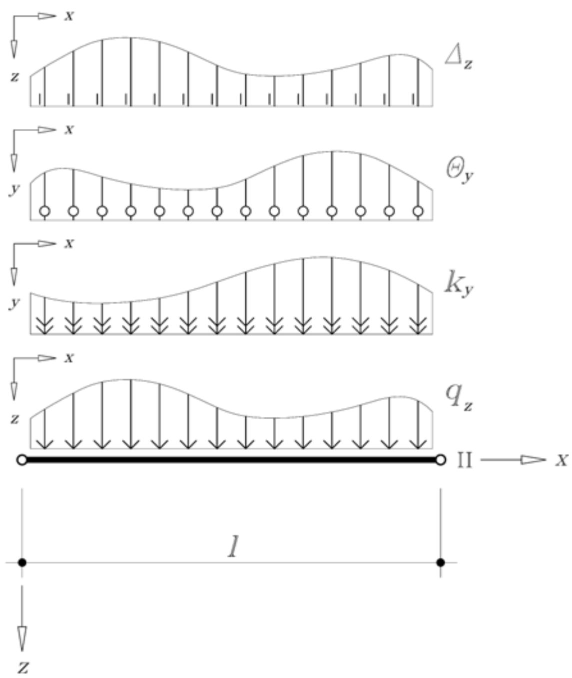

Fig. 1 A bar subjected to a flexion load

Where,

$$p_1(x) = -\int_0^x q_z(x)dx$$

$$p_2(x) = \int_0^x [p_1(x) - k_y(x)]dx$$

$$p_3(x) = \int_0^x \left[\Theta_y(x) + \frac{p_2(x)}{EI_y} \right] dx$$

$$p_4(x) = \int_0^x [\Delta_z(x) - p_3(x)]dx$$

Transfer matrix

Particularizing the general solution Eq. (20) for the initial end-point **I** on the bar in $x = 0$ and for the final end-point **II** in $x = l$ the transfer annotation is obtained

$$\begin{bmatrix} V_z(l) \\ M_y(l) \\ \theta_y(l) \\ w(l) \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ l & 1 & 0 & 0 \\ \frac{l^2}{2EI_y} & \frac{l}{EI_y} & 1 & 0 \\ -\frac{l^3}{6EI_y} & -\frac{l^3}{2EI_y} & -l & 1 \end{bmatrix} \begin{bmatrix} V_z(0) \\ M_y(0) \\ \theta_y(0) \\ w(0) \end{bmatrix} + \begin{bmatrix} p_1(l) \\ p_2(l) \\ p_3(l) \\ p_4(l) \end{bmatrix} \tag{21}$$

Stiffness matrix

The terms of Eq. (21) are rearranged in accordance with the criteria indicated in section 4.2.

$$\begin{bmatrix} 1 & 0 & 1 & 0 \\ l & 1 & 0 & 1 \\ \frac{l^2}{2EI_y} & \frac{l}{EI_y} & 1 & 0 \\ -\frac{l^3}{6EI_y} & -\frac{l^2}{2EI_y} & 0 & 0 \end{bmatrix} \begin{bmatrix} -V_z(0) \\ -M_y(0) \\ V_z(l) \\ M_y(l) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & -l & -1 & 0 \end{bmatrix} \begin{bmatrix} w(0) \\ \theta_y(0) \\ w(l) \\ \theta_y(l) \end{bmatrix} + \begin{bmatrix} p_1(l) \\ p_2(l) \\ p_3(l) \\ p_4(l) \end{bmatrix} \tag{22}$$

The stiffness matrix is determined by finding the reaction vector

$$\begin{bmatrix} -V_z(0) \\ -M_y(0) \\ V_z(l) \\ M_y(l) \end{bmatrix} = \frac{EI_y}{l^3} \begin{bmatrix} 12 & -6l & -12 & -6l \\ -6l & 4l^2 & 6l & 2l^2 \\ -12 & 6l & 12 & 6l \\ -6l & 2l^2 & 6l & 4l^2 \end{bmatrix} \begin{bmatrix} w(0) \\ \theta_y(0) \\ w(l) \\ \theta_y(l) \end{bmatrix} + \begin{bmatrix} 0 & 0 & \frac{6EI_y}{l^2} & \frac{12EI_y}{l^3} \\ 0 & 0 & -\frac{2EI_y}{l} & -\frac{6EI_y}{l^2} \\ 1 & 0 & -\frac{6EI_y}{l^2} & -\frac{12EI_y}{l^3} \\ 0 & 1 & -\frac{4EI_y}{l} & -\frac{6EI_y}{l^2} \end{bmatrix} \begin{bmatrix} p_1(l) \\ p_2(l) \\ p_3(l) \\ p_4(l) \end{bmatrix} \tag{23}$$

It should be noted that to determine the equivalent load vector the multiplication operation pointed out in the previous Eq. (23) has to be done.

