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Stochastic control approach to reliability of elasto-plastic structures

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Abstract. An importance sampling method is presented for computing the first passage probability of elasto-plastic structures under stochastic excitations. The importance sampling distribution corresponds to shifting the mean of the excitation to an 'adapted' stochastic process whose future is determined based on information only up to the present. A stochastic control approach is adopted for designing the adapted process. The optimal control law is determined by a control potential, which satisfies the Bellman's equation, a nonlinear partial differential equation on the response state-space. Numerical results for a single-degree-of freedom elasto-plastic structure shows that the proposed method leads to significant improvement in variance reduction over importance sampling using design points reported recently.

Keywords: adapted process; elasto-plastic; first passage problem; stochastic optimal control; reliability; stochastic dynamics.

1. Introduction

Determining the first passage probability of nonlinear hysteretic structures remains a challenging computational problem in stochastic dynamics (Wen 1976, Soong and Grigoriu 1993, Lin and Cai 1995, Lutes 1997, Schueller 2006). The complexity involved is featured by a large number of random variables (theoretically infinite), nonlinear/hysteretic behavior and a large number of degrees of freedom. The first renders geometric intuitions in low dimensions less useful or sometimes misleading in high dimensions (Schueller *et al.* 2004, Au and Beck 2003). Nonlinearity, hysteresis and a large number of degrees of freedom make the understanding of system behavior more involved and in many cases analytical solution almost impossible. A recent benchmark study by Schueller and co-workers (Schueller 2007) indicated that advanced Monte Carlo methods have shown promise for tackling complex systems. Subset Simulation, line sampling, and collectively methods that make use of Markov Chain Monte Carlo have demonstrated their variance reduction capability while retaining certain robustness in applicability. These methods explore the progressive failure nature of reliability problems and do not significantly rely on knowing system behavior. The benchmark study also revealed a big gap between the extent of variance reduction that can currently be achieved for linear systems versus nonlinear hysteretic systems. For example, Problem 3 in the

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benchmark study indicates that, using Subset Simulation (Au *et al.* 2007a) or line sampling (Pradlwarter *et al.* 2007) that are based on Markov Chain Monte Carlo, one can achieve a unit c.o.v. (standard error = unit c.o.v./square root of no. of samples) of 30-40 for a target failure probability of the order of 10^{-4}. This translates into about 5,000 samples to achive a 50% c.o.v. in the failure probability estimate. In contrast, using methods such as importance sampling or line sampling that incorporate information about system behavior, one can achieve for linear systems a unit c.o.v. of 2-5 regardless of failure probability level, i.e., at most 100 samples required to achieve 50% c.o.v. in the estimate. A large variance reduction has been achieved for linear systems but there is much room to improve for general nonlinear systems. Notably, linearly system behavior is analytically tractable and very efficient importance sampling methods have been developed (Au and Beck 2001, Yuen and Katafygiotis 2005, Jensen and Valdebenito 2007).

The trade-off between efficiency and robustness of a simulation method justifies methods to be more application-focused to improve efficiency at the expense of generality in application. This of course rests on the premise that the problem focused has important relevance. An effort was initiated to develop an importance sampling method for nonlinear hysteretic structures. The ultimate aim is to achieve a level of variance reduction similar to that for linear systems, or otherwise find out what methods do not work and the reasons behind. The first attempt was to determine the design points of single-degree-of-freedom (SDOF) elasto-plastic structures and use them to construct an importance sampling density, speculating significant variance reduction similar to the linear case. SDOF structures with perfect elastic-plastic behavior were focused as they exhibit important hysteretic characteristics that are shared by their MDOF (multi-degree-of-freedom) counterparts. It turned out that efficient solutions for design points are possible in this case (Au 2006a, Au 2006b). Using the design points for importance sampling, however, led to only limited variance reduction compared to the linear case (Au *et al.* 2007b). Further investigations showed that the actions of the design points were rendered ineffective by random phase shifts associated with plastic excursions as well as opposing plastic excursions that cancelled out each other. This suggested naturally a design point that could ‘adapt’ and synchronize its actions with the response. Although a ‘stochastic design point’ sounds subtle at first glance because the design point plays the role of the mean shift of the importance sampling distribution, detailed arguments support its legitimacy when its components bear a causal property. Such a process is in fact called an adapted process, and its concept is long-rooted in the celebrated Girsanov Theorem (Girsanov 1960). The use of adapted process opens up new variance reduction possibilities for hysteretic structures and, in general, complex causal systems. An attempt was previously made using heuristic rules to design the adapted process and lead to improvement over importance sampling using design points (Au 2008). The heuristic rules were developed with the aim to grow the response according to the regime it currently belongs to. Sequential pulses were applied by the adapted process whose magnitudes were determined based on approximate energy concepts. For simplicity the design formulation ignored the stochastic effect of the white noise that was combined with the adapted process during importance sampling.

This paper presents a sequel on the development with an attempt to address the stochastic effect of the white noise in the design of the adapted process. A stochastic control approach is adopted, which appears a natural choice because designing the adapted process is equivalent to designing a control law to achieve certain purposes with minimum energy under a stochastic environment where the future response is influenced not only by the control force but also by random white noise disturbances. Although stochastic control is an established area, ‘off-the-shelf’ control laws are not available for first passage problems. This paper shall discuss the design of adapted process for solving the first passage problem in the context of stochastic control. An optimal control law, in some heuristic sense, is developed and its variance reduction efficiency shall be investigated.

