Structural Engineering and Mechanics, Vol. 31, No. 6 (2009) 663-678
DOI: http://dx.doi.org/10.12989/sem.2009.31.6.663

Exact solution for forced torsional vibration of finite piezoelectric hollow cylinder

H. M. Wang†

Department of Mechanics, Zhejiang University, Hangzhou 310027, P. R. China

C. B. Liu‡ and H. J. Ding‡†

Department of Civil Engineering, Zhejiang University, Hangzhou 310027, P. R. China

(Received October 10, 2007, Accepted March 13, 2009)

Abstract. An exact solution is obtained for forced torsional vibration of a finite class 622 piezoelectric hollow cylinder with free-free ends subjected to dynamic shearing stress and time dependent electric potential at both internal and external surfaces. The solution is first expanded in axial direction with trigonometric series and the governing equations for the new variables about radial coordinate r and time t are derived with the aid of Fourier series expansion technique. By means of the superposition method and the separation of variables technique, the solution for torsional vibration is finally obtained. Natural frequencies and the transient torsional responses for finite class 622 piezoelectric hollow cylinder with free-free ends are computed and illustrated.

Keywords: exact solution; torsional vibration; finite hollow cylinder; piezoelectric.

1. Introduction

Torsional vibrations often occur in rotating machinery systems such as turbogenerators, compressors and motors etc. High torsional vibration will result in severe deformation and shaft fatigue failure. So to know the torsion vibration characteristics exactly is very important for the sake of the system's safety and reliability.

There are numerous works on the subject of torsional vibration. The torsional vibration of circular shafts can be cited in the book (Timoshenko *et al.* 1974). Mitra and Mukherji (1972) investigated the torsional vibration of a finite circular cylinder of non-homogeneous material subjected to a particular type of twist on one of its ends. Xie and liu (1998) studied the transient torsional wave propagation in a transversely isotropic tube. Wang *et al.* (2003) obtained the elastodynamic solution of finite orthotropic hollow cylinder under torsion impact. Singh *et al.* (2006) investigated the torsional vibration of functionally graded finite solid cylinder.

Some works have also been carried out on torsional vibrations for piezoelectric cylinders.

† Associate Professor, Corresponding author, E-mail: wanghuiming@zju.edu.cn
‡ Engineer, E-mail: lcb@zju.edu.cn
‡† Professor, E-mail: dinghj@zju.edu.cn

Srinivasamoorthy and Anandam (1980) investigated the torsional wave propagation in an infinite crystal class 622 piezoelectric cylinder. Lin (1996) studied the resonance frequencies of tabgentially polarized piezoelectric torsional tube. Paul and Sarma (1977) obtained the transient torsional solution subjected to prescribed shearing stress on the internal surface by applying Green's function technique. While with the limitation of computation level at that time, no numerical results for torsional responses are performed.

In this study, an exact solution is obtained for torsional vibration of a finite class 622 piezoelectric hollow cylinder with free-free ends subjected to dynamic shearing stresses and time dependent electric potentials at both internal and external surfaces. The solution procedure is operated thoroughly in time domain and the obtained solution is suitable for the finite piezoelectric hollow cylinder subjected to dynamic mechanical and electric loads with arbitrary time variation form.

2. Basic formulations

Consider a finite piezoelectric hollow cylinder. Its length, inner and outer radii are denoted as L, a and b, respectively. In the following, we refer the problem in cylindrical coordinate system (r,θ,z). The z-axis lies along the rotation axis of hollow cylinder and the ends of the hollow cylinder lies along the planes $z=0$ and $z=L$, as shown in Fig. 1.

For torsional vibration problem, both the components of displacement and electric potential are independent of θ and especially we have

$$u_r = u_z = 0, \quad u_\theta = u_\theta(r,z,t), \quad \Phi = \Phi(r,z,t) \tag{1}$$

Then the constitutive relations of class 622, axially polarized piezoelectric media are (Srinivasamoorthy and Anandam 1980, Paul and Sarma 1977)

$$\begin{aligned} \tau_{\theta z} &= c_{44}\frac{\partial u_\theta}{\partial z} + e_{14}\frac{\partial \Phi}{\partial r}, \quad \tau_{r\theta} = c_{66}\left(\frac{\partial u_\theta}{\partial r} - \frac{u_\theta}{r}\right) \\ D_{rr} &= e_{14}\frac{\partial u_\theta}{\partial z} - \varepsilon_{11}\frac{\partial \Phi}{\partial r}, \quad D_{zz} = -\varepsilon_{33}\frac{\partial \Phi}{\partial z} \end{aligned} \tag{2}$$

