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Efficient Monte Carlo simulation procedures in structural uncertainty and reliability analysis - recent advances[1]

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Abstract. The present contribution addresses uncertainty quantification and uncertainty propagation in structural mechanics using stochastic analysis. Presently available procedures to describe uncertainties in load and resistance within a suitable mathematical framework are shortly addressed. Monte Carlo methods are proposed for studying the variability in the structural properties and for their propagation to the response. The general applicability and versatility of Monte Carlo Simulation is demonstrated in the context with computational models that have been developed for deterministic structural analysis. After discussing Direct Monte Carlo Simulation for the assessment of the response variability, some recently developed advanced Monte Carlo methods applied for reliability assessment are described, such as Importance Sampling for linear uncertain structures subjected to Gaussian loading, Line Sampling in linear dynamics and Subset simulation. The numerical example demonstrates the applicability of Line Sampling to general linear uncertain FE systems under Gaussian distributed excitation.

Keywords: uncertainty propagation; stochastic; Monte Carlo Simulation; dynamics; reliability.

1. Introduction

It is a well known fact that numerical procedures are of paramount importance in modern structural mechanics. Following this trend that is taking place in traditional, deterministic structural mechanics over the last three decades, similar developments are also observed in stochastic structural mechanics. It is to be noted that pure analytical solutions, while most useful for verifying results of numerical algorithms for problems in which a reference solution exists, are in most cases - mainly for dimensional reasons - not applicable to provide solutions required by the engineering practice. Hence, further development of the methods for computational stochastic structural analysis is required to provide analysis tools that are suitable for engineering applications.

Needless to say, the complexity of procedures treating stochastic structural systems is far more involved than that of procedures developed for deterministic structures. This is because in the latter case the parameters are represented - i.e., approximated - by a single value rather than by a probability distribution representing a finite or infinite number of values the parameter may assume, each with a certain probability. Hence, in order to make numerical stochastic procedures applicable for practical applications, computationally efficient procedures have to be developed. Experience has

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shown that the Monte Carlo Simulation (MCS) procedure fulfills this requirement. MCS is the most versatile method for uncertainty quantification as well as reliability analysis. A key advantage of this method consists in the fact that it utilizes the very same mechanical model that is used in deterministic analysis. The considerably increased computational efforts associated with MCS can be reduced by applying advanced simulation techniques, as shown below, as well as parallel computation.

In this overview, recent advances in linear and non-linear structural dynamics are described. In linear structural analysis, the focus is on the efficiency of reliability evaluations in linear dynamics without limitations regarding the size of the finite element models in terms of degrees of freedom. Other advancements have been made in improving the efficiency of Direct Monte Carlo Sampling by at least one order of magnitude without reducing its generality. These procedures are applied whenever Direct Monte Carlo is the only available option, e.g., for complex non-linear, involved reliability problems.

2. Numerical modeling of structural load and resistance

A large number of quantities involving random fluctuations in time and space can be adequately described by stochastic processes, fields and waves. Typical examples of engineering interest are earthquake ground motion, sea waves, wind turbulence, road roughness, imperfections of shells, fluctuating properties in random media, etc. For many practical problems involving the above physical phenomena, probabilistic characteristics of the associated stochastic processes are known from past investigations and the associated experimental measurements and data.

However, the available probabilistic characteristics of random quantities affecting the loading of mechanical systems often cannot be utilized directly to assess the resulting randomness of the structural response, due to the complexity of the relationship between the randomness in the input and in the output. For example, in the common case of strong earthquake motion, the structural response will be generally non-linear and it might be too difficult - or in many cases even impossible - to compute the probabilistic characteristics of the response with methods other than MCS. The generated sample functions represent accurately the characteristics of the underlying stochastic process or fields and might be stationary and non-stationary, homogeneous or non-homogeneous, one-dimensional or multi-dimensional, uni-variate or multi-variate, Gaussian or non-Gaussian, depending very much on the accuracy required for the representation of the physical behavior, as well as on the available statistical data. In the context with load and resistance modeling, for example, the main requirement of the sample function is the accurate representation of the available stochastic information of the process. The most common representations are: ARMA and AR models, filtered white noise (SDE), shot noise and filtered Poisson white noise, covariance decomposition, Karhunen-Loève (KL) and polynomial chaos expansion, spectral representations and wavelet representations.

Among the various methods listed above, the spectral representation methods appear to be most widely used (see e.g., Rice 1954, Shinozuka and Jan 1972, Shinozuka 1972). In this procedure, samples with a specified power spectral density are generated. For the stationary or homogeneous case the fast Fourier transform technique is utilized for the sake of computational efficiency (see e.g., references (Yang 1972, Yang 1973)). The spectral representation method generates ergodic sample functions, each of which exactly fulfills the requirements of a target power spectrum. These

procedures can be extended to the non-stationary case, to the generation of stochastic waves (see e.g., Deodatis and Shinozuka 1989) and to incorporate non-Gaussian stochastic fields (see e.g., Yamazaki and Shinozuka 1988) by a memory-less nonlinear transformation, coupled with an iterative procedure to satisfy the target spectral density (see e.g., Cai and Lin 1996). The above spectral representation procedures for unconditional simulation of stochastic processes and fields can also be extended to conditional simulation techniques for Gaussian fields (see e.g., Kameda and Morikawa 1991, 1992, 1993) employing the conditional probability density method. The aim of this procedure is the generation of Gaussian random variates U_n under the condition that $(n - 1)$ realizations are known and that the a-priori known covariance is satisfied.

A quite general spectral representation utilized for Gaussian random processes and fields is the KL expansion of the covariance function (see e.g., Ghanem and Spanos 1991, Loève 1977). This representation is applicable to stationary (homogeneous) as well as to non-stationary (inhomogeneous) stochastic processes and fields. The expansion of a stochastic process $\varphi(\mathbf{x}, \omega)$ takes the form

$$\varphi(\mathbf{x}, \omega) = \overline{\varphi}(\mathbf{x}) + \sum_{i=1}^{\infty} \theta_i(\omega)\sqrt{\lambda_i}\phi(\mathbf{x}) = \overline{\varphi}(\mathbf{x}) + \sum_{i=1}^{\infty} \theta_i(\omega)\varphi^{(i)}(\mathbf{x}) \tag{1}$$

where the symbol ω indicates the random nature of the corresponding quantity, $\overline{\varphi}(\mathbf{x})$ denotes the mean and where $\phi(\mathbf{x})$ and λ_i are the eigenfunctions and eigenvalues, respectively, of the covariance function. In the above equation, $\varphi^{(i)}(\mathbf{x}) = \sqrt{\lambda_i}\phi(\mathbf{x})$ are deterministic functions, to which the random variables $\theta_i(\omega)$ are the associated amplitudes. The set $\{\theta_i(\omega)\}$ forms a set of orthogonal (uncorrelated), zero mean random variables with unit variance. The KL expansion is mean square convergent irrespective of its probabilistic nature, provided it possesses a finite variance. For the important special case of a Gaussian process or field, the random variables $\theta_i(\omega)$ are independent, standard normal random variables. In many practical applications where the random quantities vary smoothly with respect to time or space, only few terms are necessary to capture the largest part of the random fluctuations of the process. Its major advantage is the reduction of the set of random entities, from a large number of strongly correlated random variables to a small set of important, uncorrelated ones. Hence this representation is especially suitable for band limited colored excitation and for the stochastic finite element (SFE) representation of random media in which the random variables are usually strongly correlated. It might also be utilized to represent the correlated stochastic response of MDOF-systems by few, important variables, thus achieving a space reduction. A generalization of the above KL expansion has been proposed for applications in which the covariance function is not known a-priori (see (Cameron and Martin 1947, Ghanem and Spanos 1991, Ghanem and Kruger 1996)).