2. Importance sampling for first passage problems

Consider a SDOF elasto-plastic structure subjected to white noise excitations. The displacement response of the elasto-plastic structure follows the governing equation

$$\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + F_r = W(t), \quad x(0) = \dot{x}(0) = 0 \tag{1}$$

where ω and ζ are the natural frequency and critical damping ratio of the associated linear system at low amplitudes; F_r is the restoring force that follows an elasto-plastic hysteretic rule: $dF_r = \omega^2 dx$ if $|F_r| < \omega^2 b_0$ and $dF_r = 0$ otherwise; b_0 is the first yield displacement. The white noise excitation is modeled digitally in the time domain by

$$W(t_i) = \sqrt{\frac{2\pi S}{\Delta t}} Z_i \tag{2}$$

where Δt is the sampling time; S is the spectral intensity; $\{Z_i : i = 1, \ldots, n\}$ are independent and identically distributed (i.i.d.) standard Gaussian random variables. Failure is defined as the first passage of the displacement response over the double barrier $\pm b_F$ within a given duration of interest $n\Delta t$

$$P_F = P\left(\bigcup_{i=1}^{n} |x(t_i)| > b_F\right) = E[I(\mathbf{Z} \in F)] \tag{3}$$

The last expression in Eq. (3) views the failure probability as an expectation of the indicator function $I(\mathbf{Z} \in F)$ with the failure region $F = \left\{\mathbf{z} \in R^n : \bigcup_{i=1}^{n} |x(t_i;\mathbf{z})| > b_F\right\}$; $x(t_i;\mathbf{z})$ here denotes explicitly the response at the i-th time step to the realization of white noise corresponding to $\mathbf{z} = [z_1, \ldots, z_n]$. The expectation is taken with $\mathbf{Z} = [Z_1, \ldots, Z_n]$ distributed as a standard Gaussian vector, i.e., with probability density function (PDF)

$$\phi_n(\mathbf{z}) = (2\pi)^{-n/2} \exp\left(-\frac{1}{2}\sum_{i=1}^{n} z_i^2\right) \tag{4}$$

Importance sampling (Rubinstein 1981, Hammersley 1964) is a popular technique for variance reduction in reliability problems, especially for rare failure events where the computational effort required by direct Monte Carlo is prohibitive. The idea is to generate samples from a properly chosen distribution that leads more frequently to failure. Let $f(\mathbf{z})$, called the importance sampling density (ISD), be a PDF whose support covers the failure region. By a change of probability distribution from ϕ_n to f, the failure probability is expressed as

$$P_F = E[I(\mathbf{Z} \in F)] = \int I(\mathbf{z} \in F) \frac{\phi_n(\mathbf{z})}{f(\mathbf{z})} f(\mathbf{z}) d\mathbf{z} = E_f\left[I(\mathbf{Z}' \in F) \frac{\phi_n(\mathbf{Z}')}{f(\mathbf{Z}')}\right] \tag{5}$$

where in the last expectation $\mathbf{Z}'$ is distributed as f instead of ϕ_n. The failure probability is then estimated via statistical averaging by

$$\tilde{P}_F = \frac{1}{N}\sum_{k=1}^{N} I(\mathbf{Z}'_k \in F)\frac{\phi_n(\mathbf{Z}'_k)}{f(\mathbf{Z}'_k)} \tag{6}$$

with $\{\mathbf{Z}'_k : k = 1, \ldots, N\}$ being i.i.d. (independent and identically distributed) samples generated according to f. For a proper choice of f, $\tilde{P}_F$ is an unbiased estimator for P_F. Its statistical error is often measured by its coefficient of variation (c.o.v.), δ, defined as the ratio of its standard deviation to its mean. Due to the use of i.i.d. samples, δ is inversely proportional to $\sqrt{N}$ and can be expressed as

$$\delta = \frac{\Delta}{\sqrt{N}} \tag{7}$$

where Δ is the c.o.v. of $I(\mathbf{Z}' \in F)\phi_n(\mathbf{Z}')/f(\mathbf{Z}')$, referred as the 'unit c.o.v.' of the importance sampling estimator. The unit c.o.v. Δ is sample-size independent and is often used for measuring the efficiency of the sampling method (Engelund and Rackwitz 1993, Schueller and Pradlwarter 2007). It can be readily shown that the following identity holds, which provides insights for variance reduction (Au and Beck 2003)

$$\Delta^2 + 1 = \frac{\Delta_{R|F}^2 + 1}{Q_F} \tag{8}$$

where $\Delta_{R|F}$ is the conditional c.o.v. of the 'importance sampling quotient' $R(\mathbf{Z}') = \phi_n(\mathbf{Z}')/f(\mathbf{Z}')$ given that $\mathbf{Z}' \in F$; Q_F is the probability that $\mathbf{Z}' \in F$, referred here as the 'failure rate' (to distinguish it from the failure probability P_F). For direct Monte Carlo, $Q_F = P_F$ (no improvement in failure rate) and $\Delta_{R|F} = 0$ (since $R = \phi_n/f = \phi_n/\phi_n \equiv 1$) and so $\Delta^2 = 1/P_F - 1 = (1-P_F)/P_F$ which checks with the classical expression for unit c.o.v. of direct Monte Carlo estimator. The identity in Eq. (8) suggests two general objectives in the choice of ISD: 1) to increase Q_F by producing more samples in the failure region; 2) to reduce $\Delta_{R|F}$ by a proper choice of the functional form of f. The former has been the major consideration in the structural reliability literature, although the latter has important significance for stochastic simulation problems that are featured by a large number n (theoretically infinite) of random variables (Au and Beck 2003, Schueller *et al.* 2004). Most often, the ISD is chosen as a Gaussian PDF or a weighted sum of Gaussian PDFs with unit covariance matrix; the choice of ISD then reduces to the choice of the mean vector(s) of the Gaussian PDF(s).

2.1 Shifting distribution to a fixed point

For instruction purpose, consider first the classical case when the ISD is chosen as a Gaussian PDF centered at some fixed point $\hat{\mathbf{z}} = [\hat{z}_1, \ldots, \hat{z}_n]$, i.e.

$$f(\mathbf{z}) = \phi_n(\mathbf{z} - \hat{\mathbf{z}}) = (2\pi)^{-n/2}\exp\left[-\frac{1}{2}\sum_{i=1}^{n}(z_i - \hat{z}_i)^2\right] \tag{9}$$

The failure probability is then expressed as

$$\begin{aligned} P_F &= E_f\left[I(\mathbf{Z}' \in F)\frac{\phi_n(\mathbf{Z}')}{f(\mathbf{Z}')}\right] = E\left[I(\hat{\mathbf{z}} + \mathbf{Z} \in F)\frac{\phi_n(\hat{\mathbf{z}} + \mathbf{Z})}{\phi_n(\mathbf{Z}')}\right] \\ &= E\left[I(\hat{\mathbf{z}} + \mathbf{Z} \in F)\exp\left(-\frac{1}{2}\sum_{i=1}^{n}\hat{z}_i^2 - \sum_{i=1}^{n}\hat{z}_i Z_i\right)\right] \end{aligned} \tag{10}$$

where under $E[\cdot]$ the vector $\mathbf{Z}$ is distributed as a n-dimensional standard Gaussian vector with independent components. The failure probability is then estimated by averaging the term inside the expectation over i.i.d. samples of $\mathbf{Z}$. Note that during importance sampling the structure is subjected to the forcing corresponding to $\mathbf{Z}' = \hat{\mathbf{z}} + \mathbf{Z}$, which represents the combined action of the design point excitation and the white noise ($\mathbf{Z}$).