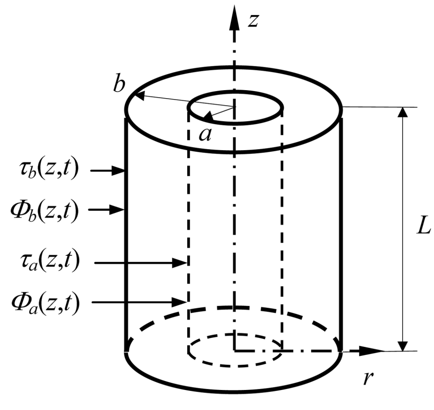

Fig. 1 Model of the finite piezoelectric hollow cylinder

where $\tau_{\theta z}$ and $\tau_{r\theta}$ are shearing stresses, D_{rr} and D_{zz} are electric displacements. c_{44} and c_{66} are elastic, e_{14} is piezoelectric and ε_{11} and ε_{33} are dielectric constants. In the absence of body force and free charge density, the equation of motion and the charge equation of electrostatics are

$$c_{66}\left(\frac{\partial^2 u_\theta}{\partial r^2}+\frac{1}{r}\frac{\partial u_\theta}{\partial r}-\frac{u_\theta}{r^2}\right)+c_{44}\frac{\partial^2 u_\theta}{\partial z^2}+e_{14}\frac{\partial^2\Phi}{\partial r\partial z} = \rho\frac{\partial^2 u_\theta}{\partial t^2} \tag{3a}$$

$$e_{14}\left(\frac{\partial^2 u_\theta}{\partial r\partial z}+\frac{1}{r}\frac{\partial u_\theta}{\partial z}-\frac{u_\theta}{r^2}\right)-\varepsilon_{11}\left(\frac{\partial^2\Phi}{\partial r^2}+\frac{1}{r}\frac{\partial\Phi}{\partial r}\right)-\varepsilon_{33}\frac{\partial^2\Phi}{\partial z^2} = 0 \tag{3b}$$

where ρ is the mass density. The boundary conditions considered here are

$$\Phi(r,0,t)=0,\quad \Phi(r,L,t)=0 \tag{4a}$$

$$\begin{aligned}\Phi(a,z,t) &= \Phi_a(z,t) \quad \text{and} \quad \Phi_a(0,t)=\Phi_a(L,t)=0\\ \Phi(b,z,t) &= \Phi_b(z,t) \quad \text{and} \quad \Phi_b(0,t)=\Phi_b(L,t)=0\end{aligned} \tag{4b}$$

$$\tau_{\theta z}(r,0,t) = 0,\quad \tau_{\theta z}(r,L,t)=0 \tag{5a}$$

$$\tau_{r\theta}(a,z,t) = \tau_a(z,t),\quad \tau_{r\theta}(b,z,t) = \tau_b(z,t) \tag{5b}$$

For dynamic problem, the initial conditions should be completed as

$$u_\theta(r,z,0) = U_0(r,z),\quad \dot u_\theta(r,z,0) = V_0(r,z) \tag{6}$$

where a dot over a quantity denotes its partial derivative with respect to time.

For the sake of simplicity, the following non-dimensional forms are introduced as

$$\begin{gathered} c_1=\frac{c_{66}}{c_{44}},\quad e_1=\frac{e_{14}}{\sqrt{c_{44}\varepsilon_{33}}},\quad \varepsilon_1=\frac{\varepsilon_{11}}{\varepsilon_{33}},\quad \xi=\frac{r}{b},\quad \eta=\frac{z}{b},\quad l=\frac{L}{b},\quad s=\frac{a}{b}\\ v=\frac{u_\theta}{b},\quad \phi=\frac{\Phi}{\Phi_0},\quad D_r=\frac{D_{rr}}{D_0},\quad D_z=\frac{D_{zz}}{D_0},\quad \sigma_{\theta z}=\frac{\tau_{\theta z}}{c_{44}},\quad \sigma_{r\theta}=\frac{\tau_{r\theta}}{c_{44}}\\ \phi_a=\frac{\Phi_a}{\Phi_0},\quad \phi_b=\frac{\Phi_b}{\Phi_0},\quad p_a=\frac{\tau_a}{c_{44}},\quad p_b=\frac{\tau_b}{c_{44}},\quad \chi_0=\frac{U_0}{b},\quad \chi_1=\frac{V_0}{c_v}\\ \Phi_0=b\sqrt{\frac{c_{44}}{\varepsilon_{33}}},\quad D_0=\sqrt{c_{44}\varepsilon_{33}},\quad c_v=\sqrt{\frac{c_{44}}{\rho}},\quad \tau=\frac{c_v}{b}t \end{gathered} \tag{7}$$