Boundary matrices

The terms of Eq. (21) are rearranged in agreement with the criteria described in section 4.3.

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ L & 1 & 0 & 0 \\ \frac{l^2}{2EI_y} & \frac{l}{EI_y} & 1 & 0 \\ -\frac{l^3}{6EI_y} & -\frac{l^2}{2EI_y} & -l & 1 \end{bmatrix} \begin{bmatrix} V_z(0) \\ M_y(0) \\ \theta_y(l) \\ w(l) \end{bmatrix} - \begin{bmatrix} V_z(l) \\ M_y(l) \\ \theta_y(l) \\ w(l) \end{bmatrix} = -\begin{bmatrix} p_1(l) \\ p_2(l) \\ p_3(l) \\ p_4(l) \end{bmatrix} \tag{24}$$

The support structure conditions are applied on this Eq. (24). Two cases of support are considered in this example: first, a bi-fixed bar under a flexion load and then a bi-articulated one.

The known end values supplied by the bi-fixed support are

$$\theta_y(0) = 0; \; w(0) = 0; \; \theta_y(l) = 0; \; w(l) = 0 \tag{25}$$

The null values are deleted from Eq. (24) thus yielding

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ l & 1 & 0 & -1 \\ \dfrac{l^2}{2EI_y} & \dfrac{l}{EI_y} & 0 & 0 \\ -\dfrac{L^3}{6EI_y} & -\dfrac{l^2}{2EI_y} & 0 & 0 \end{bmatrix} \begin{bmatrix} V_z(0) \\ M_y(0) \\ V_z(l) \\ M_y(l) \end{bmatrix} = -\begin{bmatrix} p_1(l) \\ p_2(l) \\ p_3(l) \\ p_4(l) \end{bmatrix} \tag{26}$$

Therefore, the unknown values of the effect are

$$\begin{bmatrix} V_z(0) \\ M_y(0) \\ V_z(l) \\ M_y(l) \end{bmatrix} = \begin{bmatrix} 0 & 0 & -\dfrac{6EI_y}{l^2} & -\dfrac{12EI_y}{l^3} \\ 0 & 0 & \dfrac{2EI_y}{l} & \dfrac{6EI_y}{l^2} \\ 1 & 0 & -\dfrac{6EI_y}{l^2} & -\dfrac{12EI_y}{l^3} \\ 0 & 1 & -\dfrac{4EI_y}{l} & -\dfrac{6EI_y}{l^2} \end{bmatrix} \begin{bmatrix} p_1(l) \\ p_2(l) \\ p_3(l) \\ p_4(l) \end{bmatrix} = \frac{EI_y}{l^3} \begin{bmatrix} -[lp_3(l) + 2p_4(l)] \\ -2l[lp_3(l) + 3p_4(l)] \\ 6\left[\dfrac{l^3 p_1(l)}{6EI_y} - lp_3(l) - 2p_4(l)\right] \\ 2l\left[\dfrac{l^2 p_2(l)}{2EI_y} - 2lp_3(l) - 3p_4(l)\right] \end{bmatrix} \tag{27}$$

where the boundary matrix of a bar under a flexion load with a bi-fixed is annotated.

The procedure developed in 4.3 is applied analogously with other support conditions. For instance, the known values at the endpoints given by the bi-articulated support are

$$M_y(0) = 0;\ w(0) = 0;\ M_y(l) = 0;\ w(l) = 0 \tag{28}$$

The null values are eliminated from Eq. (24) giving

$$\begin{bmatrix} 1 & 0 & -1 & 0 \\ l & 0 & 0 & 0 \\ \dfrac{l^2}{2EI_y} & 1 & 0 & -1 \\ -\dfrac{L^3}{6EI_y} & -l & 0 & 0 \end{bmatrix} \begin{bmatrix} V_z(0) \\ \theta_y(0) \\ V_z(l) \\ \theta_y(l) \end{bmatrix} = -\begin{bmatrix} p_1(l) \\ p_2(l) \\ p_3(l) \\ p_4(l) \end{bmatrix} \tag{29}$$

