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Free vibration of circular and annular membranes with varying density by the method of discrete singular convolution

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Abstract. A numerical method is developed to investigate the effects of some geometric parameters and density variation on frequency characteristics of the circular and annular membranes with varying density. The discrete singular convolution method based on regularized Shannon's delta kernel is applied to obtain the frequency parameter. The obtained results have been compared with the analytical and numerical results of other researchers, which showed well agreement.

Keywords: discrete singular convolution; circular membrane; annular membrane; non-homogeneous density.

1. Introduction

Membrane structures are frequently encountered in most practical acoustical and technological applications. Analytical and numerical studies of the free vibration of circular and annular membranes have also received a good deal of attention. Hence, many researchers in this area have been carried out. Free vibration analysis of annular and circular membrane has been solved by several authors (Laura *et al*. 1997, Jabareen and Eisenberger 2001, Buchanan and Peddieson 1999, 2005, Buchanan 2005, Casperson and Nicolet 1968, Willatzen 2002). An analysis of the free vibration of circular and annular membranes has been presented by Laura *et al.* (1997). Jabareen and Eisenberger (2001) proposed an exact method for free vibration analysis of non-homogeneous circular and annular membranes. Buchanan and Peddieson (1999, 2005) and Buchanan (2005) used Ritz and finite element method respectively, for vibration analysis of circular and elliptic membranes with variable density. An experimental study has been made for vibrations of circular

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membrane by Casperson and Nicolet (1968). Exact power series solutions for axisymmetric vibrations of circular and annular membranes with continuously varying density were presented by Willatzen (2002). Mei (1969) presented a finite element solution of free vibration problem of circular membranes under arbitrary tension. Oden and Sato (1697) applied the finite element method for static analysis of elastic membranes. Analytical solutions of the free vibration problems of arbitrarily shaped membranes have been investigated by Kang *et al.* (1999) and Kang and Lee (2000) using non-dimensional dynamic influence function. Radial basis function-based differential quadrature method was used for free vibration analysis of arbitrary shaped membrane by Wu *et al.* (2007). Some important studies concerning analysis of membranes have been carried out, namely by Leung *et al.* (2003), Houmat (2001, 2006), Masad (1996), Laura *et al.* (1997), Ho and Chen (2000), Pronsato *et al.* (1999), and Gutierrez *et al.* (1998).

In the past ten years, the method of discrete singular convolution (DSC) and differential quadrature (DQ) methods have become increasingly popular in the numerical solution of initial and boundary value problems (Wei 1999, 2001, 2002, Zhao *et al.* 2002, Lim *et al.* 2005, Civalek 2006, 2007, Wei *et al.* 2002, Xiang *et al.* 2002, Wang and Wang 2004, Shu *et al.* 2000, Shu and Richards 1992, Shu and Xue 1997, 1998, Shu 1996, 1999, Shu and Du 1997). The method of DQ and DSC can yield accurate solutions with relatively much fewer grid points. It has been also successfully employed for different plate problems (Wang and Lee 1996, Wang *et al.* 2004, Hang *et al.* 2005, Xiang *et al.* 1993, Xiang and Zhang 2005, Xiang 2003, Liew and Liu 1999, 2000, Han and Liew 1997, Liew *et al.* 1997, Liew and Yang 2000, Xiang 2003).

In this paper, we examine the discrete singular convolution method for free vibration problem of circular and annular membranes with varying density. The performance of the method is tested for free vibration analysis of membranes considering a number of problems. The results are compared, wherever possible, with the available analytical and numerical solutions. This is the first instance in which the DSC method has been adopted for free vibration analysis of circular membranes.

2. Discrete singular convolution

The discrete singular convolution (DSC) method is an efficient and useful approach for the numerical solutions of differential equations. This method introduced by Wei in 1999. In the present paper, details of the DSC method are not given; interested readers may refer to the works of (Wei 2001, Wei *et al.* 2002, Zhao *et al.* 2002, Lim *et al.* 2005, Civalek 2007). Since it was first introduced by Wei (2001), the discrete singular convolution method has been applied solutions of many problems (Civalek 2006, 2007, Wei *et al.* 2002, Xiang *et al.* 2002). Consider a distribution, T and $\eta(t)$ as an element of the space of the test function. A singular convolution can be defined by Wei (2001)

$$F(t) = (T*\eta)(t) = \int_{-\infty}^{\infty} T(t-x)\,\eta(x)dx \tag{1}$$

where $T(t-x)$ is a singular kernel. For example, singular kernels of delta type (Wei 2001)

$$T(x) = \delta^{(n)}(x); \quad (n = 0, 1, 2, \ldots) \tag{2}$$

Kernel $T(x) = \delta(x)$ is important for interpolation of surfaces and curves, and $T(x) = \delta^{(n)}(x)$ for