Then Eqs. (2) and (3) can be rewritten as

$$\begin{gathered}\sigma_{\theta z}=\frac{\partial v}{\partial\eta}+e_1\frac{\partial\phi}{\partial\xi},\quad \sigma_{r\theta}=c_1\left(\frac{\partial v}{\partial\xi}-\frac{v}{\xi}\right)\\ D_r=e_1\frac{\partial v}{\partial\eta}-\varepsilon_1\frac{\partial\phi}{\partial\xi},\quad D_z=-\frac{\partial\phi}{\partial\eta}\end{gathered} \tag{8}$$

$$c_1\left(\frac{\partial^2 v}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial v}{\partial\xi}-\frac{v}{\xi^2}\right)+\frac{\partial^2 v}{\partial\eta^2}+e_1\frac{\partial^2\phi}{\partial\xi\partial\eta} = \frac{\partial^2 v}{\partial\tau^2} \tag{9a}$$

$$e_1\left(\frac{\partial^2 v}{\partial\xi\partial\eta}+\frac{1}{\xi}\frac{\partial v}{\partial\eta}\right)-\varepsilon_1\left(\frac{\partial^2\phi}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial\phi}{\partial\xi}\right)-\frac{\partial^2\phi}{\partial\eta^2}=0 \tag{9b}$$

The boundary conditions are rewritten as

$$\phi(\xi,0,\tau)=0,\quad \phi(\xi,l,\tau)=0 \tag{10a}$$

$$\begin{aligned}&\phi(s,\eta,\tau)=\phi_a(\eta,\tau)\quad\text{and}\quad \phi_a(0,\tau)=\phi_a(l,\tau)=0\\&\phi(1,\eta,\tau)=\phi_b(\eta,\tau)\quad\text{and}\quad \phi_b(0,\tau)=\phi_b(l,\tau)=0\end{aligned} \tag{10b}$$

$$\sigma_{\theta z}(\xi,0,\tau)=0,\quad\text{and}\quad \sigma_{\theta z}(\xi,l,\tau)=0 \tag{10c}$$

$$\sigma_{r\theta}(s,\eta,\tau)=p_a(\eta,\tau),\quad\text{and}\quad \sigma_{r\theta}(1,\eta,\tau)=p_b(\eta,\tau) \tag{10d}$$

The initial conditions are rewritten as

$$v(\xi,\eta,0)=\chi_0(\xi,\eta),\quad \dot v(\xi,\eta,0)=\chi_1(\xi,\eta) \tag{11}$$

In Eq. (11) and there after, a dot over a quantity denotes its partial derivative with respect to non-dimensional time.

3. Solving technique

3.1 Series solution form

The solution is firstly written in the form as

$$v(\xi,\eta,\tau)=\sum_{i=0}^{\infty}v_i(\xi,\eta)\cos(\alpha_i\eta),\quad \phi(\xi,\eta,\tau)=\sum_{i=0}^{\infty}\phi_i(\xi,\tau)\sin(\alpha_i\eta) \tag{12}$$

where

$$\alpha_i=i\pi/l \tag{13}$$

The substitution of Eq. (12) into Eq. (9a,b) derives

$$c_1\left(\frac{\partial^2 v_i}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial v_i}{\partial\xi}-\frac{v_i}{\xi^2}\right)-\alpha_i^2 v_i+e_1\alpha_i\frac{\partial\phi_i}{\partial\xi}=\frac{\partial^2 v_i}{\partial\tau^2} \tag{14a}$$

$$e_1\alpha_i\left(\frac{\partial v_i}{\partial\xi}+\frac{v_i}{\xi}\right)-\varepsilon_1\left(\frac{\partial^2\phi_i}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial\phi_i}{\partial\xi}\right)-\alpha_i^2\phi_i=0 \tag{14b}$$