3. Direct Monte Carlo simulation

Monte Carlo methods are used successfully in many fields ranging from finance, social sciences, chemistry, medicine, mathematics to engineering, to mention the most prominent ones. In this article, Monte Carlo methods related to structural uncertainties and reliability will be discussed. One major area of application is the generation of samples, based on available probability distributions, and the propagation of the uncertainty to the response. This provides information on the uncertainty of the response, in terms of scatter plots or moment estimates such as mean, standard deviation,

correlations, etc. In other words, MCS is used for simulating and understanding the effects of uncertainty and variability in structural design.

In engineering applications, MCS based procedures have advantages over alternative procedures in several respects:

1. a considerably smaller growth rate of the computational effort with dimensionality than analytical procedures;
2. MCS procedures are generally applicable, well suited for parallel processing and computationally straightforward;
3. complex, non-linear behavior does not pose complications for the procedure;
4. the procedures remain manageable for complex systems.

Contrary to the numerical solution of analytical procedures, the employed structural model and the type of stochastic loading do not play a decisive role for MCS. For this reason, MCS procedures might be classified according to their purpose, i.e., whether the sample functions are generated for the estimation of the response distribution, or for the generation of rare, adverse events, which are needed for reliability assessment.

The most basic application of Monte Carlo procedures consists in the generation of independent samples of the structural properties and of the loading, by making use of a random number generator. Since each call to the random number generator is supposed to return an independent random number, the stochastic process needs to be represented as a function of independent random quantities.

In the simplest cases, loads and structural properties can be modelled by independent random variables, each following a pre-defined probability distribution. For commonly used distribution types, available mathematical packages will immediately return independent random realizations of each quantity. For non-standard distributions, realizations can be computed with the inverse transformation method, the composition method and the acceptance-rejection method. The incorporation of correlations is in most cases not as simple. In fact, the injection of correlations is in general quite difficult; the exception are Gaussian distributed variables, which can always be transformed into independent random variables with a linear transformation. Non-Gaussian, correlated random variables are therefore often transformed into a set of correlated Gaussian variables, which can then be represented as a linear combination of independent quantities. Eventually, these are then transformed back to the original set of non-Gaussian random variables (Liu and Der Kiureghian 1986).

Gaussian distributed random fields and stochastic processes can be represented by a truncated Karhunen-Loève representation, in which the random variables are independent. Similarly as for non-Gaussian random variables, non-Gaussian stochastic processes are first transformed to Gaussian processes, which are then easily sampled, since they can be seen as functions of independent random variables, and then again transformed back to non-Gaussian properties (Grigoriu 1995).

For cases in which the complexity of the problem does not permit an explicit formulation of the joint probability function, but only the definition of ratios of the density for various random quantities, the Metropolis-Hastings algorithm (Metropolis 1953, Hastings 1970) can be applied. Another alternative to generate random samples of correlated random variables is represented by Gibbs-sampling (Geman and Geman 1988). Both procedures are computationally demanding and therefore only applied if simpler, alternative procedures are not applicable.

The ability to generate independent samples of the structural properties and of the loading

provides the basis for studying the associated variability of the response. In the following it is assumed that all structural properties are uniquely specified as a function of a set of k independent standard normal variates $\boldsymbol{\theta} = \{\theta_l\}_{l=1}^{k}$ and all parameters defining the loading are specified as a function of m independent standard normal variates $\boldsymbol{\xi} = \{\xi_l\}_{l=1}^{m}$. These functions are in general non-linear, unless all properties are Gaussian distributed. As a consequence, the equation of motion for linear structural analysis has the form

$$\mathbf{M}(\boldsymbol{\theta})\ddot{\mathbf{u}}(t;\boldsymbol{\theta},\boldsymbol{\xi}) + \mathbf{D}(\boldsymbol{\theta})\dot{\mathbf{u}}(t;\boldsymbol{\theta},\boldsymbol{\xi}) + \mathbf{K}(\boldsymbol{\theta})\mathbf{u}(t;\boldsymbol{\theta},\boldsymbol{\xi}) = \mathbf{f}(t;\boldsymbol{\xi}) \tag{2}$$

where **M**, **D** and **K** denote the mass, damping and stiffness matrix, respectively. For static problems only the time-independent displacement response $\mathbf{u}(\boldsymbol{\theta}, \boldsymbol{\xi})$ is considered. For nonlinear structural analysis, the structural matrices are usually a function of the static or dynamic response, as well as of other parameters. In the context of Monte Carlo simulation, it is essential to note that any point $\boldsymbol{\zeta} = (\boldsymbol{\theta}, \boldsymbol{\xi})$ in a $(k + m)$-dimensional space uniquely specifies the equation of motion.

Let $\{\boldsymbol{\zeta}^{(i)}\}_{i=1}^{N}$ be a set of independent random realizations, drawn from the distribution of $(\boldsymbol{\theta}, \boldsymbol{\xi})$. Each realization $\boldsymbol{\zeta}^{(i)}$ leads then to a deterministically defined structural analysis problem. The static/dynamic, linear/non-linear response can be determined using a variety of commercial and open-source FE analysis tools. In the following, the responses of interest, e.g., extreme stresses, displacements or acceleration, $r_l^{(i)}$, associated with the realization $\boldsymbol{\zeta}^{(i)}$, are represented by the components of the vector $\mathbf{r}^{(i)}$. Statistical estimates for the mean value $\hat{\mu}_l$, for the standard deviation $\hat{\sigma}_l$ and for the correlation coefficients $\hat{\rho}_{lj}$ between different response quantities, are then given by

$$\hat{\mu}_l = \frac{1}{N}\sum_{i=1}^{N} r_l^{(i)} \tag{3}$$

$$\hat{\sigma}_l^2 = \frac{1}{N-1}\sum_{i=1}^{N} (r_l^{(i)} - \hat{\mu}_l)^2 \tag{4}$$

$$\hat{\rho}_{lj} = \frac{1}{N\hat{\sigma}_l\hat{\sigma}_j}\sum_{i=1}^{N} (r_l^{(i)} - \hat{\mu}_l)(r_j^{(i)} - \hat{\mu}_j) \tag{5}$$

These estimates provide information on the variability of the response. So-called scatter plots, where pairs of response quantities $\{(r_l^{(i)}, r_j^{(i)})\}_{i=1}^{N}$, or pairs of input and output variables, are plotted as dots in a two-dimensional coordinate system, may reveal additional information beyond the first two moments provided by the mean, standard deviation and correlation coefficient. Exact first-, second- and higher-order moments of the stochastic response could only be obtained with an infinitely large sample size, i.e., where $N \to \infty$. Characteristic for all estimates obtained by Monte Carlo Simulation is the rather slow convergence rate, which is proportional to $1/\sqrt{N}$. This law implies that the sample size has to increase four times, in order to reduce the standard deviation of the estimate by a factor of two.