$n > 1$ are essential for numerically solving differential equations. With a sufficiently smooth approximation, it is more effective to consider a discrete singular convolution (Wei 2001)

$$F_{\alpha}(t) = \sum_{k} T_{\alpha}(t - x_k) f(x_k) \tag{3}$$

where $F_{\alpha}(t)$ is an approximation to $F(t)$ and $\{x_k\}$ is an appropriate set of discrete points on which the DSC (32) is well defined. Note that, the original test function $\eta(x)$ has been replaced by $f(x)$. Recently, the use of some new kernels and regularizer such as delta regularizer (Wei *et al.* 2002, Zhao *et al.* 2002, Lim *et al.* 2005, Civalek 2007) was proposed to solve applied mechanics problem. The Shannon's kernel is regularized as Zhao *et al.* (2002)

$$\delta_{\Delta,\sigma}(x - x_k) = \frac{\sin[(\pi/\Delta)(x - x_k)]}{(\pi/\Delta)(x - x_k)} \exp\left[-\frac{(x - x_k)^2}{2\sigma^2}\right]; \quad \sigma > 0 \tag{4}$$

where Δ is the grid spacing, σ is a regularization parameter. It is also known that the truncation error is very small due to the use of the Gaussian regularizer, the above formulation given by Eq. (4) is practically and has an essentially compact support for numerical interpolation. In the DSC method, the function $f(x)$ and its derivatives with respect to the x coordinate at a grid point x_i are approximated by a linear sum of discrete values $f(x_k)$ given by Wei (2001)

$$\left.\frac{d^n f(x)}{dx^n}\right|_{x = x_i} = f^{(n)}(x) \approx \sum_{k=-M}^{M} \delta^{(n)}_{\Delta,\sigma}(x_i - x_k) f(x_k); \quad (n = 0, 1, 2, \ldots) \tag{5}$$

where $\delta_{\Delta}(x_i - x_k) = \Delta\delta_{\alpha}(x_i - x_k)$ and superscript (n) denotes the n*th*-order derivative, and $2M + 1$ is the computational bandwidth which is centered around x and is usually smaller than the whole computational domain. For example the second order derivative at $x = x_i$ of the DSC kernels for directly given (Wei *et al.* 2002)

$$f^{(2)}(x) = \left.\frac{d^2 f}{dx^2}\right|_{x = x_i} \approx \sum_{k=-M}^{M} \delta^{(2)}_{\Delta,\sigma}(k\Delta x_N) f_{i+k,j} \tag{6}$$

Second-order derivative in Eq. (6) is given as Wei (2001)

$$\begin{aligned}
\delta^{(2)}_{\pi/\Delta,\sigma}(x_m - x_k) = {} & -\frac{(\pi/\Delta)\sin(\pi/\Delta)(x - x_k)}{(x - x_k)} \exp[-(x - x_k)^2/2\sigma^2] \\
& -2\frac{\cos(\pi/\Delta)(x - x_k)}{(x - x_k)^2} \exp[-(x - x_k)^2/2\sigma^2] \\
& -2\frac{\cos(\pi/\Delta)(x - x_k)}{\sigma^2} \exp[-(x - x_k)^2/2\sigma^2] + 2\frac{\sin(\pi/\Delta)(x - x_k)}{\pi(x - x_k)^3/\Delta} \exp[-(x - x_k)^2/2\sigma^2] \\
& + \frac{\sin(\pi/\Delta)(x - x_k)}{\pi(x - x_k)\sigma^2/\Delta} \exp[-(x - x_k)^2/2\sigma^2] + \frac{\sin(\pi/\Delta)(x - x_k)}{\pi\sigma^4/\Delta}(x - x_k)\exp[-(x - x_k)^2/2\sigma^2]
\end{aligned} \tag{7}$$

3. Governing equations

Following the same notation given by Laura *et al*. (1997), consider a circular, annular membrane of outer radius b, inner radius a and the radial coordinate r as shown in Fig. 1. The non-dimensional governing differential equation for free vibration can be given as Laura *et al*. (1997)

$$r\frac{\partial^2 W}{\partial r^2}+\frac{\partial W}{\partial r}+\Omega^2 f(r)rW = 0 \tag{8}$$

Where W is the transverse deflection, ρ is the mass per unit area, ω is the circular frequency, and T is the tension per unit length. The density of the membrane is the linear function of the x and given in non-dimensional form written as follows

$$\rho(r) = \rho_0[1+\alpha(r^n)] \tag{9}$$

The related dimensionless quantities

$$r_0 = a/b, \quad \Omega^2 = \omega^2 b^2 \rho_0/T \tag{10}$$

Applying the discrete singular convolution to the governing equation yields

$$r\sum_{j=-M}^{M}\delta_{\Delta,\sigma}^{(2)}(k\Delta r)W_{i,j}+\sum_{k=-M}^{M}\delta_{\Delta,\sigma}^{(1)}(k\Delta r)W_{i,j}+\Omega^2 f(r)r_i W_i = 0 \tag{11}$$