By inspecting Eq. (12), we find that the boundary conditions at the ends, Eqs. (10a) and (10c), are satisfied automatically. By virtue of Eq. (12), the boundary conditions at internal and external surfaces Eqs. (10b) and (10d), can be rewritten as

$$\begin{aligned}\phi(s,\eta,\tau)&=\sum_{i=0}^{\infty}\phi_i(s,\tau)\sin(\alpha_i\eta)=\phi_a(\eta,\tau)\\ \phi(1,\eta,\tau)&=\sum_{i=0}^{\infty}\phi_i(1,\tau)\sin(\alpha_i\eta)=\phi_b(\eta,\tau)\end{aligned} \tag{15a}$$

$$\begin{aligned}\sigma_{r\theta}(s,\eta,\tau) &= c_1\sum_{i=0}^{\infty}\left[\frac{\partial v_i(\xi,\tau)}{\partial\xi}-\frac{v_i(\xi,\tau)}{\xi}\right]_{\xi=s}\cos(\alpha_i\eta)=p_a(\eta,\tau)\\ \sigma_{r\theta}(1,\eta,\tau) &= c_1\sum_{i=0}^{\infty}\left[\frac{\partial v_i(\xi,\tau)}{\partial\xi}-\frac{v_i(\xi,\tau)}{\xi}\right]_{\xi=1}\cos(\alpha_i\eta)=p_b(\eta,\tau)\end{aligned}\tag{15b}$$

By employing Fourier series expansion technique, we have

$$\phi_a(\eta,\tau)=\sum_{i=0}^{\infty}\phi_{ai}(\tau)\sin(\alpha_i\eta),\quad \phi_b(\eta,\tau)=\sum_{i=0}^{\infty}\phi_{bi}(\tau)\sin(\alpha_i\eta)\tag{16a}$$

$$p_a(\eta,\tau)=\frac{1}{c_1}\sum_{i=0}^{\infty}p_{ai}(\tau)\cos(\alpha_i\eta),\quad p_b(\eta,\tau)=\frac{1}{c_1}\sum_{i=0}^{\infty}p_{ai}(\tau)\cos(\alpha_i\eta)\tag{16b}$$

where

$$\begin{aligned}\phi_{a0}(\tau)&=0,\quad \phi_{ai}(\tau)=\frac{1}{l}\int_0^l\phi_a(\eta,\tau)\sin(\alpha_i\eta)\mathrm{d}\eta\quad (i=1,2,\ldots,\infty)\\ \phi_{b0}(\tau)&=0,\quad \phi_{bi}(\tau)=\frac{1}{l}\int_0^l\phi_b(\eta,\tau)\sin(\alpha_i\eta)\mathrm{d}\eta\quad (i=1,2,\ldots,\infty)\end{aligned}\tag{17a}$$

$$\begin{aligned}p_{a0}(\tau)&=\frac{2}{l}\int_0^l p_a(\eta,\tau)\mathrm{d}\eta,\quad p_{ai}(\tau)=\frac{1}{l}\int_0^l p_a(\eta,\tau)\cos(\alpha_i\eta)\mathrm{d}\eta\quad (i=1,2,\ldots,\infty)\\ p_{b0}(\tau)&=\frac{2}{l}\int_0^l p_b(\eta,\tau)\mathrm{d}\eta,\quad p_{bi}(\tau)=\frac{1}{l}\int_0^l p_b(\eta,\tau)\cos(\alpha_i\eta)\mathrm{d}\eta\quad (i=1,2,\ldots,\infty)\end{aligned}\tag{17b}$$

By comparing Eqs. (15a,b) with Eqs. (16a,b), we then obtain

$$\phi_i(s,\tau)=\phi_{ai}(\tau),\quad \phi_i(1,\tau)=\phi_{bi}(\tau)\tag{18a}$$

$$\left[\frac{\partial v_i(\xi,\tau)}{\partial\xi}-\frac{v_i(\xi,\tau)}{\xi}\right]_{\xi=s}=p_{ai}(\tau),\quad \left[\frac{\partial v_i(\xi,\tau)}{\partial\xi}-\frac{v_i(\xi,\tau)}{\xi}\right]_{\xi=1}=p_{bi}(\tau)\tag{18b}$$

By utilizing Eq. (12), the initial conditions (11) are rewritten as

$$\begin{aligned}v(\xi,\eta,0)&=\sum_{i=0}^{\infty}v_i(\xi,0)\cos(\alpha_i\eta)=\chi_0(\xi,\eta)\\ \dot v(\xi,\eta,0)&=\sum_{i=0}^{\infty}\dot v_i(\xi,0)\cos(\alpha_i\eta)=\chi_1(\xi,\eta)\end{aligned}\tag{19}$$