Therefore, the unknown values of the effect at the end-points for this support are

$$\begin{bmatrix} V_z(0) \\ \theta_y(0) \\ V_z(l) \\ \theta_y(l) \end{bmatrix} = \begin{bmatrix} 0 & -\dfrac{1}{l} & 0 & 0 \\ 0 & \dfrac{l}{6EI_y} & 0 & \dfrac{1}{l} \\ 1 & -\dfrac{1}{l} & 0 & 0 \\ 0 & -\dfrac{l}{3EI_y} & 1 & \dfrac{1}{l} \end{bmatrix} \begin{bmatrix} p_1(l) \\ p_2(l) \\ p_3(l) \\ p_4(l) \end{bmatrix} = \begin{bmatrix} -\dfrac{p_2(l)}{l} \\ \dfrac{l}{6EI_y}\left[p_2(l) + \dfrac{6EI_y}{l^2} p_4(l)\right] \\ p_1(l) - \dfrac{p_2(l)}{l} \\ \dfrac{l}{3EI_y}\left[-p_2(l) + \dfrac{3EI_y}{l^2} p_4(l)\right] \end{bmatrix} \tag{30}$$

where the boundary matrix of a bar under a flexion load with a bi-articulated support is annotated.

For each support structure there is a boundary matrix, which, multiplied by the load transmission vector offers the unknown values of the effect in the end-points of the curved element. This is a useful procedure for the determination of the statically indeterminate unknown values.

5.2 Structural matrices of the semi-elliptic arch

A semi-elliptic arch has been selected to obtain the structural matrices. It has an axis z perpendicular to its plane, its centre is the origin of the coordinates, minor axis a and major axis b Fig. 2.

The parametric equations of the ellipse directrix are

$$x = a\cos\lambda; \quad y = b\sin\lambda; \quad z = 0$$

The derivative of the arc length s with respect to parameter λ is

$$\frac{ds}{d\lambda} = \sqrt{a^2\sin^2\lambda + b^2\cos^2\lambda}$$

The Frenet-Serret reference system vectors are

$$\mathbf{t} = \frac{(-a\sin\lambda, b\cos\lambda, 0)}{\sqrt{a^2\sin^2\lambda + b^2\cos^2\lambda}}; \quad \mathbf{n} = \frac{(-a\cos\lambda, b\sin\lambda, 0)}{\sqrt{a^2\sin^2\lambda + b^2\cos^2\lambda}}; \quad \mathbf{b} = (0, 0, 1)$$

The flexion curvature of the elliptic arch directrix is

$$\chi(\lambda) = \frac{ab}{(a^2\sin^2\lambda + b^2\cos^2\lambda)^{3/2}}$$

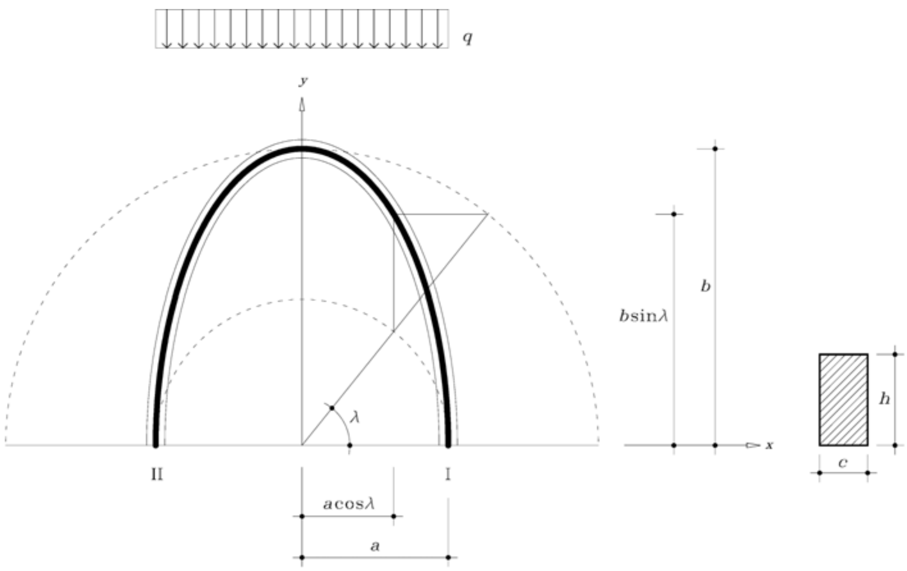

Fig. 2 Diagram of a semi-elliptic arch

The torsion curvature is null $\tau(\lambda) = 0$.