The boundary conditions are as follows

$$W = 0 \text{ at edges} \tag{12}$$

In the present study, we can't obtain reliable results for standard grid distributions. In this study we use the below formula for grid points in radial directions as proposed Wang and Wang (2004)

$$r_i = b+\frac{a-b}{2}\left[\frac{i-1}{N-1}\right] \tag{13}$$

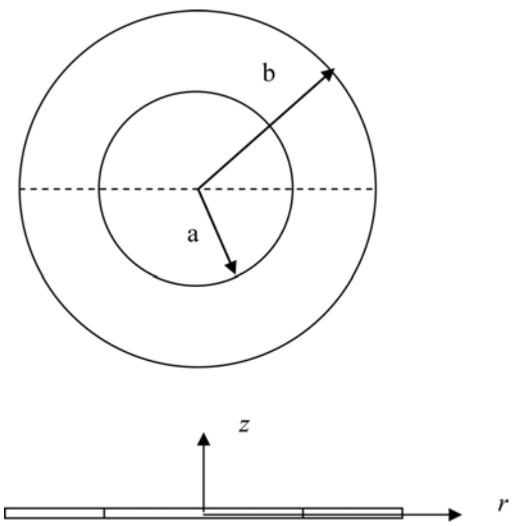

Fig. 1 Geometry of annular membrane

4. Numerical results

To validate the accuracy and applicability of the present formulation, the numerical results for circular and annular membrane are compared to the results of Gutierrez (1998) by the Ritz, differential quadrature and the exact solution (Jabareen and Eisenberger 2001). The obtained frequency values are given in Table 1 for different radius ratio of the circular (a/b = 0) and annular membranes. It is very clear from Table 1 that the rate of convergence is very good for both annular and circular membranes with the increase in the grid numbers.

Table 2 summarizes numerical results of fundamental frequency of circular (a/b = 0) membranes by DSC with different ratio of density. Frequency values obtained by DSC method are presented in Table 2 together with the finite element solutions (Wei 1999), differential quadrature (Wei 1999) and exact solution (Jabareen and Eisenberger 2001). The DSC results are generally in agreement with the results produced from the analytical Jabareen and Eisenberger (2001) and the DQ results (Wei 1999). It is seen in these two tables that the present method yields accurate results.

Figs. 2-5 show the variation of fundamental frequency versus a/b for different density. In these figures different values of n are taken into consideration. Namely, linear, parabolic, cubic and

Table 1 Convergence of fundamental frequency of circular and annular homogeneous membranes

Methods	a/b = 0	a/b = 0.4	a/b = 0.6	a/b = 0.8
Gutierrez *et al.* (1998) (Exact)	2.4048	5.1831	7.8284	15.6981
Gutierrez *et al.* (1998) (DQ)	2.4048	5.1830	7.8284	15.6981
Gutierrez *et al.* (1998) (FEM)	2.4049	5.1867	7.8337	15.7085
Jabareen and Eisenberger (Jabareen and Eisenberger (2001)	2.4048	-	-	-
Present DSC Results $N = M = 11$	2.4058	5.1902	7.8305	15.7003
Present DSC Results $N = M = 13$	2.4049	5.1856	7.8288	15.6986
Present DSC Results $N = M = 15$	2.4048	5.1833	7.8285	15.6981
Present DSC Results $N = M = 17$	2.4048	5.1830	7.8285	15.6981
Present DSC Results $N = M = 19$	2.4048	5.1830	7.8285	15.6981

Table 2 Comparison of fundamental frequency of circular (a/b = 0) membranes [$\rho(r) = \rho_0(1 + \alpha r)$]

α	Ref. 2 Exact	Ref.21 DQ	Ref.21 FEM	Present DSC results	
				N = 13	N = 15
0	2.4048	2.4048	2.4049	2.4048	2.4048
0.5	2.1827	2.1827	2.1828	2.1828	2.1827
1.0	2.0108	2.0108	2.0109	2.0111	2.0108
1.5	1.8731	1.8731	1.8732	1.8732	1.8730
2.0	1.7598	1.7598	1.7600	1.7599	1.7598

reverse proportional density cases are taken into account. In general, the values of frequency increase with an increase in the radius ratio for membranes with different value of α. Fig. 6

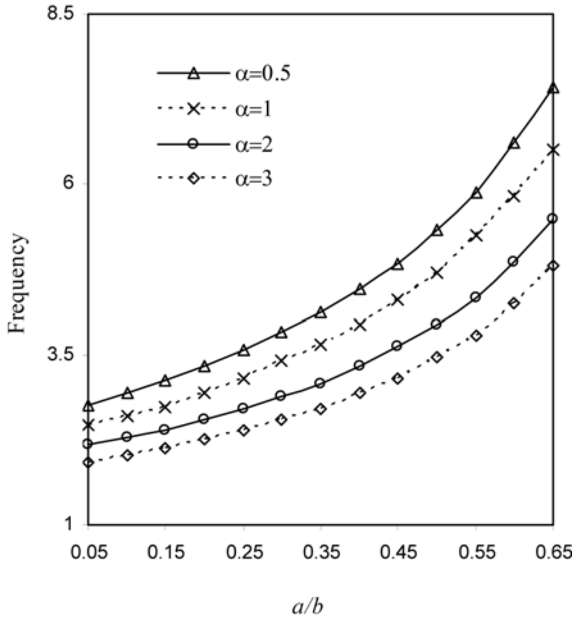

Fig. 2 Variation of fundamental frequency versus *a/b* for linear density $(1 + \alpha r)$