2.2 Shifting distribution to an adapted process

Consider now allowing $\hat{\mathbf{z}} = [\hat{z}_1, \ldots, \hat{z}_n]$ to depend on $\mathbf{Z} = [Z_1, \ldots, Z_n]$ and viewing $\{\hat{z}_i : i = 1, 2, \ldots\}$ as a discrete-time stochastic process. Specifically, $\hat{z}_1$ is fixed, and for every $i = 1, 2, \ldots, \hat{z}_{i+1}$ depends only on $\{Z_1, \ldots, Z_i\}$ but not $\{Z_{i+1}, Z_{i+2}, \ldots\}$. That is, the next future value depends on the information only up to the present. In the theory of stochastic process $\hat{\mathbf{z}} = [\hat{z}_1, \ldots, \hat{z}_n]$ is called an adapted (or predictable) process. In our context, $\hat{\mathbf{z}}$ may be viewed as an adaptive control force in a stochastic environment. It should be designed by the user depending on the intended purpose. In contrast to a design point that is a fixed vector, the choice of $\hat{\mathbf{z}}$ as an adapted process involves the design of a set of rules (control law) defining how a realization of the process is generated.

For a given adapted process $\hat{\mathbf{z}}$, consider a transformation from $\mathbf{Z}$ to $\mathbf{Z}'$ defined by

$$\mathbf{Z}' = \hat{\mathbf{z}}(\mathbf{Z}') + \mathbf{Z} \tag{11}$$

where the sequence $\{Z_1', Z_2', \ldots\}$ is generated sequentially from $\{Z_1, Z_2, \ldots\}$ as follows

$$Z_1' = Z_1 + \hat{z}_1; Z_2' = Z_2 + \hat{z}_2(Z_1'); Z_3' = Z_3 + \hat{z}_3(Z_1', Z_2'); Z_4' = Z_4 + \hat{z}_4(Z_1', Z_2', Z_3'); \ldots \tag{12}$$

In general, for $i \geq 1$,

$$Z_{i+1}' = Z_{i+1} + \hat{z}_{i+1}(Z_1', \ldots, Z_i') = Z_{i+1} + \hat{z}_{i+1}(\mathbf{Z}'^{(i)}) \tag{13}$$

where $\mathbf{Z}'^{(i)} = [Z_1', \ldots, Z_i']$ denotes the components of $\mathbf{Z}'$ up to the i-th time step; a similar notation applies for other processes. Note that given $\{Z_1', Z_2', \ldots\}$, $\{Z_1, Z_2, \ldots\}$ can be obtained sequentially in a reversed manner based on Eq. (12).

To obtain the PDF for $\mathbf{Z}'$, note that during importance sampling, $\{Z_1, \ldots, Z_n\}$ are i.i.d. standard Gaussian. Given $\{Z_1', \ldots, Z_i'\}$, $\hat{z}_{i+1}(\mathbf{Z}'^{(i)})$ is already determined, and hence Z_{i+1}' in Eq. (13) is Gaussian with mean $\hat{z}_{i+1}(\mathbf{Z}'^{(i)})$ and unit variance. The conditional PDF of Z_{i+1}' given $\mathbf{Z}'^{(i)} = [z_1', \ldots, z_i']$ is thus given by

$$p(z_{i+1}' \,|\, \mathbf{z}'^{(i)}) = \phi_1(z_{i+1}' - \hat{z}_{i+1}(\mathbf{z}'^{(i)})) \tag{14}$$

where $\phi_1(\cdot)$ denotes the one-dimensional standard Gaussian PDF. Here, the lower-case letters denote the input argument of the PDF for the random vectors denoted by upper-case letters. We also use $p(\cdot)$ to denote a generic probability density function, and $p(\cdot|\cdot)$ for the conditional counterpart. The joint PDF for the random vector $\mathbf{Z}'$ can be obtained by sequentially conditioning on $\mathbf{Z}'^{(i)}$ $(i = n-1, n-2, \ldots)$

$$p(\mathbf{z}') = p(\mathbf{z}'^{(n)}) = p(z'_n|\mathbf{z}'^{(n-1)})p(\mathbf{z}'^{(n-1)}) = \ldots = p(z'_1)\prod_{i=2}^{n} p(z'_i|\mathbf{z}'^{(i-1)}) \tag{15}$$

Using Eq. (14), this becomes (abbreviating $\hat{z}_i(\mathbf{z}'^{(i-1)})$ as $\hat{z}_i$)

$$p(\mathbf{z}') = \phi_1(z'_1 - \hat{z}_1)\prod_{i=2}^{n}\phi_1(z'_i - \hat{z}_i) = \phi_n(\mathbf{z}' - \hat{\mathbf{z}}) = (2\pi)^{-n/2}\exp\left[-\frac{1}{2}\sum_{i=1}^{n}(z'_i - \hat{z}_i)^2\right] \tag{16}$$

which is the same expression as if $\hat{\mathbf{z}}$ were fixed.

In summary, it is legitimate to make a change of distribution to a Gaussian PDF centered at a stochastic vector $\hat{\mathbf{z}}$ when it has a causal structure; the corresponding PDF takes the same form as the one with a shift of a deterministic mean vector. Note, however, that the resulting ISD is no longer Gaussian in $\mathbf{z}'$ because $\hat{\mathbf{z}}$ now depends on $\mathbf{z}'$. Provided that $\|\hat{\mathbf{z}}\| < \infty$ almost surely, the importance sampling estimator is valid with a finite variance. The continuous-time or infinite dimensional ($n \to \infty$) analogue of this result is the Girsanov Theorem (Girsanov 1960, Protter 1990).

3. Design of adapted process for variance reduction

In the time domain, the adapted process acts as a feed-forward control force. The design of adapted process involves the design of a stochastic process, or a control law, in contrast to the choice of design point excitation that only involves specification of a fixed vector. Two issues need to be resolved: 1) to specify a practical design goal and 2) to design an algorithm that meets the specified goal.