Recall the Fourier series expansion technique, we obtain

$$\chi_0(\xi,\eta)=\sum_{i=0}^{\infty}\chi_{0i}(\xi)\cos(\alpha_i\eta),\quad \chi_1(\xi,\eta)=\sum_{i=0}^{\infty}\chi_{1i}(\xi)\cos(\alpha_i\eta)\tag{20}$$

where

$$\begin{aligned}\chi_{00}(\xi)&=\frac{2}{l}\int_0^l\chi_0(\xi,\eta)\mathrm{d}\eta,\quad \chi_{0i}(\xi)=\frac{1}{l}\int_0^l\chi_0(\xi,\eta)\cos(\alpha_i\eta)\mathrm{d}\eta\quad (i=1,2,\ldots,\infty)\\ \chi_{10}(\xi)&=\frac{2}{l}\int_0^l\chi_1(\xi,\eta)\mathrm{d}\eta,\quad \chi_{1i}(\xi)=\frac{1}{l}\int_0^l\chi_1(\xi,\eta)\cos(\alpha_i\eta)\mathrm{d}\eta\quad (i=1,2,\ldots,\infty)\end{aligned}\tag{21}$$

Then the following relations can be derived from Eqs. (19) and (20).

$$v_i(\xi,0)=\chi_{0i}(\xi),\quad \dot v_i(\xi,0)=\chi_{1i}(\xi)\tag{22}$$

3.2 The homogenization for boundary conditions

In order to homogenize the boundary conditions (18a,b), $v_i(\xi,\tau)$ and $\phi_i(\xi,\tau)$ are assumed as

$$v_i(\xi,\tau)=v_i^s(\xi,\tau)+v_i^d(\xi,\tau),\quad \phi_i(\xi,\tau)=\phi_i^s(\xi,\tau)+\phi_i^d(\xi,\tau) \tag{23}$$

in which $v_i^s(\xi,\tau)$ and $\phi_i^s(\xi,\tau)$ are named as the quasi-static parts and should satisfy the following equations.

$$e_1\alpha_i\left(\frac{\partial v_i^s}{\partial\xi}+\frac{v_i^s}{\xi}\right)+\varepsilon_1\left(\frac{\partial^2\phi_i^s}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial\phi_i^s}{\partial\xi}\right)-\alpha_i^2\phi_i^s=0 \tag{24a}$$

$$\phi_i^s(s,\tau)=\phi_{ai}(\tau),\quad \phi_i^s(1,\tau)=\phi_{bi}(\tau) \tag{24b}$$

$$\left[\frac{\partial v_i^s(\xi,\tau)}{\partial\xi}-\frac{v_i^s(\xi,\tau)}{\xi}\right]_{\xi=s}=p_{ai}(\tau),\quad \left[\frac{\partial v_i^s(\xi,\tau)}{\partial\xi}-\frac{v_i^s(\xi,\tau)}{\xi}\right]_{\xi=1}=p_{bi}(\tau) \tag{24c}$$

While the dynamic parts $v_i^d(\xi,\tau)$ and $\phi_i^d(\xi,\tau)$ are the solution of the following equations.

$$c_1\left(\frac{\partial^2 v_i^d}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial v_i^d}{\partial\xi}-\frac{v_i^d}{\xi^2}\right)-\alpha_i^2 v_i^d+e_1\alpha_i\frac{\partial\phi_i^d}{\partial\xi}-\frac{\partial^2 v_i^d}{\partial\tau^2}=\frac{\partial^2 v_i^s}{\partial\tau^2}-e_1\alpha_i\frac{\partial\phi_i^s}{\partial\xi}+\alpha_i^2 v_i^s-c_1\left(\frac{\partial^2 v_i^s}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial v_i^s}{\partial\xi}-\frac{v_i^s}{\xi^2}\right) \tag{25a}$$

$$e_1\alpha_i\left(\frac{\partial v_i^d}{\partial\xi}+\frac{v_i^d}{\xi}\right)-\varepsilon_1\left(\frac{\partial^2\phi_i^d}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial\phi_i^d}{\partial\xi}\right)-\alpha_i^2\phi_i^d=0 \tag{25b}$$

$$\phi_i^d(s,\tau)=0,\quad \phi_i^d(1,\tau)=0 \tag{25c}$$

$$\left[\frac{\partial v_i^d(\xi,\tau)}{\partial\xi}-\frac{v_i^d(\xi,\tau)}{\xi}\right]_{\xi=s}=0,\quad \left[\frac{\partial v_i^d(\xi,\tau)}{\partial\xi}-\frac{v_i^d(\xi,\tau)}{\xi}\right]_{\xi=1}=0 \tag{25d}$$

$$v_i^d(\xi,0)=\chi_{0i}(\xi)-v_i^s(\xi,0),\quad \dot v_i^d(\xi,0)=\chi_{1i}(\xi)-\dot v_i^s(\xi,0) \tag{25e}$$