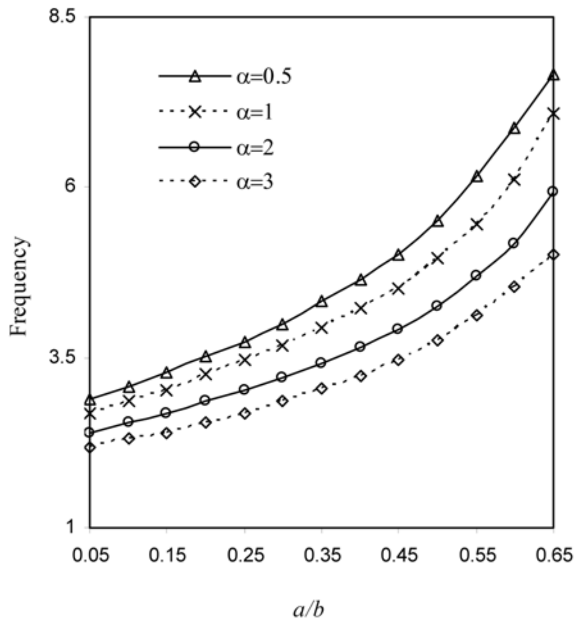

Fig. 3 Variation of fundamental frequency versus *a/b* for parabolic density $(1 + \alpha r^2)$

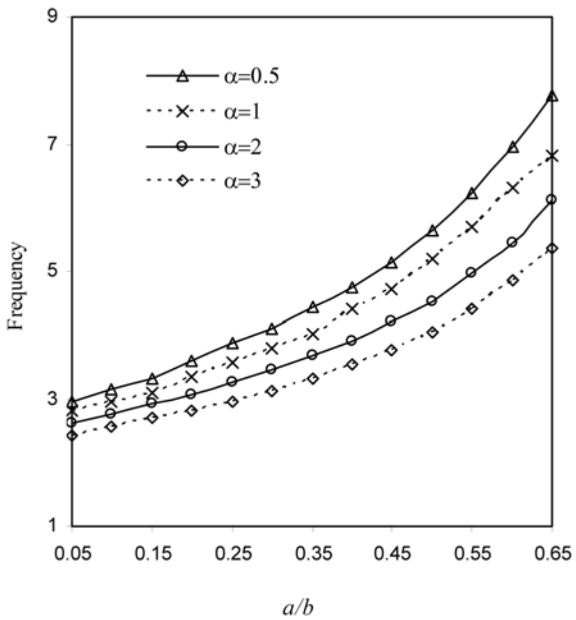

Fig. 4 Variation of fundamental frequency versus *a/b* for cubic density $(1 + \alpha r^3)$

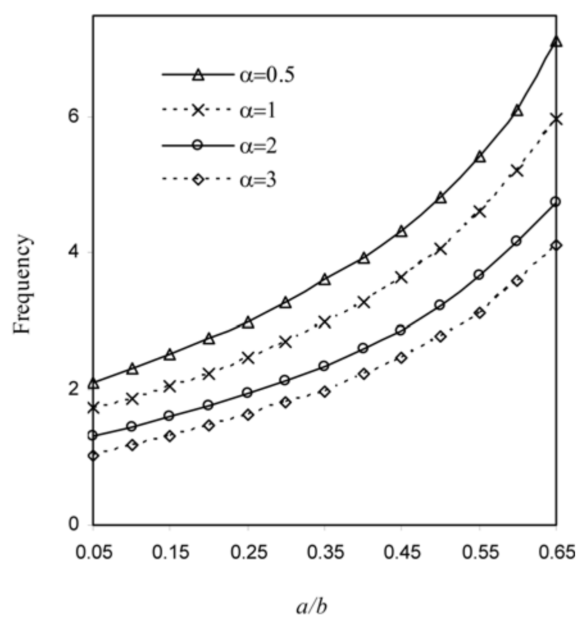

Fig. 5 Variation of fundamental frequency versus *a/b* for reverse proportional density $(1 + \alpha r^{-1})$

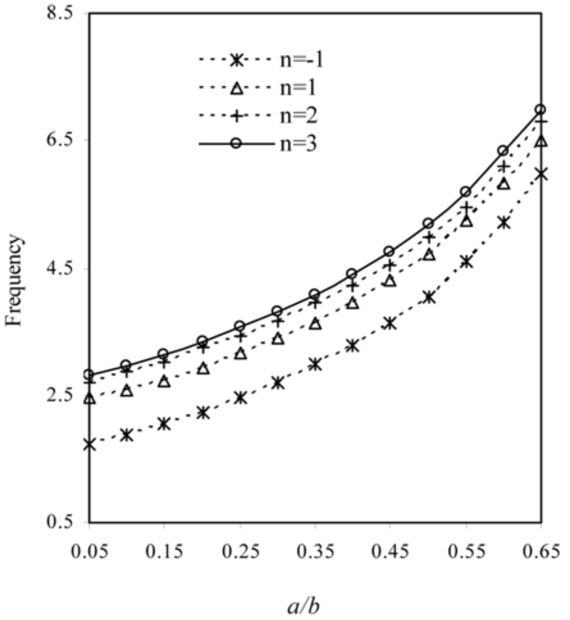

Fig. 6 Variation of fundamental frequency versus *a/b* for different value of n $(1 + r^n)$