The second major area of application of Monte Carlo simulation is the evaluation of higher dimensional integrals, which is needed in reliability analysis and, more specifically, for the identification of the failure domains by random search algorithms. In order to compute the failure probability, a high-dimensional integral needs to be evaluated

$$p_f = E[g(\theta,\xi)\le 0] = \int\!\dots\!\int I_f(g(\theta,\xi))q(\theta)q(\xi)\,d\theta_1\,\dots\,d\theta_k\cdot d\xi_1\,\dots\,d\xi_m \tag{6}$$

where $I_f(g)$ is an indicator function, assuming the value of one for $g \le 0$ and zero otherwise, and $E[\cdot]$ denotes the mean or mathematical expectation. In the above representation, it is assumed that the performance function $g = g(\theta, \xi)$ is formulated such that it is positive whenever all response quantities $r_l(\theta, \xi)$ of interest are in the safe domain, i.e., when they fulfill the associated performance criteria.

Two basic difficulties are associated with the above integral. Firstly, the dimension $d = m + k$ is usually high, precluding any available deterministic integration scheme. Secondly, the performance function is usually not known explicitly as a function of the random variables (θ, ξ); instead it must be computed point-wise by applying a suitable deterministic FE-analysis tool. These two difficulties assign a competitive edge to Monte Carlo simulation as a fundamental approach to cope with the above integral. Applying Direct MCS, the above integral is estimated by using the following approximation

$$\hat{p}_f = \frac{1}{N}\sum_{i=1}^{N} I_f(g(\zeta^{(i)})), \quad \zeta^{(i)} = (\theta^{(i)}, \xi^{(i)}) \tag{7}$$

Its convergence rate, which in the mean square sense is most appropriately measured by the coefficient of variation of $\hat{p}_f$

$$\mathrm{CoV}(\hat{p}_f) = \sqrt{(1-p_f)/(Np_f)} \tag{8}$$

is independent of the dimensionality of the random vector ζ. Its main disadvantage is its inefficiency in estimating small failure probabilities p_f due to the large number N of samples - and hence of FE runs - needed to achieve an acceptable level of accuracy ($N \propto 1/p_f$).

To remove or at least alleviate this drawback of Monte Carlo simulation, several advanced simulation methods have been developed, such as Importance Sampling, Directional Sampling and Adaptive Importance Sampling (Ditlevsen and Madsen 2005). Specifically for the reliability assessment of high dimensional problems, i.e., problems with large numbers of uncertain parameters, recently two methodologies evolved, namely Subset Simulation, which has been shown to be very general in terms of the class of problems it can handle (Au and Beck 2001b, Au *et al.* 2007, Katafygiotis and Cheung 2007), and Line Sampling (Koutsourelakis *et al.* 2004, Pradlwarter *et al.* 2007). These methods are described briefly in the following section.

4. Advanced algorithms

4.1 General remarks

For the estimation of the failure probability of a system, simulation is frequently applied, in particular, when direct integration becomes impractical. Theoretically, simulation methods -which also can be considered as a particular type of numerical integration - can yield accurate solutions if the number of simulations is sufficiently large. However, considerable computational effort is inevitable if the failure probability is very low. In this case efficient variance reduction techniques are a suitable alternative to reduce the number of simulations.

Variance reduction techniques can be most advantageously applied if prior information about the problem - i.e., about the region of simulation - is utilized. Instead of simulating the samples randomly over the entire range of each variable, the samples are concentrated in the important regions. Therefore, the highest efficiency in variance reduction can be accomplished if prior information about the problem is included in the simulation process. This information can be obtained for instance from direct simulation or other approximate methods.

Among variance reduction techniques the Importance Sampling technique is most frequently applied in structural reliability analysis (Kahn 1956, Schuëller *et al.* 1989, Fishman 1996, Neal 2001, Au and Beck 2003, Olsen and Naess 2006, Katafygiotis and Zuev 2007). Other well known techniques such as Stratified Sampling (Ding *et al.* 1998, Ding *et al.* 1996), Latin Hypercube Sampling (Huntington and Lyrintzis 1998, Helton and Davis 2003, Olsson *et al.* 2003), Control Variates and Antithetic Variates (Yang and Liou 1996), aim at achieving a better representation with the finite set of samples. Stratified Sampling is mainly used to represent different sub-populations according to their size, in order to reduce the variability induced by random sampling for small sample sizes N. Latin Hypercube Sampling is useful to reduce the variability of moment estimates by making sure that the relevant domain is more uniformly covered. Control variates are used to correlate known results of the control problem with the estimate of the problem of interest to reduce its variance. Antithetic Variates induce negative correlations between two samples. The estimates resulting from both samples are averaged and lead to a reduced variance, depending on the induced negative correlation.

4.2 Importance sampling

The basic idea of this technique (Hammersley and Handscomb 1964) is to concentrate the distribution of the sampling points in the region of greatest importance, i.e., the part of the sample space which mainly contributes to the failure probability, instead of spreading out the samples evenly over the entire range of definition of the involved parameters. In other words, the samples are simulated according to the importance sampling density function $h(\zeta)$ instead of the original joint probability density function $q(\zeta)$. Therefore, the integral of Eq. (6) can be estimated numerically as follows

$$\hat{p}_f = \frac{1}{N}\sum_{i=1}^{N} I_f(g(\zeta^{(i)}))\frac{q(\zeta^{(i)})}{h(\zeta^{(i)})} \tag{9}$$

where N is the number of simulations. It can be shown that the optimal importance sampling density is proportional to the original density $q(\zeta)$ within the failure domain $g(\zeta) \le 0$ and zero elsewhere. In this case the ratio $q(\zeta^{(i)})/h(\zeta^{(i)})$ assumes the constant value p_f, i.e., the variance reduces to zero and the exact result is obtained. However, since the importance sampling density function $h(\zeta)$ must integrate to one, p_f is simply the unknown normalizing constant. If this constant was known, then there would be no need for importance sampling. Although p_f is usually not known in advance, the variance of $\hat{p}_f$ can be essentially reduced if the importance sampling density function is chosen to be partially similar to the shape of the original probability density function. In general, however, the identification of the failure domain and the construction of a suitable importance sampling distribution is rather difficult. The first difficulty to overcome is the identification of the failure domain, which is straightforward only for fairly regular performance functions $g(\zeta^{(i)})$. In the simplest case, the performance depends linearly on ζ, which leads to a plain limit state surface on

which $g(\boldsymbol{\zeta}^{(i)}) = 0$, separating the safe from the failure domain. For this simplest case, the failure probability is known analytically to be

$$p_f = \Phi(-\beta), \quad \beta = g(\mathbf{0})/\|\nabla(g(\mathbf{0}))\| \tag{10}$$

where $\nabla(\cdot)$ denotes the gradient and β - which is called the reliability index - has the geometric interpretation of the shortest distance from the plain limit state surface to the origin in standard normal space.