3.3 Solution for quasi-static part

We first find the solutions for quasi-static part $v_i^s(\xi,\tau)$ and $\phi_i^s(\xi,\tau)$. The governing equations are presented in Eqs. (24a-c). Fortunately, the governing equations can be separated into two groups as

$$e_1\alpha_i\left(\frac{\partial v_i^s}{\partial\xi}+\frac{v_i^s}{\xi}\right)=0 \tag{26a}$$

$$\left[\frac{\partial v_i^s(\xi,\tau)}{\partial\xi}-\frac{v_i^s(\xi,\tau)}{\xi}\right]_{\xi=s}=p_{ai}(\tau),\quad \left[\frac{\partial v_i^s(\xi,\tau)}{\partial\xi}-\frac{v_i^s(\xi,\tau)}{\xi}\right]_{\xi=1}=p_{bi}(\tau) \tag{26b}$$

$$\varepsilon_1\left(\frac{\partial^2\phi_i^s}{\partial\xi^2}+\frac{1}{\xi}\frac{\partial\phi_i^s}{\partial\xi}\right)-\alpha_i^2\phi_i^s=0 \tag{27a}$$

$$\phi_i^s(s,\tau)=\phi_{ai}(\tau),\quad \phi_i^s(1,\tau)=\phi_{bi}(\tau) \tag{27b}$$

Then $v_i^s(\xi,\tau)$ and $\phi_i^s(\xi,\tau)$ can be solved independently. The general solutions of Eqs. (26a) and (27a) can be written as

$$v_i^s(\xi, \tau) = A_{1i}(\tau)\xi^{-1} \tag{28a}$$

$$\phi_i^s(\xi, \tau) = A_{2i}(\tau)I_0(\beta_i\xi) + B_{2i}(\tau)K_0(\beta_i\xi) \tag{28b}$$

where $A_{1i}(\tau), A_{2i}(\tau)$ and $B_{2i}(\tau)$ are undetermined function. $I_0(\,)$ and $K_0(\,)$ are modified Bessel functions of zero-order and

$$\beta_i = \alpha_i/\sqrt{\varepsilon_1} \tag{29}$$

Substituting Eq. (28a) into the first of Eq. (26b), we obtain

$$A_{1i}(\tau) = -s^2 p_{ai}(\tau)/2 \tag{30}$$

The substitution of Eq. (28b) into Eq. (27b) derives.

$$A_{2i}(\tau) = \frac{K_0(\beta_i)\phi_{ai}(\tau) - K_0(\beta_i s)\phi_{bi}(\tau)}{I_0(\beta_i s)K_0(\beta_i) - I_0(\beta_i)K_0(\beta_i s)}, \quad B_{2i}(\tau) = \frac{I_0(\beta_i s)\phi_{bi}(\tau) - I_0(\beta_i)\phi_{ai}(\tau)}{I_0(\beta_i s)K_0(\beta_i) - I_0(\beta_i)K_0(\beta_i s)} \tag{31}$$

It should be particularly mentioned here that the prescribed shear stress at the internal surface $p_{ai}(\tau)$ and that at the external surfaces $p_{bi}(\tau)$ are connected each other with the equilibrium of torque about z-axis. We write it in a formula form as

$$p_a(\eta, \tau)\cdot 2\pi\cdot s\cdot l\cdot s = p_b(\eta, \tau)\cdot 2\pi\cdot 1\cdot l\cdot 1 \tag{32}$$

That is

$$s^2 p_a(\eta, \tau) = p_a(\eta, \tau) \tag{33}$$

Similarly, by utilizing the Fourier series expansion technique, we further have

$$s^2 p_{ai}(\tau) = p_{bi}(\tau) \tag{34}$$

With the aid of (34), we learn that Eq. (28a) satisfy simultaneously the two equations in Eq. (26b).

3.4 Solution for dynamic part

3.4.1 The eigenequation and eigenfunctions

Dropping the right hand side of Eq. (25a) and ignoring the initial conditions (25e), we then obtain a homogeneous system. In this system, if we assume

$$v_i^d(\xi, \tau) = R_i(\xi)e^{i\omega\tau}, \quad \phi_i^d(\xi, \tau) = T_i(\xi)e^{i\omega\tau} \tag{35}$$

where ω is angular frequency. Utilizing Eq. (35), the homogeneous system then can be transformed as

$$c_1\left[\frac{\mathrm{d}^2 R_i(\xi)}{\mathrm{d}\xi^2} + \frac{1}{\xi}\frac{\mathrm{d}R_i(\xi)}{\mathrm{d}\xi} - \frac{R_i(\xi)}{\xi^2}\right] + (\omega^2 - \alpha_i^2)R_i(\xi) + e_1\alpha_i\frac{\mathrm{d}T_i(\xi)}{\mathrm{d}\xi} = 0 \tag{36a}$$