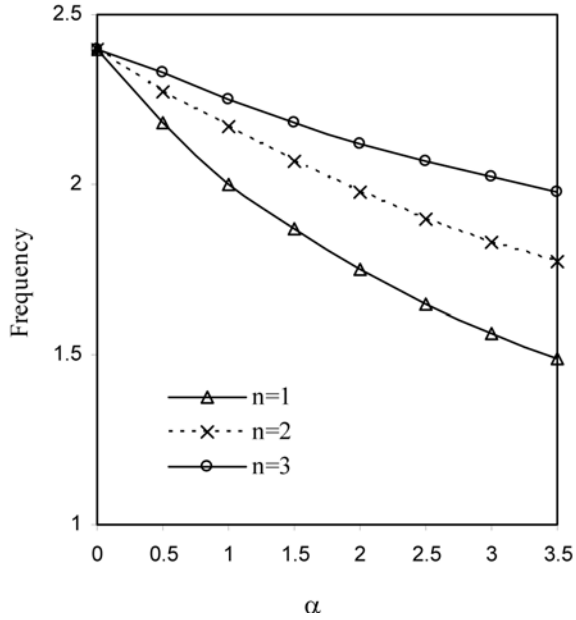

Fig. 7 Variation of fundamental frequency of circular ($a/b = 0$) membrane versus α for different value of n $(1 + \alpha r^n)$

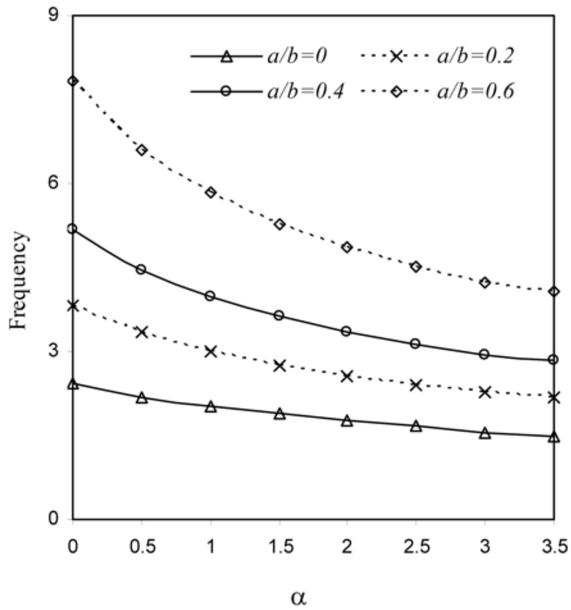

Fig. 8 Variation of fundamental frequency of annular and circular membrane versus α for different value of *a/b*

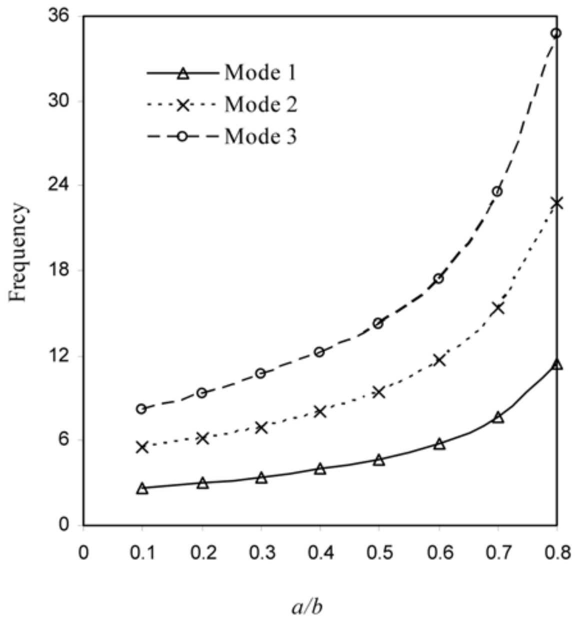

Fig. 9 Variation of first three frequency values of annular and circular membrane versus *a/b* for linear density ($\alpha = 1$; $n = 1$)