Assuming the product of inertia and the shearing deformations to be null, the system expressed by the mechanical behaviour of the elliptic arch is

$$\begin{aligned}
&\frac{dN}{d\lambda} - \alpha_1 V_n + \alpha_2 q_t = 0\\
&\alpha_1 N + \frac{dV_n}{d\lambda} + \alpha_2 q_n = 0\\
&\alpha_2 V_n + \frac{dM_b}{d\lambda} + \alpha_2 k_b = 0\\
&-\alpha_2 \frac{M_b}{EI_b} + \frac{d\theta_b}{d\lambda} - \alpha_2 \Theta_b = 0\\
&-\alpha_2 \frac{N}{EA} + \frac{du}{d\lambda} - \alpha_1 v - \alpha_2 \Delta_t = 0\\
&-\alpha_2 \theta_b + \alpha_1 u + \frac{dv}{d\lambda} - \alpha_2 \Delta_n = 0
\end{aligned} \tag{31}$$

Where, $\alpha_1 = \dfrac{ab}{a^2\sin^2\lambda + b^2\cos^2\lambda}$ and $\alpha_2 = \sqrt{a^2\sin^2\lambda + b^2\cos^2\lambda}$

Fig. 2 presents the calculus of a semi-elliptic arch with a constant section and material, with a uniformly distributed load force in projection $q = -1$ kip/ft. The starting data are as follows:

Axes of the ellipse directrix $a = 17.9$ ft, $b = 26$ ft.

Rectangular section with width $c = 6.6$ ft and thickness $h = 2$ ft.

Modulus of longitudinal elasticity of the material $E = 10^5$ kip/sq ft.

Transfer matrix

Following the approximate procedure described in section 3.2, the transfer-matrix expression is obtained

$$\begin{bmatrix} N_{\mathrm{II}} \\ V_{n\mathrm{II}} \\ M_{b\mathrm{II}} \\ \theta_{b\mathrm{II}} \\ u_{\mathrm{II}} \\ v_{\mathrm{II}} \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 35.8 & 0 & 1 & 0 & 0 & 0 \\ 0.00283 & -0.00246 & 0.00016 & 1 & 0 & 0 \\ 0.02298 & -0.04401 & 0.00283 & 35.8 & -1 & 0 \\ 0.04401 & -0.04858 & 0.00246 & 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} u_{\mathrm{I}} \\ v_{\mathrm{I}} \\ \theta_{b\mathrm{I}} \\ u_{\mathrm{II}} \\ v_{\mathrm{II}} \\ \theta_{b\mathrm{II}} \end{bmatrix} + \begin{bmatrix} 17.90 \\ 6.73 \\ 31.84 \\ -17.90 \\ 6.73 \\ -31.84 \end{bmatrix} \tag{32}$$

Stiffness matrix

Using the procedure given in section 4.2 on Eq. (32), the stiffness matrix is obtained

$$\begin{bmatrix} -N_{\mathrm{I}} \\ -V_{n\mathrm{I}} \\ -M_{b\mathrm{I}} \\ N_{\mathrm{II}} \\ V_{n\mathrm{II}} \\ M_{b\mathrm{II}} \end{bmatrix} = \begin{bmatrix} 36.15 & 0 & -647.10 & 36.15 & 0 & -647.10 \\ 0 & 96.82 & 1506.06 & 0 & 96.82 & -1506.06 \\ -647.10 & 1506.06 & 41337.94 & -647.10 & 1506.06 & -18171.74 \\ 36.15 & 0 & -647.10 & 36.15 & 0 & -647.10 \\ 0 & 96.82 & 1506.06 & 0 & 96.82 & -1506.06 \\ -647.10 & -1506.06 & -18171.74 & -647.10 & -1506.06 & 41337.94 \end{bmatrix} \begin{bmatrix} u_{\mathrm{I}} \\ v_{\mathrm{I}} \\ \theta_{b\mathrm{I}} \\ u_{\mathrm{II}} \\ v_{\mathrm{II}} \\ \theta_{b\mathrm{II}} \end{bmatrix} + \begin{bmatrix} 17.90 \\ 6.73 \\ 31.84 \\ -17.90 \\ 6.73 \\ -31.84 \end{bmatrix} \tag{33}$$