$$e_1\alpha_i\left[\frac{\mathrm{d}R_i(\xi)}{\mathrm{d}\xi} + \frac{R_i(\xi)}{\xi}\right] - \varepsilon_1\left[\frac{\mathrm{d}^2 T_i(\xi)}{\mathrm{d}\xi^2} + \frac{1}{\xi}\frac{\mathrm{d}T_i(\xi)}{\mathrm{d}\xi}\right] - \alpha_i^2 T_i(\xi) = 0 \tag{36b}$$

and the boundary conditions (25c,d) are rewritten as

$$T_i(s) = 0, \quad T_i(1) = 0 \tag{36c}$$

$$\left[\frac{\mathrm{d}R_i(\xi)}{\mathrm{d}\xi} - \frac{R_i(\xi)}{\xi}\right]_{\xi=s} = 0, \quad \left[\frac{\mathrm{d}R_i(\xi)}{\mathrm{d}\xi} - \frac{R_i(\xi)}{\xi}\right]_{\xi=1} = 0 \tag{36d}$$

By inspecting Eqs. (36a,b), the general solutions for $R_i(\xi)$ and $T_i(\xi)$ can be written as

$$R_i(\xi) = R_{1i}J_1(k_i\xi) + R_{2i}Y_1(k_i\xi), \quad T_i(\xi) = T_{1i}J_0(k_i\xi) + T_{2i}Y_0(k_i\xi) \tag{37}$$

where k_i is undetermined constant. $J_0(\,)$ and $J_1(\,)$ denote Bessel functions of the first kind of order 0 and 1 and $Y_0(\,)$ and $Y_1(\,)$ are Bessel functions of the second kind of order 0 and 1. Substituting Eq. (37) into Eqs. (36a,b), we obtain

$$\begin{bmatrix} -c_1k_i^2 - \alpha_i^2 + \omega^2 & -e_1\alpha_i k_i \\ e_1\alpha_i k_i & \varepsilon_1 k_i^2 - \alpha_i^2 \end{bmatrix} \begin{Bmatrix} R_{1i} \\ T_{1i} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{38a}$$

$$\begin{bmatrix} -c_1k_i^2 - \alpha_i^2 + \omega^2 & -e_1\alpha_i k_i \\ e_1\alpha_i k_i & -\varepsilon_1 k_i^2 - \alpha_i^2 \end{bmatrix} \begin{Bmatrix} R_{2i} \\ T_{2i} \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{38b}$$

The existence of nontrivial solution for Eqs. (38a,b) leads to the following auxiliary equation.

$$c_1\varepsilon_1 k_i^4 + \alpha_i^2\left[c_1 + e_1^2 + \varepsilon_1\left(1 - \frac{\omega^2}{\alpha_i^2}\right)\right]k_i^2 + \alpha_i^4\left(1 - \frac{\omega^2}{\alpha_i^2}\right) = 0 \tag{39}$$

Obviously, four roots $k_{1i}, k_{2i}, -k_{1i}$ and $-k_{2i}$, can be obtained from Eq. (39). Then $R_i(\xi)$ and $T_i(\xi)$ can be written in a form as

$$R_i(\xi) = \sum_{j=1}^{2}[R_{1ji}J_1(k_{ji}\xi) + R_{2ji}Y_1(k_{ji}\xi)] \tag{40a}$$

$$T_i(\xi) = \sum_{j=1}^{2}[T_{1ji}J_0(k_{ji}\xi) + T_{2ji}Y_0(k_{ji}\xi)] \tag{40b}$$

By means of Eqs. (38a,b), for each $k_{ji}(j = 1, 2)$, we have the relations as

$$\frac{T_{1ji}}{R_{1ji}} = \frac{T_{2ji}}{R_{2ji}} = -\frac{c_1k_{ji}^2 + \alpha_i^2 - \omega^2}{e_1\alpha_i k_{ji}} = E_{ji} \tag{41}$$

Then Eq. (40b) is rewritten as

$$T_i(\xi) = \sum_{j=1}^{2}E_{ji}[R_{1ji}J_0(k_{ji}\xi) + R_{2ji}Y_0(k_{ji}\xi)] \tag{42}$$