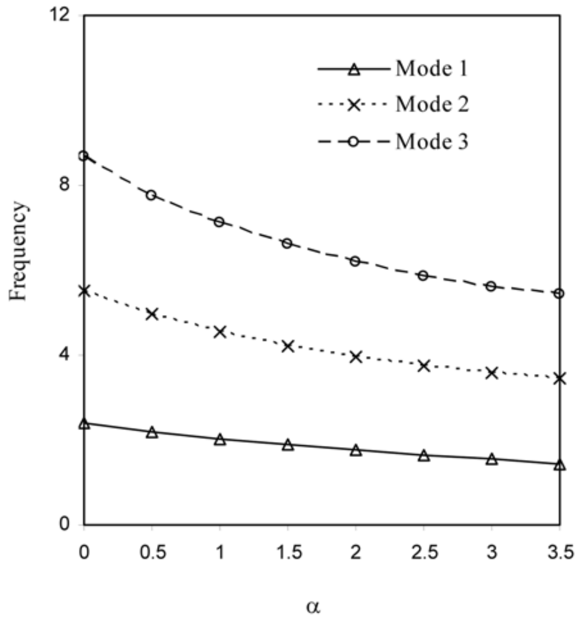

Fig. 10 Variation of first three frequency values of circular membrane ($a/b = 0$) versus α for linear density ($n = 1$)

illustrates the fundamental frequency of annular membrane versus a/b for different value of n. It is shown that the frequencies for the annular membrane increase rapidly with radius ratio. With the increase of a/b ratio the effect of the n value on the frequency parameter is insignificant. The influence of the effect of the parameter α on the fundamental frequency is studied by comparing the results for circular membranes with varying density. The result is depicted in Fig. 7. It is concluded that, the frequency parameter is uniformly decreased when the parameter α increases. It is also shown in this figure that the frequency increases with the increasing value of n. Fig. 8 shows the effect of inhomogeneity parameter α on frequency for different radius ratio. As expected, the frequency value is minimum for the circular membrane ($a/b = 0$). In other words, it is concluded that the frequency parameter generally increases as radius ratio increase. It may be also noticed that with increasing density parameter, the frequency decreases. Fig. 9 displays the effects of inner-to-outer radius ratio on the frequency value. Linear density is considered. From Fig. 9, it is clear that the inner-to-outer radius ratio is an effective magnitude on frequency value. The frequencies for the annular membrane increase quickly with inner-to-outer radius ratio at any mode numbers. This increased in the frequencies with the a/b ratio is due to an increased in the inner radius of the annulus. Fig. 10 describes the relationship between frequency and α for first three axisymmetric modes. The frequency parameter decreases rapidly for small inhomogeneity parameter ($\alpha \leq 1$). Natural frequencies and corresponding mode shapes for circular membrane are depicted in Figs. 11-13 for different value of tension. It is shown from these figures that, the frequency value are directly related with the value of applied tension. The frequency values are rapidly increased with the increasing value of T. Also, the applied tension is more significant effect on related mode shapes.