Ideally, the design goal is to minimize the unit c.o.v. Δ of the failure probability estimate. This is not practical, however, because Δ is analytically intractable; computationally the estimation of Δ is even more difficult than the failure probability. A practical goal is one that can be readily analyzed so that specific algorithms can be devised. According to Eq. (8), Δ can be reduced by increasing Q_F and reducing $\Delta_{R|F}$. As Q_F and $\Delta_{R|F}$ are difficult to analyze and control directly, we shall work with their proxies instead. In this work, we take the energy of adapted process as a proxy for $\Delta_{R|F}$. This can be reasoned from the variance of the exponential term in Eq. (10). The design goal is to grow the response to excursion with as less energy as possible. These two objectives are often conflicting. For a given class of control strategy one does not know a priori the optimal combination of Q_F and $\Delta_{R|F}$ that minimizes Δ. A prudent choice should trade-off between Q_F and $\Delta_{R|F}$ in order to reduce Δ.

4. Stochastic control law for adapted process

In this section we design the adapted process based on a stochastic control approach. We first focus on the up-crossing problem and then extend the results to apply for a double-barrier problem. Previous work (Au 2008) employed a heuristic rule for assigning the future excitation that is three-folded: 1) when the response is small, apply a suitable force to grow the amplitude; 2) when the amplitude is close to the yield limit and the structure is accelerating towards the target, apply a pulse to cause plastic deformation in the direction of failure; 3) when the amplitude is close to the yield limit but the response is accelerating away from the target, let the structure to go through free vibration until it reaches the other extreme position. The rule was updated whenever the displacement response has just passed a stationary point.

The algorithm presented in Au (2008) did not consider the stochastic effect of the white noise excitation. In particular, intuition suggests that when the response is small the effect of the white noise dominates. In this case the response will grow even in the absence of the adapted process and so addressing the stochastic effect may help save energy in the adapted process whenever suitable. In this work we adopt a stochastic control approach to design the adapted process. The method naturally takes into account the stochastic effect of white noise. The theory is more generic and allows for further development.

We consider the stochastic control problem of driving the response to have large positive plastic excursion, assuming the current state is in the linear regime. The objective is to drive the response from the current state in the linear regime to yielding. This objective does not directly address the original objectives in the first passage problem but it simplifies the design of controller.

For the purpose of deriving the control law we shall work in continuous-time as the derivation is more elegant. Under importance sampling the governing equation in the linear regime is given by

$$\ddot{x}(t) + 2\zeta\omega\dot{x}(t) + \omega^2 x(t) = W(t) + u(t) \tag{17}$$

where $x(t)$ is measured relative to the current neutral axis; $u(t)$ is the adapted process in the time domain, related to that in the standard Gaussian space by $u(t_k) = \sqrt{2\pi S/\Delta t}\,\hat{z}_k$. Interpreting in the Ito sense, $\dot{x}$ follows the stochastic differential equation

$$d\dot{x} = [u - \mu(x, \dot{x})]dt + \sigma dB \tag{18}$$

where $\mu(x, \dot{x}) = \omega^2 x + 2\zeta\omega\dot{x}$, $\sigma = \sqrt{2\pi S}$ and $dB = Wdt$ is the standard Brownian motion increment ($E[(dB)^2] = dt$).

Consider designing a control force that depends on the current state $(x(t), \dot{x}(t))$. For a given future realization define the objective function as

$$J(u) = \frac{1}{2}\int_t^\tau u(s)^2 ds + \lambda_1\tau - \lambda_2[\dot{x}(\tau)]_+ \tag{19}$$

where $\lambda_1, \lambda_2 > 0$ are positive constants to be specified; $\tau = \inf\{s > t : |x(s)| > b_0\}$ is the time at which the response exits the linear-elastic regime, assuming $(x(t), \dot{x}(t))$ is in the linear-elastic regime; $[\cdot]_+ = \max(\cdot, 0)$ gives the value of its argument when positive and zero otherwise. The objective function $J(u)$ increases with the energy of the adapted process u until yielding. It also increases with the time until yielding, through the second term in Eq. (19). To understand the third term, note that it decreases with the exit velocity at positive yielding (with positive plastic displacement). On the other hand, it is zero if yielding occurs with negative exit velocity (and hence causing negative plastic displacement). The third term is thus smaller if positive yielding occurs, and zero if negative yielding occurs.

Since $J(u)$ depends on the stochastic future the actual objective function for design shall be expressed through its expectation conditional on the current state

$$E[J(u)|x(t), \dot{x}(t)] = E\left[\frac{1}{2}\int_t^\tau u(s)^2 ds + \lambda_1\tau - \lambda_2[\dot{x}(\tau)]_+ \Big| x(t), \dot{x}(t)\right] \tag{20}$$

Minimizing $E[J(u)|x(t),\dot{x}(t)]$ will therefore achieve three objectives through the three terms discussed, respectively: 1) reducing the control effort to yielding; 2) reducing the time to yielding; and 3) promoting positive yielding. Relative importance of these three objectives are controlled by a proper choice of the parameters λ_1 and λ_2. We shall come back to this after we have derived the control law.

4.1 Bellman's equation

The minimization should be performed on the space of all admissible control strategies that depend only on the information up to the present. The stochastic control problem corresponds to one of an indefinite time horizon, i.e., the time span over which the performance is evaluated is not fixed but is dependent on the stopping time τ. With a time invariant system it can be argued that the control law and the minimum of the objective function depend only on the current state but not explicitly on time t (Kushner and Dupuis 2001). Let $V(x,\dot{x})$ be the minimum, i.e.

$$V(x,\dot{x}) = \min_u E[J(u)|x,\dot{x}] \tag{21}$$

Assuming V is sufficiently smooth to apply Ito Calculus, it satisfies the Bellman's equation (Kushner 1967), which in our case is given by

$$\min_u \left\{ \frac{\partial V}{\partial x}\dot{x} + \frac{\partial V}{\partial \dot{x}}[u - \mu(x,\dot{x})] + \frac{1}{2}\frac{\partial^2 V}{\partial \dot{x}^2}\sigma^2 + \frac{1}{2}u^2 + \lambda_1 \right\} = 0 \tag{22}$$

where the minimization is point-wise for any given time $0 < t < \tau$. Solving for the minimum gives the optimal control force

$$u^*(x,\dot{x}) = -\frac{\partial V(x,\dot{x})}{\partial \dot{x}} \tag{23}$$

and the resulting Bellman's equation reads

$$\frac{\partial V}{\partial x}\dot{x} - \frac{\partial V}{\partial \dot{x}}\mu(x,\dot{x}) + \frac{1}{2}\frac{\partial^2 V}{\partial \dot{x}^2}\sigma^2 - \frac{1}{2}\left(\frac{\partial V}{\partial \dot{x}}\right)^2 + \lambda_1 = 0 \tag{24}$$