Since for the above case, the failure probability is known right away, there is of course no need for importance sampling. Importance sampling is applied in cases where slight non-linearities are present. For linear static problems, non-linearities occur whenever the loading is not jointly Gaussian distributed, or if there are uncertainties in the properties of the structure, i.e., if the system matrices are uncertain. Non-plain limit state surfaces arise also if several performance criteria are formulated simultaneously, as it is the case for series and parallel systems. Non-linear structural analysis leads to non-linear performance functions and limit state surfaces. For non-linear structures the applicability of Importance Sampling depends mainly on the degree of non-linearity and on the dimension of the problem, i.e., on the number of independent random variables (Au and Beck 2003). To identify the important failure domain, algorithms have been developed to compute the so-called design point, which is the point on the limit state surface with the highest probability density. Geometrically, it is the point $\boldsymbol{\zeta}^*$, satisfying $g(\boldsymbol{\zeta}^*) = 0$ and where $\|\boldsymbol{\zeta}\|$ is a minimum (Schuëller and Stix 1987, Schuëller *et al.* 2004). Besides the necessity to cover all important failure domains, the sampling density must be designed such that it can be readily used for sampling. Because of this requirement, Gaussian and exponential distributed densities are most often used.

In dynamics, the important failure domain is not concentrated around one single failure point. This is most evident by considering deterministic linear systems, subjected to Gaussian excitation, for which the critical domain is well known for any time t (Au and Beck 2001a). At each considered instant t_k, the design point is analytically known and the failure domain is separated by a plain limit state surface. The importance sampling density is then defined as the union of all failure domains. Solving the reliability problem in terms of the first excursion probability, where the crossing of the response of one of the thresholds must be counted only once, the correlation of crossings is considered. Alternatively, the "average probability flow" concept can be used (Pradlwarter and Schuëller 2004) to determine the first excursion probability. The considered deterministic structure corresponds to the case in Eq. (2) where only the excitation $\mathbf{f}(t;\boldsymbol{\xi})$ has random properties, i.e., the random structural properties - collected in the vector $\boldsymbol{\theta}$ - are neglected.

In more realistic situations, the uncertainty in the structural properties is also accounted for and the structural properties are viewed as functions of the vector $\boldsymbol{\theta}$ as expressed in Eq. (2). It is proposed to evaluate the reliability integral in two steps. In the first step, the failure probability $p_f(\boldsymbol{\theta})$ conditioned on $\boldsymbol{\theta}$ is determined, i.e., the structural properties are fixed

$$p_f(\boldsymbol{\theta}) = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} I_f(g(\boldsymbol{\theta},\boldsymbol{\xi}))q(\boldsymbol{\xi})\, d\xi_1 \dots d\xi_m \tag{11}$$

The unconditional total failure probability is obtained by integrating over the space of uncertain structural parameters.

$$p_f = \int_{-\infty}^{\infty} \dots \int_{-\infty}^{\infty} p_f(\boldsymbol{\theta})q(\boldsymbol{\theta})\, d\theta_1 \dots d\theta_k \tag{12}$$

Clearly, the design point in the space of the structural random variables, $\boldsymbol{\theta}^*$, with the largest contribution to the total failure probability, is the one which satisfies

$$p_f(\boldsymbol{\theta}^*)q(\boldsymbol{\theta}^*) \geq p_f(\boldsymbol{\theta})q(\boldsymbol{\theta}) \tag{13}$$

An iterative procedure to estimate the structural design point $\boldsymbol{\theta}^*$ has been developed recently (Pradlwarter and Schuëller 2008, submitted), and provides the basis for applying standard Importance Sampling to compute the total unconditional first excursion probability. It is proposed to use a shifted standard normal sampling density $h(\boldsymbol{\theta})$, with the design point $\boldsymbol{\theta}^*$ as its center

$$h(\boldsymbol{\theta}) = \prod_{l=1}^{k} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{1}{2}(\theta_l - \theta_l^*)^2\right) \tag{14}$$

Importance Sampling then provides the estimate

$$\bar{p}_f = \frac{1}{N}\sum_{i=1}^{N} \tilde{p}_f(\boldsymbol{\theta}^{(i)}); \quad \tilde{p}_f(\boldsymbol{\theta}^{(i)}) = \frac{q(\boldsymbol{\theta}^{(i)})}{h(\boldsymbol{\theta}^{(i)})} p_f(\boldsymbol{\theta}^{(i)}) \tag{15}$$

$$\frac{q(\boldsymbol{\theta}^{(i)})}{h(\boldsymbol{\theta}^{(i)})} = \prod_{l=1}^{k} \exp\left[\frac{1}{2}(\theta_l^*)^2 - \theta_l^{(i)}\theta_l^*\right] \tag{16}$$

with the approximate variance of $\bar{p}_f$

$$\mathrm{Var}[\bar{p}_f] = \frac{1}{N(N-1)}\sum_{i=1}^{N} (\tilde{p}_f(\boldsymbol{\theta}^{(i)}) - \bar{p}_f)^2 \tag{17}$$

The required sample size N to obtain a robust estimate depends on the variability of $\tilde{p}_f(\boldsymbol{\theta}^{(i)})$, which in turn depends on the quality of the design point estimate and on the variability of the structural input parameters. If the structural parameters have a coefficient of variation not larger than 10%, a sample size N of a few hundred will be sufficient to obtain a robust estimate with a coefficient of variation of less than 20%.

4.3 Line sampling

In this section, a simple, novel procedure based on Line Sampling (LS), is shown. Line sampling (Koutsourelakis *et al*. 2004, Schuëller *et al.* 2004) is particularly efficient if a so called important direction can be computed, which points towards the portion of the failure domain that is closest to the origin (see Fig. 1). Most importantly, it is not required that this vector $\boldsymbol{\alpha}$, shown in Fig. 1, points exactly towards the center of the domain, nor are any assumptions made regarding the shape of the limit state function $g(\boldsymbol{\theta}, \boldsymbol{\xi}) = 0$ (Koutsourelakis *et al.* 2004). LS is robust, since unbiased reliability estimates are obtained irrespectively of the important direction $\boldsymbol{\alpha}$, and any deviation from the optimal direction merely increases the variance of the estimate. Because of this property, LS is also applicable for slightly non-linear structures where local gradients information is used to approximate the important direction.