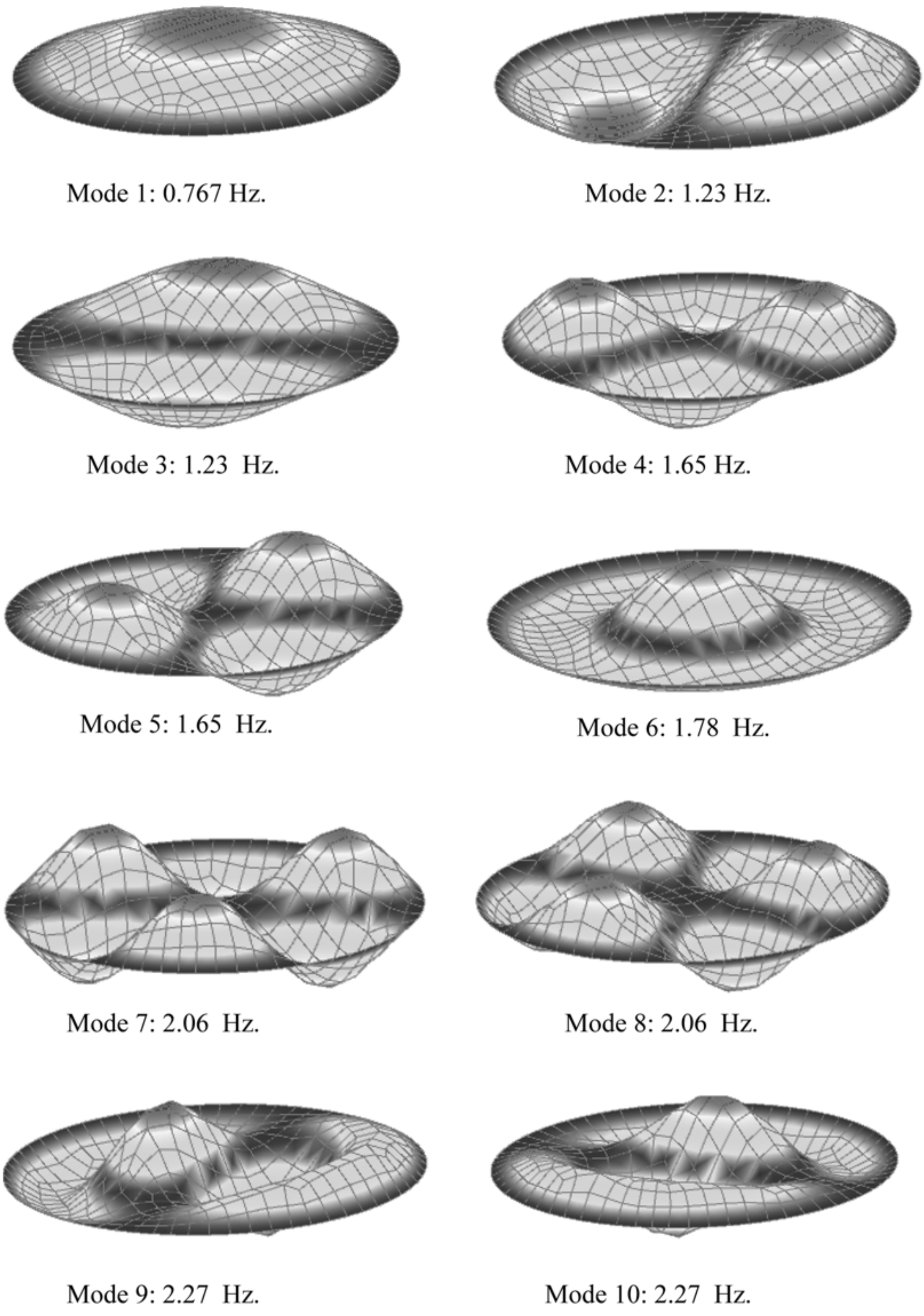

Fig. 11 Natural frequencies and corresponding mode shapes for circular membrane (T = 1 N/mm^2)

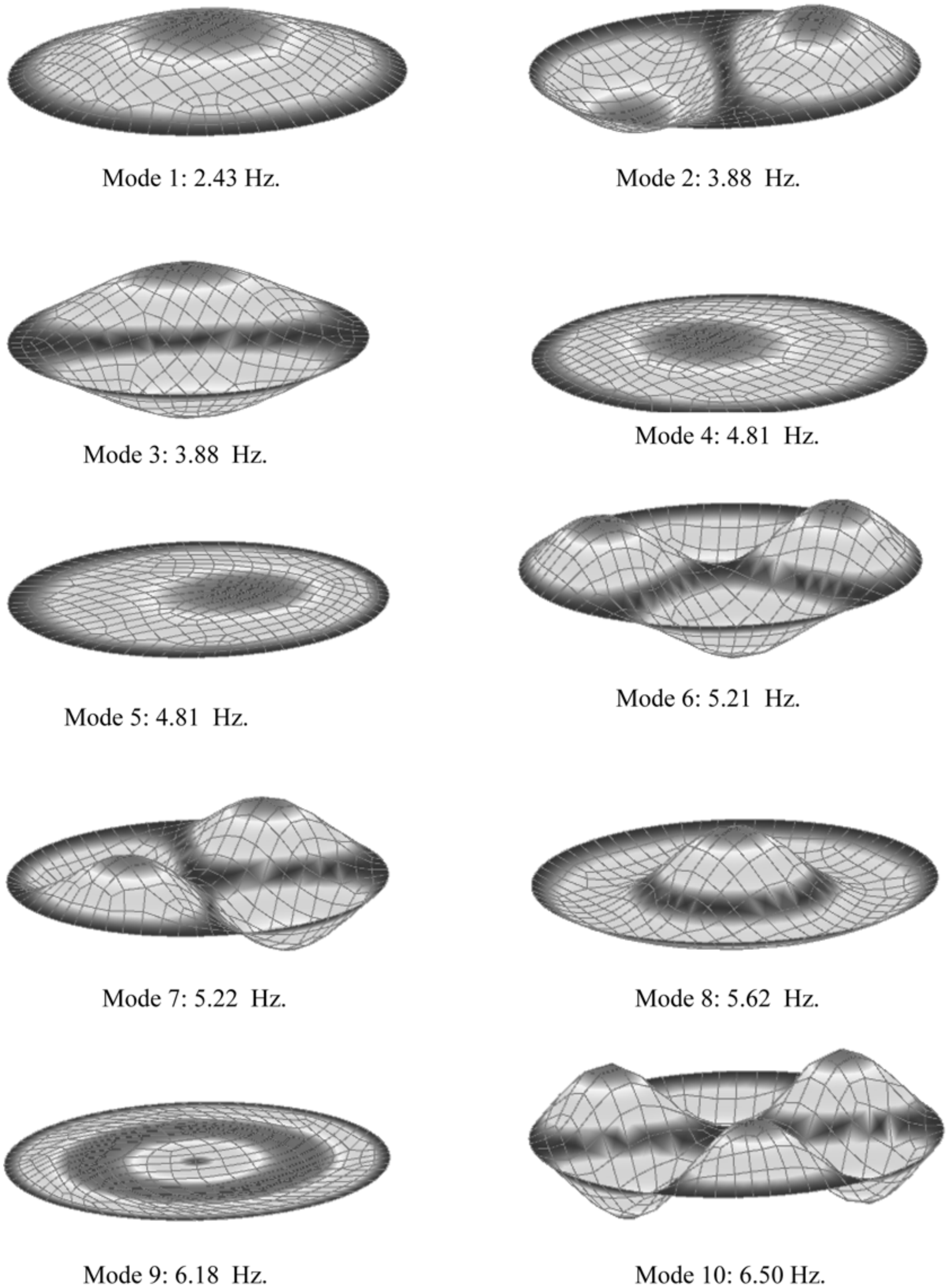

Fig. 12 Natural frequencies and corresponding mode shapes for circular membrane (T = 10 N/mm^2)