Boundary matrices

With the transfer-matrix expression Eq. (32) as a starting point, considering the *bi-fixed* support structure and using the procedure in 4.3, the annotation of the boundary matrix is

$$\begin{bmatrix} N_{\mathrm{I}} \\ V_{n\mathrm{I}} \\ M_{b\mathrm{I}} \\ N_{\mathrm{II}} \\ V_{n\mathrm{II}} \\ M_{b\mathrm{II}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & -647.10 & 36.151 & 0 \\ 0 & 0 & 0 & -1506.06 & 0 & 96.82 \\ 0 & 0 & 0 & -18171.74 & -647.10 & 1506.06 \\ 1 & 0 & 0 & 647.10 & -36.15 & 0 \\ 0 & 1 & 0 & 1506.06 & 0 & -96.82 \\ 0 & 0 & 1 & -41337.94 & 647.10 & 1506.06 \end{bmatrix} \begin{bmatrix} -35.8 \\ 0 \\ 640.82 \\ 0.039 \\ 0.205 \\ 0.539 \end{bmatrix} = \begin{bmatrix} 17.900 \\ 6.727 \\ 31.840 \\ 17.900 \\ -6.727 \\ 31.840 \end{bmatrix} \tag{34}$$

In this case of *bi-fixed* support structure, the unknown values of the effect in the endpoints of the curved element coincide, in an absolute value, with the reaction values obtained by means of the stiffness matrix, which are

$$\mathbf{f} = \{17.900, 6.727, 31.840, -17.900, 6.727, -31.840\}^T \tag{35}$$

Having found all the values of the effect (stress and deformation) at the ends of the bi-fixed semi-elliptic arch, Eq. (9) can be employed to obtain the solution to the problem at any other point on the directrix of the curved element.

The graphs of the effect are plotted in the following Fig. 3:

Following the same process in 4.3 for a *bi-articulated* support, the expression of the boundary matrix is obtained

$$\begin{bmatrix} N_{\mathrm{I}} \\ V_{n\mathrm{I}} \\ \theta_{b\mathrm{I}} \\ N_{\mathrm{II}} \\ V_{n\mathrm{II}} \\ \theta_{b\mathrm{II}} \end{bmatrix} = \begin{bmatrix} 0 & 0 & -0.02793 & 0 & 0 & 0 \\ 0 & 0 & -0.02531 & 0 & 0 & 20.58582 \\ 0 & 0 & -0.00001 & 0 & -0.02793 & 0.02531 \\ 1 & 0 & 0.02793 & 0 & 0 & 0 \\ 0 & 1 & 0.02531 & 0 & 0 & -20.58582 \\ 0 & 0 & -0.00003 & 1 & -0.02793 & -0.02531 \end{bmatrix} \begin{bmatrix} -35.8 \\ 0 \\ 640.82 \\ 0.039 \\ 0.205 \\ 0.539 \end{bmatrix} = \begin{bmatrix} 17.900 \\ 5.116 \\ 0.001 \\ 17.900 \\ -5.116 \\ -0.001 \end{bmatrix} \tag{36}$$

The graphs of the effect are shown in Fig. 4, as follows:

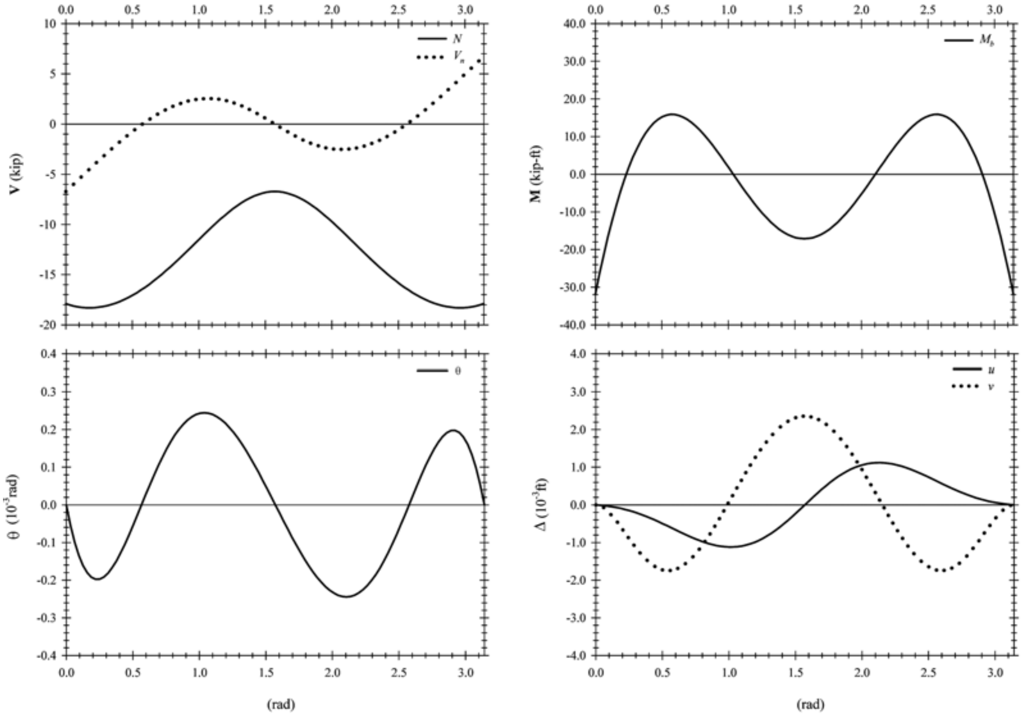

Fig. 3 Graphs of the effect components of a bifixed semi-elliptic arch

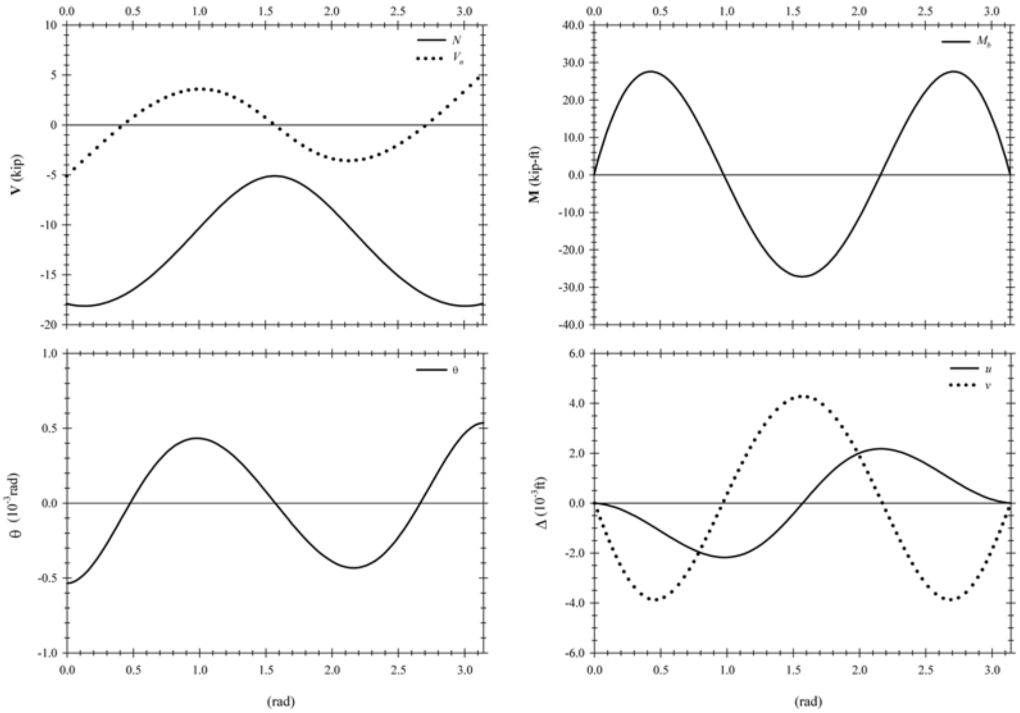

Fig. 4 Graphs of the effect components of a bi-articulated semi-elliptic arch

6. Conclusions

The structure behaviour of a curved element has been expressed in a system of twelve linear ordinary differential equations. This is the most efficient expression for applying the resolution procedures described in this article. The law of recurrence used in the numerical procedure relates the two end-points of an increase in the curve by means of the fourth order Runge-Kutta approximation. By applying this law of recurrence, the internal forces and displacements values of the end-points of the curved element can be related and the transfer matrix obtained directly, regardless of the number of increases used in the calculus. The stiffness matrix and equivalent load vector are obtained directly by some algebraic operations starting from the transfer matrix, this being an advantage over other methods. Similarly, using the transfer matrix expression and applying the support structure conditions, facilitates the determination of boundary matrices. These have the characteristic of being able to supply the unknown values of internal forces and displacements at the two supported endpoints. The method shown does not distinguish between determinate or indeterminate structures. The procedure evolved is considered to be suitable for matrices determination of a curved-beam element, as well as illustrative for educational purposes.

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