Because of the way V is related to the control force, it is often called the 'control potential'. The Bellman's equation in Eq. (24) is a nonlinear partial differential equation (PDE) on the state-space domain $(-b_0, b_0) \times (-\infty, +\infty)$. Note that only the first two terms of J are reflected in Eq. (24). The third term of J, which is related to the exit velocity, is reflected in the boundary conditions for V

$$V(b_0,\dot{x}) = -\lambda_2\dot{x} \quad \text{for} \quad \dot{x} > 0 \quad \text{and} \quad V(-b_0,\dot{x}) = 0 \quad \text{for} \quad \dot{x} < 0 \tag{25}$$

4.2 Jacobi iteration

The Bellman's equation belongs to a class of PDEs that can often be solved numerically by iteration, despite its nonlinearity. We next present the numerical solution for V by means of a method called Jacobi iteration (Kushner and Dupuis 2001). The partial derivatives are approximated by their finite difference on a mesh of the domain. For this purpose it is important to recognize the flow of information from the boundary conditions to affect the solution within the domain. For

$\dot{x} > 0$, the right boundary $x = b_0$ is constrained and so information should flow leftward, suggesting a forward difference for $\partial V/\partial x$. Similarly, for $\dot{x} < 0$, V is constrained on the left boundary $x = -b_0$ and so a backward difference should be used for $\partial V/\partial x$. For $\dot{x} = 0$, $\partial V/\partial x$ does not appear in the equation. By similar reasoning, we approximate $\partial V/\partial \dot{x}$ by forward, central and backward difference for $\dot{x} > 0$, $\dot{x} = 0$ and $\dot{x} < 0$, respectively. Thus, we take

$$\left.\frac{\partial V}{\partial x}\right|_{i,j} \cong \begin{cases} \dfrac{V_{i+1,j} - V_{i,j}}{\Delta x} & \text{for} \quad \dot{x} > 0 \\ \dfrac{V_{i,j} - V_{i-1,j}}{\Delta x} & \text{for} \quad \dot{x} < 0 \end{cases} \tag{26}$$

$$\left.\frac{\partial V}{\partial \dot{x}}\right|_{i,j} \cong \begin{cases} \dfrac{V_{i,j+1} - V_{i,j}}{\Delta \dot{x}} & \text{for} \quad \dot{x} > 0 \\ \dfrac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{2\Delta \dot{x}} & \text{for} \quad \dot{x} = 0 \\ \dfrac{V_{i,j} - V_{i,j-1}}{\Delta \dot{x}} & \text{for} \quad \dot{x} < 0 \end{cases} \tag{27}$$

and

$$\left.\frac{\partial^2 V}{\partial \dot{x}^2}\right|_{i,j} \cong \frac{V_{i,j+1} - 2V_{i,j} + V_{i,j-1}}{\Delta \dot{x}^2} \tag{28}$$

Approximating the partial derivatives in Eq. (22) by their finite difference and minimizing, one obtains the optimal control force as the finite difference approximation of the control potential. The resulting Bellman's equation reads

$$V_{i,j} = p_1 V_{i,j+1} + p_2 V_{i,j-1} + p_3 V_{i+1,j} + B_{i,j}(\hat{u}_{i,j}^2 + \lambda_1) \tag{29}$$

where

$$p_k = p_k' / \sum_{r=1}^{3} p_r', \quad B_{i,j} = 1/\sum_{r=1}^{3} p_r' \tag{30}$$

and

$$p_1' = \frac{\sigma^2}{2\Delta \dot{x}_{i,j}^2}, \quad p_2' = \frac{\sigma^2}{2\Delta \dot{x}_{i,j}^2} - \frac{\hat{u}_{i,j} - \mu_{i,j}}{\Delta \dot{x}_{i,j}}, \quad p_3' = \frac{|\dot{x}_j|}{\Delta x_i} \quad \text{for} \quad \dot{x}_{i,j} > 0 \tag{31}$$

$$p_1' = \frac{\sigma^2}{2\Delta \dot{x}_{i,j}^2} + \frac{\hat{u}_{i,j} - \mu_{i,j}}{2\Delta \dot{x}_{i,j}}, \quad p_2' = \frac{\sigma^2}{2\Delta \dot{x}_{i,j}^2} - \frac{\hat{u}_{i,j} - \mu_{i,j}}{2\Delta \dot{x}_{i,j}}, \quad p_3' = 0 \quad \text{for} \quad \dot{x}_{i,j} = 0 \tag{32}$$

$$p_1' = \frac{\sigma^2}{2\Delta \dot{x}_{i,j}^2} + \frac{\hat{u}_{i,j} - \mu_{i,j}}{\Delta \dot{x}_{i,j}}, \quad p_2' = \frac{\sigma^2}{2\Delta \dot{x}_{i,j}^2}, \quad p_3' = \frac{|\dot{x}_j|}{\Delta x_i} \quad \text{for} \quad \dot{x}_{i,j} < 0 \tag{33}$$

We have dropped the dependence on (i,j) in p_k and p_k' to simplify notations. Note that $V_{i,j}$ in Eq. (29) cannot be solved explicitly because $\hat{u}_{i,j}$ depends on $V_{i,j}$. Nevertheless, the equation suggests solution for $V_{i,j}$ through the following iteration

$$V_{i,j}^{(n+1)} = p_1^{(n)} V_{i,j+1}^{(n)} + p_2^{(n)} V_{i,j-1}^{(n)} + p_3^{(n)} V_{i+1,j}^{(n)} + B_{i,j}^{(n)}(\hat{u}_{i,j}^2 + \lambda_1) \tag{34}$$

where a superscript '(n)' denotes that the quantity is calculated using value at the n-th iteration. Convergence of this iterative scheme is approximately ensured by requiring that $p_1^{(n)} + p_2^{(n)} + p_3^{(n)} = 1$ and $p_1^{(n)}, p_2^{(n)}, p_3^{(n)} > 0$, since then the mapping from the matrix $\underline{V}^{(n)} = [V_{i,j}^{(n)}]_{i,j}$ to $\underline{V}^{(n+1)} = [V_{i,j}^{(n+1)}]_{i,j}$ after linearization is a contraction. The first requirement is automatically satisfied by the definition of $p_k^{(n)}$. The second requirement can be enforced by limiting $\hat{u}_{i,j}$ to lie within $(\mu_{i,j}^{(n)} - \sigma^2/\Delta\dot{x}, \mu_{i,j}^{(n)} + \sigma^2/\Delta\dot{x})$. The control law obtained in this manner is optimal only among the space of all those that make the iterative scheme convergent. The bounds on $\hat{u}_{i,j}$ are inactive when $\Delta\dot{x} \to 0$. This means that when the bounds are found to be active the converged solution is only suboptimal and it can be improved by refining the mesh on $\dot{x}$ (reducing $\Delta\dot{x}$).