For structures with assumed fixed properties $\boldsymbol{\theta}$ and excited by a Gaussian distributed loading process, the critical response has a Karhunen-Loève representation of the form

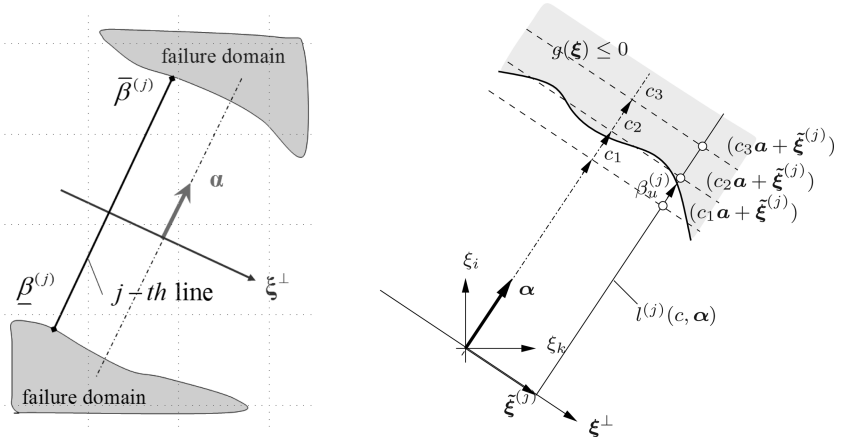

Fig. 1 Limit state and important direction α for Line Sampling

$$y(t;\theta,\xi) = y^{(0)}(t;\theta) + \sum_{j=1}^{m} \xi_j y^{(j)}(t;\theta) \tag{18}$$

The important direction is well known for all times t to be proportional to the response $y^{(j)}(t;\theta)$. Let $\gamma(t;\theta)$ be the standard deviation of the response $y(t;\theta)$

$$\gamma^2(t;\theta) = \mathrm{Var}[y(t;\theta)] = \sum_{j=1}^{m} [y^{(j)}(t;\theta)]^2 \tag{19}$$

Then, the unit vector $\alpha(t;\theta)$ pointing towards the important direction is specified by the components

$$\alpha_j(t;\theta) = \frac{y^{(j)}(t;\theta)}{\gamma(t;\theta)} \tag{20}$$

It is also important to note that the important direction is a continuous function of the time t. Moreover, the linearity implies that any excitation $\xi^{\perp}$ which is orthogonal to $\alpha(t;\theta)$, will have a zero response at time t. Fig. 1 shows a schematic representation of LS, where the vector α denotes the important direction and the region in which $g(\xi;\theta) \le 0$ corresponds to the failure domain. Each point ξ in the standard normal space is decomposed into the one dimensional space $c\alpha$ and the $(m-1)$ dimensional subspace $\xi^{\perp}(\alpha)$ orthogonal to the direction α

$$\xi = c\alpha(\theta) + \xi^{\perp}(\alpha) \tag{21}$$

In the following it is assumed that the direction α is normalized to unit length, i.e., $\|\alpha\| = 1$, and that $\{\tilde{\xi}^{(i)}\}_{i=1}^{N}$ are independent sample points of the subspace $\xi^{\perp}(\alpha)$ drawn by direct Monte Carlo simulation. The random sample $\tilde{\xi}^{(i)}$ is generated by simulating first m independent standard normally distributed components of a vector $\bar{\xi}^{(i)}$ and then extracting the components which point in the direction of α

$$\tilde{\xi}^{(i)} = \bar{\xi}^{(i)} - (\alpha^T \bar{\xi}^{(i)})\alpha \tag{22}$$

For each independent random realization $\tilde{\xi}^{(i)}$, the probability of failure conditioned on the fixed value $\tilde{\xi}^{(i)}$ can be determined

$$p_f^{(i)}(\theta) = \int_{-\infty}^{+\infty} \mathcal{I}_f(\xi^{(i)}(c)) \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{c^2}{2}\right) dc = \Phi(\underline{\beta}^{(i)}(\theta)) + \Phi(-\overline{\beta}^{(i)}(\theta)) \tag{23}$$

where $\mathcal{I}_f(\cdot)$ is an indicator function of failure and $p_f^{(i)}(\theta)$ denotes the failure probability conditioned on the i-th randomly selected line $\xi^{(i)}(c) = \tilde{\xi}^{(i)} + c\alpha$ and on the structural parameter set θ. The safe domain lies within the bounds $[\underline{\beta}^{(i)}(\theta), \overline{\beta}^{(i)}(\theta)]$. In dynamics, when estimating the conditional first excursion probability, not a single direction, but several directions are used, since the important direction changes with respect to time. The contribution from each direction can be simply added, while ensuring that no overlapping of failure points is accepted. The important directions are determined by a sequential orthogonalization algorithm of the Karhunen-Loève representation. The independent estimates $p_f^{(i)}(\theta)$ of the failure probability allow to compute an unbiased estimate of the conditional failure probability and of its variance

$$\hat{p}_f(\theta) = \frac{1}{N} \sum_{i=1}^{N} p_f^{(i)}(\theta), \quad \mathrm{Var}[\hat{p}_f(\theta)] = \frac{1}{N(N-1)} \sum_{i=1}^{N} (p_f^{(i)}(\theta) - \hat{p}_f(\theta))^2 \tag{24}$$

which can be used for deriving further confidence intervals.

An alternative approach is based on a novel Line Sampling procedure where the lines are drawn in the subspace of the structural parameters θ. This procedure requires the knowledge of the structural design point θ^*. In order to identify this point, composed by the standard normal variables $\{\xi, \theta\}$, the point $\{\xi^*, \theta^*\}$ at time t^*-where either $\overline{\beta}(t, \theta)$ or $\underline{\beta}(t, \theta)$ assumes the smallest reliability index - is considered. Since the vectors ξ and θ are independent and therefore orthognal in the standard normal space, this distance is specified by the rule of Pythagoras

$$\beta^2(\xi, \theta) = \|\xi\|^2 + \|\theta\|^2 \tag{25}$$

Suppose $\underline{b}$ and $\overline{b}$ represent the lower and upper threshold, respectively, the threshold is reached for

$$\|\xi\|^2 = \frac{d^2}{\gamma^2(t^*; \theta)}, \quad d = \min[\overline{b} - y^{(0)}(t^*; \theta), -\underline{b} + y^{(0)}(t^*; \theta)] \tag{26}$$

The failure point (θ^*, ξ^*) is defined, in geometric terms, as the closest point to the origin in the standard normal space, and hence can be derived by imposing the condition

$$\frac{\partial \beta^2(\xi, \theta)}{\partial \theta_k} = 0, \quad k = 1, 2, \ldots, K \tag{27}$$

This leads to the so called design point in the structural parameter space

$$\theta_k^* = \frac{b^2}{\gamma^3(t^*, \theta^*)} \cdot \frac{\partial \gamma(t^*, \theta^*)}{\partial \theta_k} \tag{28}$$

The above relation can be solved accurately only in an iterative manner, where it converges fast towards the exact solution for $s > S$ with $\varepsilon^{(S)}$ smaller than the tolerance

$$\theta_k^{*(s+1)} = (1 - w)\, \theta_k^{*(s)} + w \frac{b^2}{\gamma^3(t^*, \theta_k^{*(s)})} \cdot \frac{\partial \gamma(t^*, \theta_k^{*(s)})}{\partial \theta_k} \tag{29}$$

$$\varepsilon^{(s+1)} = \|\theta^{*(s+1)} - \theta^{*(s)}\| / \|\theta^{*(s+1)}\| \tag{30}$$

where $w = 0.5$ leads to stable and fast convergence. This leads to robust results, even for large uncertainties of the structural parameters. The procedure can be outlined as follows:

1. Determine the "design point" θ^* with acceptable accuracy and compute

$$\beta_S = \|\theta^*\|, \quad \alpha_S = \frac{\theta^*}{\beta_S} \tag{31}$$

   where the index S denotes the subspace of the structural uncertainties.
2. Generate samples $\{\theta^{\perp(i)}\}_{i=1}^{N}$ using direct MCS in the subspace of θ, which are perpendicular to the vector α_S.
3. Compute for each parallel line five discrete value $\{c_l\}_{l=1}^{5}$ of c

$$\theta^{(i)}(c_l) = \theta^{\perp(i)} + c_l \cdot \alpha_S \tag{32}$$

$$c_l = \beta_S + (l-3)\Delta, \quad (\Delta \approx 0.6) \quad l = 1, 2, \ldots, 5 \tag{33}$$

   And estimate for each point $\theta^{(i)}(c_l)$ the associated failure probability $p_f^{(i)}(c_l)$ corresponding to a deterministic system.
4. Estimate the conditional failure probability $p_f^{(i)}$ along each line

$$p_f^{(i)} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-c^2/2} p_f^{(i)}(c)\, dc \tag{34}$$

   This estimation will be discussed subsequently in more detail.
5. Estimate the mean and variance of the failure probability $p_f(t)$ according to Eq. (24).

It should be noted that contributions to the conditional failure probability $p_f^{(i)}(c)$ come from all positions c along the line in Eq. (34), since the position along each line defines uniquely the properties of a structure subjected to random excitation.

Since $p_f^{(i)}(c)$ is a strictly positive quantity, it is proposed to represent $p_f^{(i)}(c)$ as the exponential of a linear or quadratic polynomial

$$p_f^{(i)}(c) = \exp[a_0 + a_1 c + a_2 c^2/2] \tag{35}$$

where the coefficients are obtained by solving a least square problem,

$$a_0 + a_1 c_l + a_2 c_l^2/2 = \ln[p_f^{(i)}(c_l)], \quad l = 1, 2, \ldots, 5 \tag{36}$$

The advantage of this approximation is that the infinite integral in Eq. (34) can be represented in closed form for $a_2 < 1$

$$p_f^{(i)} = \frac{1}{\sqrt{1-a_2}} \exp\left(a_0 + \frac{a_1^2}{2(1-a_2)}\right) \tag{37}$$

A linear approximation of $\ln[p_f^{(i)}(c)]$ in the least square solution should be applied ($a_2 = 0$), whenever $a_2 > 0.4$, which might be due to random fluctuation of the estimates $p_f^{(i)}(c_l)$. Experience showed that the results using either a linear or quadratic approximation are quite similar. However,

a linear approximation is less sensitive to random fluctuations of the estimate $p_f^{(i)}(c_l)$. Hence the integration over all points along the line is efficiently approximated, requiring only three to five FE runs per line.

4.4 Subset simulation

A procedure called *Subset Simulation* was proposed by Au and Beck (2001b) for structural reliability problems. It was further developed and investigated in the context of seismic risk analysis (Au and Beck 2003). Subset Simulation overcomes the inefficiency of Direct Monte Carlo in estimating small probabilities, by expressing p_f as a product of larger, conditional probabilities. This is achieved by defining a decreasing sequence of events (subsets) $\{F_i\}_{i=1}^{m}$ such that $F_m = F$ and $F_1 \supset F_2 \supset \ldots \supset F_m = F$. Due to the latter property, $\cup_{i=1}^{k} F_i = F_k \quad \forall k \le m$. As a result, the probability of failure p_f can be written as

$$p_f = P(F_m) = P(F_1)\prod_{i=1}^{m-1} P(F_{i+1}|F_i) \tag{38}$$

It is obvious that by an appropriate selection of $\{F_i\}_{i=1}^{m-1}$, the probabilities $P(F_1)$ and $P(F_{i+1}|F_i)$ $\forall i \ge 1$ can be made sufficiently large so that their estimation can be performed efficiently by Direct Monte Carlo estimators. Hence, the original problem is broken up into a series of m intermediate problems; for each one of these a solution can be obtained with a relatively small number of simulations. Distinct estimators $\hat{p}_1$ and $\hat{p}_{i+1}$ are used for each of the factors $P(F_1)$ and $P(F_{i+1})$, $\forall i \ge 1$ appearing in (38).

Suppose that the vector ζ in standard normal space represents all independent random quantities, and that the performance function $g(\zeta)$ is specified such that $g(\zeta) > 0$ denotes the safe domain and $g(\zeta) \le 0$ the failure domain. Moreover, it is implied that $g(\zeta)$ is continuous over all ζ.

In the first step, direct MCS is applied to generate N independent samples $\{\zeta^{(j)}\}_{j=1}^{N}$ for which a deterministic (FE) analysis leads to the associated performance function values $\{g^{(j)} = g(\zeta^{(j)})\}_{j=1}^{N}$. The performance function values are then ordered such that

$$g^{(j_1)} \le g^{(j_2)} \le \ldots \le g^{(j_N)} \tag{39}$$

Then the samples

$$\{\zeta^{(j_k)}\}_{k=1}^{pN} \in F_1 \tag{40}$$

are regarded as independent samples in the subset F_1. The value p is usually chosen to be in the range of [0.1, 0.2], which implies $P(F_1) = P[\zeta \in F_1] = p$.

In the following it is assumed that $\{\zeta^{(j_k)}\}_{k=1}^{pN} \in F_i$ are the samples belonging to the subset F_i; the sample size is increased from the number pN to N, i.e., $(1-p)N$ new samples are generated. These new samples must also belong to the domain F_i, which is specified by the condition

$$g(\zeta) < b_i = (g_{i-1}^{(pN)} + g_{i-1}^{(pN+1)})/2 \tag{41}$$

where $g_{i-1}^{(k)}$ denotes the increasingly ordered samples in the subset F_{i-1}. Moreover, the samples should be distributed according to the conditional probability density function (PDF) $q(\zeta|F_i)$ in this sub-domain F_i

$$q(\zeta|F_i) = q(\zeta)I_{F_i}(\zeta)/P(F_i) \tag{42}$$

where $q(\zeta)$ denotes the probability density function and $I_{F_i}(\zeta)$ denotes the indicator function which is one in case $\zeta \in F_i$ and zero otherwise. To obtain such additional samples, Markov Chain MC sampling (MCMCS) is applied, by using either the original Metropolis (Metropolis and Ulam 1949) or the Metropolis-Hastings (Hastings 1970) algorithm in a modified form. Both algorithms produce a random chain of samples $\{\zeta^{(j_k)}\}_{k=0}^{m}$, starting from the sample $\zeta^{(j_0)}$ and proceeding from $\zeta^{(j_k)}$ to $\zeta^{(j_{k+1})}$

$$\zeta^{(j_k)} \xrightarrow{MCMCS} \zeta^{(j_{k+1})} \tag{43}$$

based on a proposal density p^* for each component of the vector $\zeta^{(j_k)} = (\zeta_1^{(j_k)}, \zeta_2^{(j_k)}, \ldots, \zeta_n^{(j_k)})$ of the n-dimensional vector. Before MCMCS is applied for each subset F_i, a sample size of pN is available. For each of these samples a chain of $1/p$ samples is generated by MCMCS, which increases the sample size in the subset F_i again to the number N. Since each newly generated sample depends on a previous sample, the samples are not independent and the correlation depends significantly on the standard deviation of the proposal density. When estimating the conditional probability $P(F_{i+1}|F_i)$ these samples are regarded nonetheless as independent.