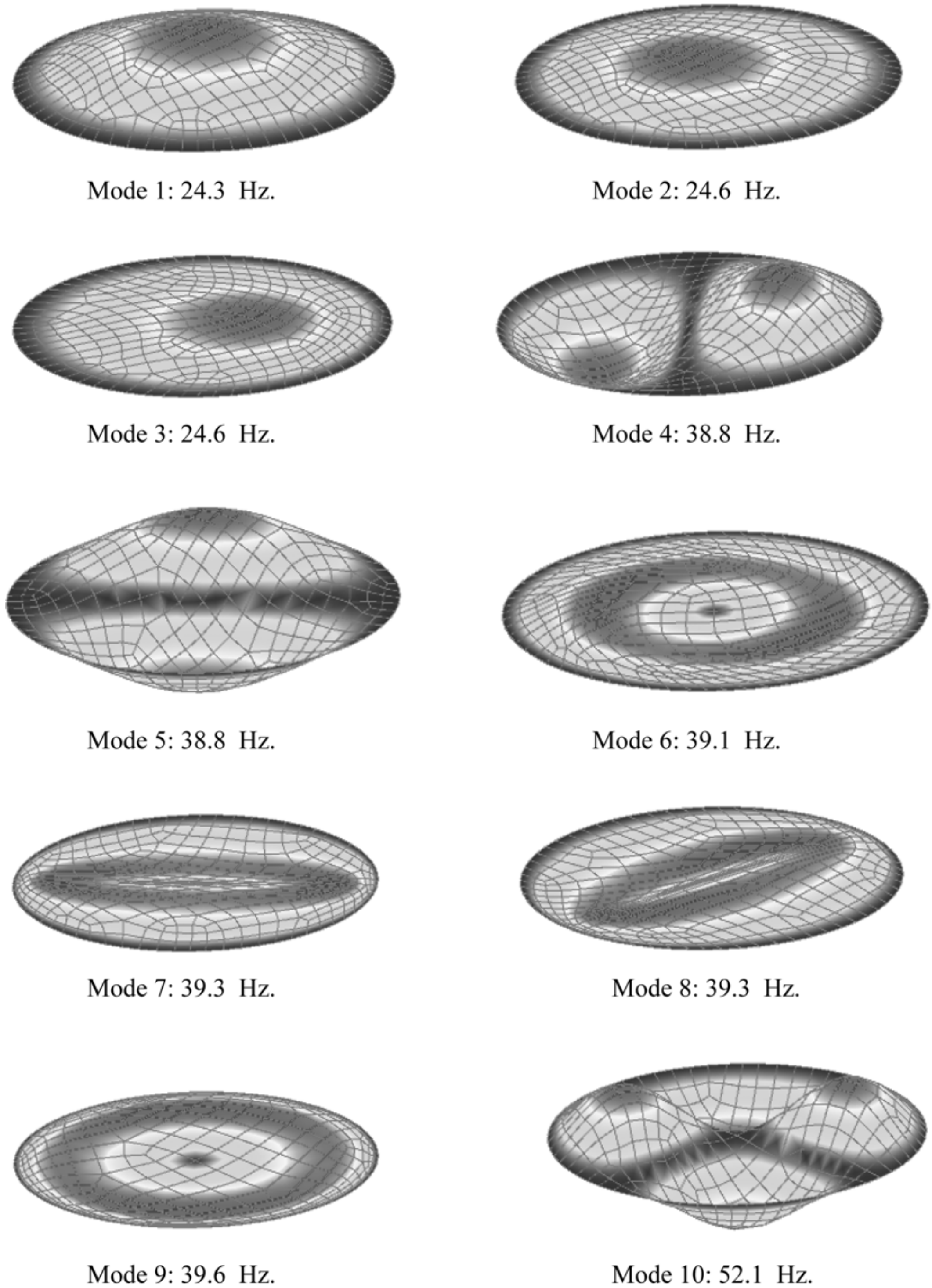

Fig. 13 Natural frequencies and corresponding mode shapes for circular membrane (T = 100 N/mm^2)

5. Conclusions

The discrete singular convolution method is successfully applied to free vibration problem for circular and annular membranes with varying density. The effects played by inner-to-outer radius ratio, variation of density, inhomogeneity parameter, and mode number are studied. Numerical examples illustrating the accuracy and convergence of the DSC method for free vibration problem of circular and annular membranes are presented. It is found that the convergence of the DSC approach is very good and the results agree well with those obtained by other researchers. In addition, the new numerical DSC algorithm has been examined and found to be simple, accurate and efficient.

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