4.3 Choice of λ_1 and λ_2

The parameters λ_1 and λ_2 control the relative importance of the three objectives in the design of the optimal control law. In principle they can be chosen to maximize variance reduction, although such choice is not trivial and may not be worthwhile to pursue. For effective variance reduction it is sufficient to assign their values to the right order of magnitude. As a simple choice, we may require the three terms in $E[J(u)|x, \dot{x}]$ to be of the same order of magnitude, which suggests

$$\lambda_1 = O\left(\frac{E}{t_0}\right) \text{ and } \lambda_2 = O\left(\frac{E}{\omega b_0}\right) \tag{35}$$

where E is the energy of the linear-elastic design point that pushes the structure from rest to the level $x = b$ at time t_0. The time t_0 may be taken as some reasonable time at which the response is expected to go from the linear regime to yielding. Note that the linear-elastic design point can be obtained easily from the unit impulse response function. The assignment of λ_1 and λ_2 need not be precise, as the aim is to specify them just to the right order of magnitude.

4.4 Double barrier problem

For single-barrier problems, e.g., up-crossing failure, one can use an adapted process designed for up-crossing and then apply Eq. (10). For double-barrier crossing problems, one can use an average of ISD based on the up-crossing and down-crossing adapted process to account for failures due to up-crossing and down-crossing, respectively

$$f(\mathbf{z}) = \frac{1}{2}\phi(\mathbf{z} - \hat{\mathbf{z}}^+) + \frac{1}{2}\phi(\mathbf{z} - \hat{\mathbf{z}}^-) \tag{36}$$

where $\hat{\mathbf{z}}_+$ and $\hat{\mathbf{z}}_-$ denote the adapted process for up- and down-crossing respectively. Note that for each realization $\hat{\mathbf{z}}^+ \neq -\hat{\mathbf{z}}^-$, even though the algorithm for generating $\hat{\mathbf{z}}_+$ can be modified to be used for generating $\hat{\mathbf{z}}_-$. By symmetry the up-crossing and down-crossing adapted process in the time domain are given by

$$u^+ = -\frac{\partial V(x, \dot{x})}{\partial \dot{x}}, \quad u^- = -\frac{\partial V(-x, -\dot{x})}{\partial \dot{x}} \tag{37}$$

Using the ISD in Eq. (36), the failure probability is given by

$$P_F = 2E_f[I(\mathbf{Z}' \in F)R(\mathbf{Z}')] \tag{38}$$

where the expectation is taken with $\mathbf{Z}'$ distributed according to f and

$$R(\mathbf{z}') = \left[\frac{\phi(\mathbf{z}'-\hat{\mathbf{z}}^+)}{\phi(\mathbf{z}')} + \frac{\phi(\mathbf{z}'-\hat{\mathbf{z}}^-)}{\phi(\mathbf{z}')}\right]^{-1} \tag{39}$$

Evaluating the expectation in Eq. (38) to account for the bimodal nature of f gives

$$P_F = E_+[I(\mathbf{Z}' \in F)R(\mathbf{Z}')] + E_-[I(\mathbf{Z}' \in F)R(\mathbf{Z}')] = 2E_+[I(\mathbf{Z}' \in F)R(\mathbf{Z}')] \tag{40}$$

where the subscript '+' in the first expectation denotes that $\mathbf{Z}'$ is distributed as $\phi(\mathbf{z}-\hat{\mathbf{z}}^+)$; similar notation applies for the second expectation. The second equality makes use of the observation that the up-crossing and down-crossing expectations are the same, due to symmetry. By importance sampling, the failure probability is estimated by averaging the term under expectation over i.i.d. samples.

5. Numerical investigation

Consider an SDOF elasto-plastic structure with $\omega = 2\pi$, $\zeta = 1\%$ and yield displacement $b_0 = 1$. Failure is defined as first passage over the double barrier $x = \pm b_F$, $b_F = 3$, within a duration of interest $t_F = 20$ sec. The spectral intensity of the white noise is assumed to be $S = 0.86$, for which the response standard deviation at 20s is approximately equal to 1.

5.1 Control law

We first study the characteristics of the control law. Here we set $\lambda_1 = \lambda_2 = 2$, which is found to yield a reasonable failure rate. Other values of λ_1 and λ_2 shall be considered later when variance reduction is investigated. Fig. 1 shows the contour of the control potential $V(x,\dot{x})$ and the corresponding control law $u(x,\dot{x})$. These are computed by Jacobi iteration on a 40×400 $(x,\dot{x})$-grid shown in the figure. As enforced by the boundary conditions, V is identically zero along the lower-left boundary and it decreases with $\dot{x}$ along the upper-right boundary. It is seen that V behaves differently in the upper and lower quadrants, partly due to the different boundary conditions imposed in the first and third quadrant. In the lower quadrants (third and fourth) it is essentially zero, while in the upper quadrants (first and second) it roughly decays radially with x and $\dot{x}/\omega$, which is akin to the total energy. The control law $u(x,\dot{x})$ shown in Fig. 1(b), which is just $-\partial V/\partial\dot{x}$, shows a peak in the upper quadrants and a trough in the lower quadrants, although the former is more pronounced. The contour shows that the control law is not directly proportional to $\dot{x}$. It does not increase monotonically with $\dot{x}$; even within the increasing region the rate is not constant. This suggests that a constant gain feed-forward rule can be far from being optimal. When $\dot{x}$ is small the control law is small regardless of x. This is the region where the stochastic effect of the white noise dominates and so saving control effort is a good strategy.

We next investigate the manner in which the control law drives the response to first passage failure. As a reference Fig. 2 shows the design point excitation that targets to drive the response to first passage failure at 20 sec, computed using an efficient algorithm documented in Au (2006a).