After completion of the sampling in the subset F_i, the next subset F_{i+1} is approximated. Among the available N random realizations, a portion of pN samples with the smallest values of the performance function is associated to the next subset F_{i+1}, following the reasoning in the first step. The whole procedure is repeated $(m - 1)$ times, until a sufficiently large portion ($> p$) yields negative performance function values $g(\zeta) < 0$, which correspond to samples in the failure domain.

Based on the original Subset Simulation procedure, various modifications have been suggest for an improvement of the performance and efficiency. In a recent publication (Katafygiotis and Zuev 2007), Subset Simulation is discussed further from a broader perspective where Subset Simulation is seen as a special case of Linked Importance Sampling. Moreover, in (Katafygiotis and Zuev 2008) the challenges related with high dimensional reliability problems, addressed in (Au and Beck 2003, Schuëller *et al.* 2004), are described in detail.

Other variants of Subset Simulation are presented in (Ching *et al.* 2005) in which splitting is proposed for increasing the efficiency of the algorithm. Splitting is applicable in case of white and filtered white noise excitation. It reduces the computational effort, but does not explore the failure region as well as the original version. Moreover, the original Subset Simulation is applicable for systems with uncertain structural properties (modeled by random parameter θ) while splitting can not be used for this case. To overcome this restriction Subset Simulation with a hybrid strategy has been developed (Ching *et al.* 2005). It explores the failure region prior to first-passage times using an MCMC method while taking advantage of causality by using trajectory splitting after the first-passage time.

The most appealing feature of subset simulation is its wide applicability, even for difficult non-linear problems. The procedure is applicable whenever the performance function is continuous, regardless of whether the problem is non-linear or very complex. Its wide applicability approaches that of direct MCS, since the restriction that the performance function must be continuous, is satisfied in almost all applications of practical interest. Although the efficiency is improved significantly over direct MCS, the procedure still requires at least several thousand performance evaluations (FE-runs) to obtain an estimate for failure probabilities in the range of $p_f < 10^{-5}$. To

asses the variance, at least 10-20 independent results by subset simulation would be required, which implies several hundred thousands performance evaluations to obtain a reliable estimate. In other words, the general applicability of SS, comes at a price if comparing its efficiency with that of highly specialized methods, which are e.g., applicable only for linear structures.

4.5 Auxiliary domain method

The Auxiliary Domain method, proposed in (Katafygiotis *et al.* 2007), can be used for cases where the failure domain can be covered approximately by an associated linear system. Hence, it is expected that this approach has advantages over the original subset simulation approach for cases where the non-linearities are moderate and weak.

Suppose the reliability evaluation of a non-linear deterministic structural system is carried out in standard normal space ξ and the structure is excited by dynamic forces representable by a Gaussian stochastic process. Let F be the failure domain which is specified by nonlinear relations $g(\xi \in F) \le 0$. Suppose further that an Auxiliary Domain F_A, defined as the union $\cup_{i=1}^{n} F_A(i)$ of failure domains specified by linear limit state functions $g_A^{(i)}(\xi) \le 0$, can be established. The Auxiliary Domain F_A is further assumed to have a significant overlap with the non-linear failure domain F, i.e., $P(F|F_A)/P(F) \ge 0.5$. Then, the probability of failure can be estimated by

$$p_f = P(F) = P(F_A) \times \frac{P(F)}{P(F_I)} \times \frac{P(F_I)}{P(F_A)} \tag{44}$$

with $F_I = F \cap F_A$. The main advantage of the Auxiliary Domain method is the fact that $P(F_A)$ can be determined very efficiently when compared to subset simulation as discussed in the previous section. The second term $P(F)/P(F_I)$ can be estimated by simulating N points in F distributed according $q(\xi|g(\xi) \le 0)$ by the ratio N/N_I, where N_I denotes the number of these N points belonging also to F_A. Similarly, the third term $P(F_I)/P(F_A)$ can be estimated by simulating N' points in F_A distributed according to $q(\xi|g_A^{(i)}(\xi) \le 0)$ by the ratio N_I'/N', where N_I' denotes the number of these N' points belonging to the non-linear failure domain F. One of the key steps is the generation of N samples points which should be distributed according to $q(\xi|g(\xi) \le 0)$. Markov chain Monte Carlo sampling is applied for this purpose. The procedures require some initial failure points, which might be even unlikely events ($||\xi||$ large), which are used as starting points for the Markov chains. It can be shown, that the Markov chain converges - after some burn in samples - to samples drawn from $q(\xi|F)$ in case the failure domain is connected. The generation of independent samples in F_A, needed to estimate the ratio $P(F_I)/P(F_A)$, can be carried out in a straight forward manner.

5. Numerical example

A twelve story building, made of reinforced concrete, is considered. The building is equipped with an additional cellar floor. The FE-model has 4046 nodes and 5972 elements consisting of shells and 3-D beams for modeling the girders, resulting in 24.276 degrees of freedom. The ground motion is in this case modelled by filtered white noise using a fourth order linear filter for the motion in one horizontal direction. The established covariance matrix is represented with sufficient accuracy by using $m = 80$ deterministic KL-terms. Fig. 2 shows the floor plan and a perspective view of the building showing the first mode shape of the displacements. All structural uncertainties

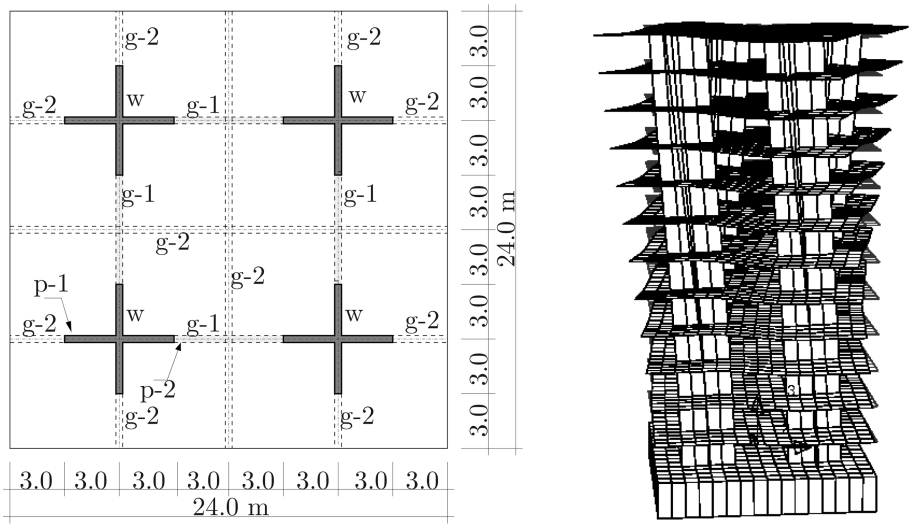

Fig. 2 Left: Floor plan of twelve story building supported by concrete walls (w) and girders of type g-1 and type g-2. Critical strains are considered at positions p-1 and p-2. Right: First mode shape of displacements

are represented as a function of 244 independent standard normal variables. These random variables model the stiffness of confined reinforced concrete via the Young's modulus which is in the range of 2.8-3.2 $\times 10^{10}$ Pa and coefficient of variation in the range of 15-22%. Under dynamic earthquake loading, two reinforced structural components are endangered to exceed the acceptable limit. The strain in vertical direction of the walls in the basement, forming a cross and indicated by p-1 in Fig. 2, might be critical, since these walls support the weight of the structure and are in addition highly strained by bending due to the horizontal earthquake ground motion. The other critical part are the girders, denoted as g-1 in Fig. 2, which connect the walls forming a cross. These girders transmit large shear forces from one group of walls to the counterpart, and contribute essentially to the bending stiffness of the twelve story building. The critical part is the curvature of the girder at the connection to the walls. The position is indicated by p-2 in Fig. 2. Allowing a vertical strain of 0.0016 for the walls without having to expect serious damage and a maximum acceptable curvature of 0.004 for being in the linear range, the curvatures in the girder are more likely to be exceeded. Hence, the reliability analysis in the form of a first excursion problem will focus on the girders not exceeding the curvature of 0.004. The girder in the fifth floor showed the largest standard deviation of the curvature. Considering the curvature due to the static dead load of –0.000041, leads to the lower threshold $\underline{b} = -0.004 + 0.000041 = -0.003959$ and the upper threshold $\overline{b} = 0.004 + 0.000041 = 0.004041$, respectively.

Since in this example the structural system is large in terms of degrees of freedom and linear elastic, Subset Simulation and modification of it are not recommended because more efficient methods exist for linear structures. Subset Simulation is recommended as efficient alternative to Direct Monte Carlo Simulation when facing non-linear reliability problems. Its efficiency is well documented in the recent benchmark study (Au *et al.* 2007) dealing mainly with non-linear MDOF-systems.

Fig. 3 shows a histogram of 1600 independent conditional first excursion probabilities. These

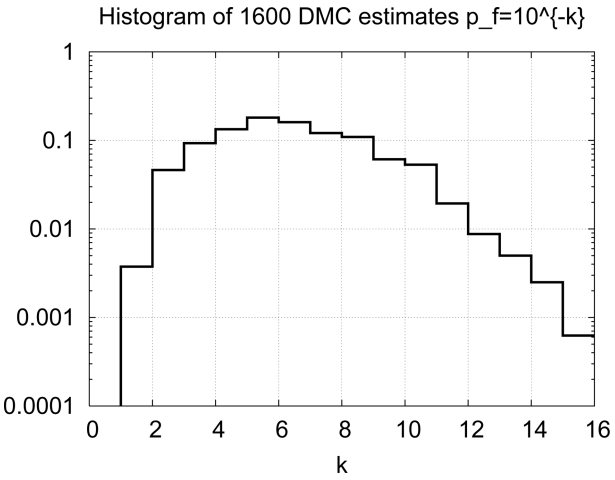

Fig. 3 Conditional failure probabilities by direct Monte Carlo sampling using a sample size of $N = 1600$ lead to the mean $\hat{p}_f$ = 0.00023 with a standard deviation $\sigma_{\hat{p}_f}$ = 0.000034

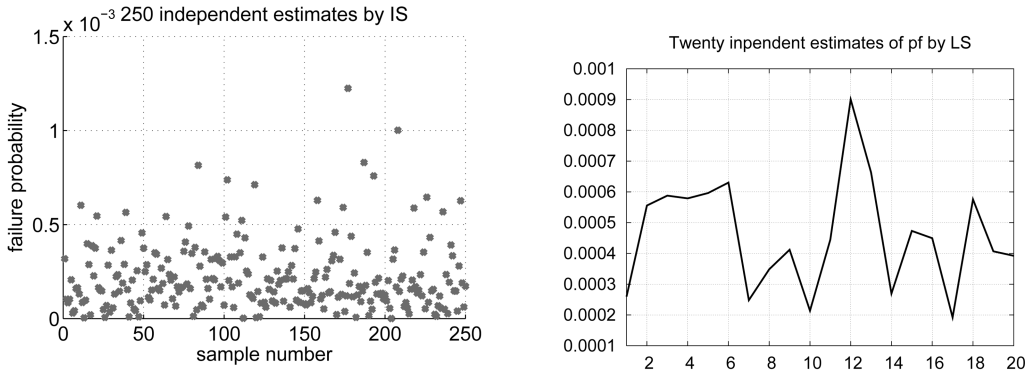

Fig. 4 Left: Conditional failure probabilities by Importance Sampling using a sample size of $N = 250$ lead to the mean $\hat{p}_f$ = 0.00023 with a standard deviation $\sigma_{\hat{p}_f}$ = 0.000012. Right: Twenty independent estimates of the total first excursion probability of the uncertain structure lead to the mean $\hat{p}_f$ = 0.00046 with a standard deviation $\sigma_{\hat{p}_f}$ = 0.000040

estimates are obtained by direct Monte Carlo sampling in the parameter space of uncertain structural parameters, which required 1600 FE analyses and 1600 estimations of the first excursion probability. It can be seen that the variability in the structural parameters has a tremendous influence on the first excursion probability, which lies in the following interval, $10^{-1} < p_f(\boldsymbol{\theta}^{(j)}) < 10^{-16}$. The left part of Fig. 4 shows the results of Importance Sampling using for the sampling density a shifted standard normal density centered at the structural design point and required 250 FE-runs and conditional first excursion evaluations. The right part of Fig. 4 shows the results of Line Sampling in the uncertain parameter space for 20 independent lines. Each line required the evaluation of five conditional first excursion probabilities, which results in 100 FE analyses and 100 estimations of the first excursion probability. This demonstrates clearly the gain in efficiency in case the structural design point can be determined.

Although the conditional first excursion probabilities have been determined accurately by Line

Sampling, its estimation required less than 5% of the total CPU time. Hence the response evaluation by Finite element analysis resulting in deterministic Karhunen-Loève functions of the critical response is the decisive factor for the efficiency.

6. Conclusions

The presented developments and the obtained results lead to the following conclusions:

1. Monte Carlo methods provide a versatile and efficient basis for uncertainty propagation to the structural response and its quantification in terms of variability.
2. Advanced Monte Carlo procedures are needed to estimate efficiently the reliability.
3. Stochastic Gaussian excitation represented by the Karhunen-Loève expansion provides an efficient basis for calculating the stochastic response of linear deterministic structural systems.
4. The proposed reliability evaluation of systems with structural uncertainties is based on the integration of the conditional reliability over the uncertain parameter space. In this way, fast and efficient solution schemes of the first excursion probability for deterministic structures subjected to stochastic excitation can be exploited.
5. Importance Sampling using the estimated structural design point and an adapted Line Sampling procedures involves only approximately one hundred conditional reliability evaluations for a robust estimation of the total unconditional reliability.

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