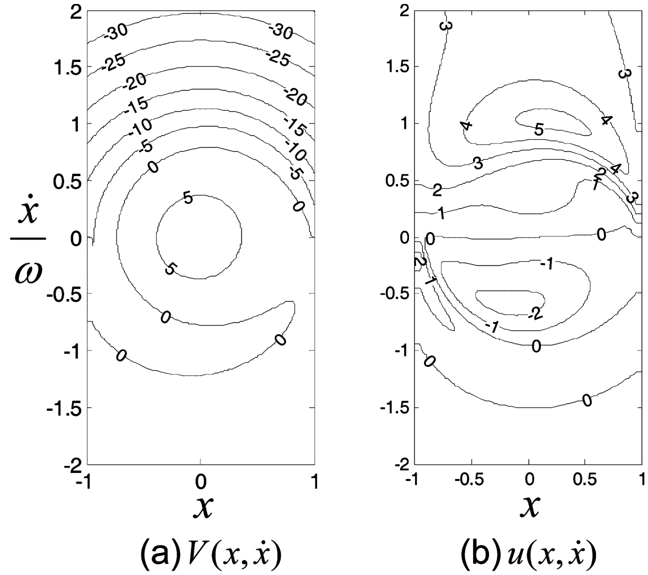

Fig. 1 Contour of control potential $V(x,\dot{x})$ and control force $u(x,\dot{x})$

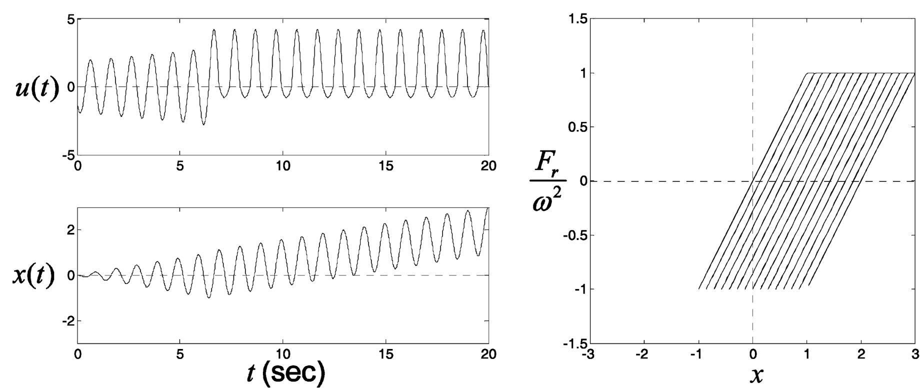

Fig. 2 Design point excitation (first passage failure at 20 sec)

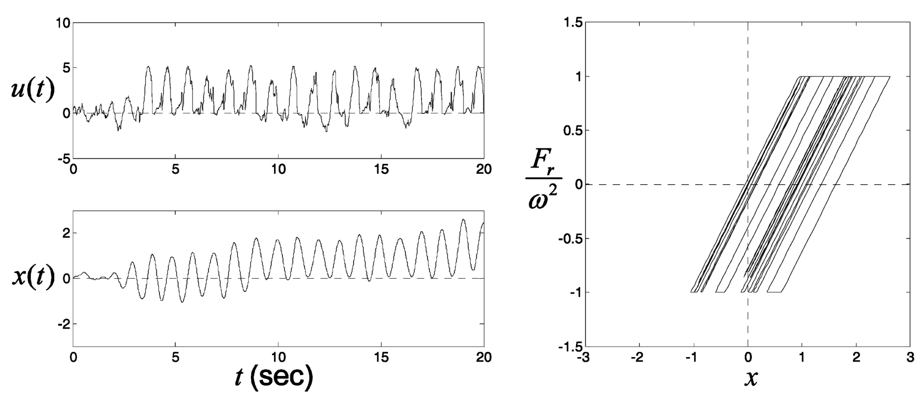

Fig. 3 Random sample of $u(t)$ and corresponding $x(t)$ (not failed)

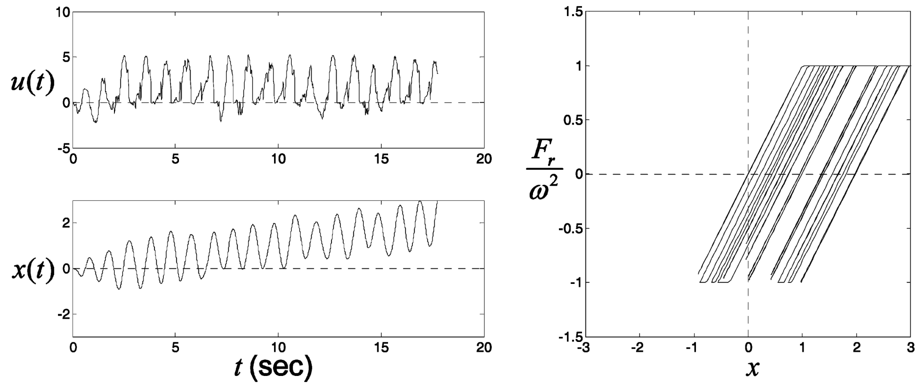

Fig. 4 Random sample of $u(t)$ and corresponding $x(t)$ (failed at 17.5 sec)

The design point excitation initially grows the response by resonance. It then applies positive pulses to create positive plastic displacements that accumulate until failure while avoiding negative plastic displacements. The design point excitation is effective in generating large response only in the absence of white noise. During importance sampling where the excitation consists of the design point excitation and white noise, previous studies reported that it actions are not effective, leading to a small failure rate (Au *et al.* 2007b).

Fig. 3 and Fig. 4 show two typical samples of the control force $u(t)$, the corresponding response $x(t)$ and hysteretic history of restoring force. Note that the response $x(t)$ is due to the combined action of the control force and white noise (not shown). The samples of control force exhibit two properties that are important to generating response effectively: 1) resonance and 2) asymmetry that avoids negative plastic displacements. Their adapted stochastic nature promotes synchronization with the response and maintains their effectiveness. The samples of control force and the design point excitation in Fig. 2 have the same order of magnitude. Previous heuristic design of the adapted process (Au 2008) also generated similar samples but it was achieved through a set of decision rules. It is interesting to note that the control force derived from a single control potential function exhibits features that mimics the decision rules while accounting for stochastic effects.

5.2 Variance reduction

To investigate the variance reduction efficiency of the designed adapted process, we perform importance sampling using the adapted process. As a reference the failure probability has been estimated by direct Monte Carlo with one million samples to be 2.4×10^{-5} (c.o.v. = 20%). The failure probability has been previously estimated by importance sampling with design points (Au *et al.* 2007b) and importance sampling with adapted process designed heuristically with updating at stationary points (Au 2008). Table 1 summarizes the reported variance reduction efficiency.

To investigate the robustness of the control force (for the purpose of importance sampling) with respect to the choice of λ_1 and λ_2, we consider different combinations of their values and perform importance sampling in each case. The combinations covered λ_1 = 1, 1.5 and λ_2 = 1.5, 2, 2.5. In all cases the contour of control law was found to be qualitatively similar. The unit c.o.v. was estimated with one hundred thousand (100,000) samples. The results are summarized in Table 2. For each combination of λ_1 and λ_2 the values of the unit c.o.v. Δ, failure rate Q_F and conditional c.o.v. $\Delta_{R|F}$

Table 1 Previous reports on variance reduction

Method	Failure rate Q_F	Unit c.o.v. Δ	Conditional c.o.v. $\Delta_{R\|F}$
Direct MCS	2.4×10^{-5}	204	0
IS-design point (Au *et al.* 2007b)	5%	38	8.4
IS-adapted (Au 2008)	14%	20	7.4

Table 2 Variance reduction for different choice of λ_1 and λ_2

Q_F, Δ, $\Delta_{R\|F}$		λ_2 ($\times E/\omega b_0$)		
		1.5	2	2.5
λ_1 ($\times E/t_0$)	1	5%, 9.8, 2.0	12%, 9.7, 3.2	24%, 13.6, 6.6
	1.5	9%, 9.0, 2.5	21%, 7.6, 3.4	38%, 12.8, 7.9

are shown. Variance reduction is assessed in term of the unit c.o.v. Δ, whereas its mechanism can be analyzed with Q_F and $\Delta_{R|F}$, through the identity in Eq. (8). Here, the values of λ_1 and λ_2 chosen, in multiples of their respective scales, are of the order of 1. The values chosen are intended to illustrate the effect of λ_1 and λ_2 on efficiency, and the trade-off between trade-off of Q_F and $\Delta_{R|F}$ that possibly leads to a minimum Δ.

As shown in Table 2, the failure rate Q_F increases with both λ_1 and λ_2. This is expected because in the objective function λ_1 reflects the importance of the time to next yield, while λ_2 reflects the importance of exit velocity. The higher the value of λ_1, the more important it is to shorten the time to next yield and hence the higher the power the control law will spend. On the other hand, a high value of λ_2 targets a high exit velocity, which again increases the power.

The conditional c.o.v. $\Delta_{R|F}$ increases with the failure rate because it roughly increases with the energy of the adapted process (control force). The rate of increase is not uniform and can be quite drastic for high failure rates. The net effect is that variance reduction is better achieved with a balance between the failure rate and the conditional c.o.v. As shown in Table 2, the value of unit c.o.v. is similar for different combinations of λ_1 and λ_2. No optimization with regard to (λ_1, λ_2) is intended in this work, although it is observed that variance reduction is highest around $\lambda_1 = 1.5$ and $\lambda_2 = 2$. It is more important for the variance reduction capability to be robust to the choice of (λ_1, λ_2) than to locate the optimal value of (λ_1, λ_2) because the optimum is likely to be problem dependent.

The unit c.o.v. in other range of values of λ_1 and λ_2 can be expected from their roles in the method. Essentially, as their values initially increase from zero they tend to decrease Δ because in that range they can increase the failure rate significantly while overweighing the accompanying increase in Q_F. At the other extreme if they are very large too much effort is spent on driving the response to plastic excursions and their large energy will lead to exponentially large $\Delta_{R|F}$. The resulting detrimental increasing effect on Δ can hardly be compensated by the increase in the failure rate Q_F. The range of values of λ_1 and λ_2 shown in the Table 2 is the 'interesting' region where the effects of Q_F and $\Delta_{R|F}$ are trading off.

The values of unit c.o.v. reported in Table 2 are generally smaller than those reported previously, reflecting a progress in variance reduction. In particular, for a similar failure rate the conditional c.o.v. of the proposed method is significantly smaller than that of the adapted process developed in the previous work (Table 1, last row), demonstrating the effectiveness of the stochastic control law. To make a comparison in terms of computational efforts, suppose a δ = 30% c.o.v. in the target failure probability of 2.4×10^{-5} is desired. This means the required average number of samples required is $N = \Delta^2/\delta^2 \sim 10\Delta^2$. Based on the values of Δ reported in Table 1 and Table 2, the number of samples for direct MCS, importance sampling using design points (Au *et al.* 2007b), importance sampling using previous heuristic rules (Au *et al.* 2008) and the present work are 416×10^3, 14×10^3, 4×10^3 and 1×10^3. Here, for the present work we have used a representative value of Δ = 10. These numbers show a progressive improvement in variance reduction using importance sampling technique.

6. Conclusions

Within the framework of importance sampling using adapted processes this paper has presented an approach for designing the adapted process as a stochastic optimal control law. The objective function reflects the expected energy needed to next yield as well as the exit velocity. This objective is heuristic but it simplifies design of the controller. Determining the optimal controller involves solution of a control potential that satisfies the Bellman's equation. The Bellman's equation is a nonlinear PDE on the state-space of response, and it is numerically solved by Jacobi iteration method in this paper. When the control potential cannot be solved analytically, as is often the case, it implies that the method will suffer from the curse of the state-space dimension for MDOF structures. In view of this, a viable strategy is to use control laws that only depend on partial observation of the response state. Alternatively, one may develop heuristic rules as control law for MDOF structures, so that it does not involve solving PDEs. This of course is not trivial but the development may leverage on understanding of the response generating mechanisms of optimal controllers in lower state-space dimensions.

The proposed method can be applied to elastic-nonlinear SDOF systems. In this case the nonlinearity will be reflected in the drift term of the stochastic differential equation of system dynamics; the same Jacobi iteration can be applied for solving the Bellman's equation to obtain the control potential. Of course, similar problems with increasing state-space dimensions as in elasto-plastic structures will be encountered, which can possibly be resolved by reducing the state-space dimension in the design of the adapted process.

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