Utilizing the properties of Bessel function, we have the relations as

$$\frac{\mathrm{d}J_1(x)}{\mathrm{d}x} - \frac{J_1(x)}{x} = -J_2(x), \quad \frac{\mathrm{d}Y_1(x)}{\mathrm{d}x} - \frac{Y_1(x)}{x} = -Y_2(x) \tag{43}$$

Substituting Eqs. (40a) and (42) into the boundary conditions (36c,d) and utilizing Eq. (43), we then obtain

$$[G_i]_{4\times4}\{X_i\}_{4\times1} = \{0\} \tag{44}$$

where

$$[G_i]_{4\times4} = \begin{bmatrix} k_{1i}J_2(k_{1i}s) & k_{1i}J_2(k_{1i}s) & k_{2i}J_2(k_{2i}s) & k_{2i}J_2(k_{2i}s) \\ k_{1i}J_2(k_{1i}) & k_{1i}Y_2(k_{1i}) & k_{2i}J_2(k_{2i}) & k_{2i}Y_2(k_{2i}) \\ E_{1i}J_0(k_{1i}s) & E_{1i}Y_0(k_{1i}s) & E_{2i}J_0(k_{2i}s) & E_{2i}Y_0(k_{2i}s) \\ E_{1i}J_0(k_{1i}) & E_{1i}Y_0(k_{1i}) & E_{2i}J_0(k_{2i}) & E_{2i}Y_0(k_{2i}) \end{bmatrix} \tag{45a}$$

$$\{X_i\}_{4\times1} = [R_{11i}\ \ R_{21i}\ \ R_{12i}\ \ R_{22i}]^T \tag{45b}$$

Also, the existence of nontrivial solution for Eq. (44) leads to

$$\mathrm{Det}[G_i] = g_i(\omega) = 0 \tag{46}$$

Eq. (46) is just the eigenequation from which a series of $\omega_{im}(m = 1, 2, \ldots, \infty)$ can be determined. Then for each $\omega_{im}(m = 1, 2, \ldots, \infty)$, the eigenfunctions $R_i(\xi)$ and $T_i(\xi)$ can be determined completely as

$$R_{im}(\xi) = g_{11i}^m J_1(k_{1i}^m\xi) + g_{12i}^m Y_1(k_{1i}^m\xi) + g_{13i}^m J_1(k_{2i}^m\xi) + g_{14i}^m Y_1(k_{2i}^m\xi) \tag{47a}$$

$$T_{im}(\xi) = E_{1i}^m g_{11i}^m J_1(k_{1i}^m\xi) + E_{1i}^m g_{12i}^m Y_1(k_{1i}^m\xi) + E_{2i}^m g_{13i}^m J_1(k_{2i}^m\xi) + E_{2i}^m g_{14i}^m Y_1(k_{2i}^m\xi) \tag{47b}$$

where $g_{1j}^m(j = 1, 2, 3, 4)$ are the cofactors of $[G_i]$ of the first row for $\omega_{im}(m = 1, 2, \ldots, \infty)$. We also can verify $R_{im}(\xi)$ has the following orthogonal property (Paul and Sarma 1977)

$$\begin{aligned} &\int_s^1 \xi R_{im}(\xi)R_{il}(\xi)\mathrm{d}\xi = 0 \quad (m \neq l) \\ &\int_s^1 \xi R_{im}^2(\xi)\mathrm{d}\xi = N_{im} \neq 0 \end{aligned} \tag{48}$$

3.4.2 The inhomogeneous solution

By means of the separation of variables method, the dynamic part $v_i^d(\xi, \tau)$ and $\phi_i^d(\xi, \tau)$ can be assumed as

$$v_i^d(\xi, \tau) = \sum_{m=1}^{\infty} R_{im}(\xi)\Omega_{im}(\tau), \quad \phi_i^d(\xi, \tau) = \sum_{m=1}^{\infty} T_{im}(\xi)\Omega_{im}(\tau) \tag{49}$$

The substitution of Eq. (49) into Eq. (25a) derives

$$\sum_{m=1}^{\infty} R_{im}(\xi)\left[\frac{\mathrm{d}^2\Omega_{im}(\tau)}{\mathrm{d}\tau^2} + \omega_{im}^2\Omega_{im}(\tau)\right] = F_i(\xi, \tau) \tag{50}$$

where

$$F_i(\xi, \tau) = -\frac{\partial^2 v_i^s}{\partial\tau^2} - \alpha_i^2 v_i^s + c_1\left(\frac{\partial^2 v_i^s}{\partial\xi^2} + \frac{1}{\xi}\frac{\partial v_i^s}{\partial\xi} - \frac{v_i^s}{\xi^2}\right) + e_1\alpha_i\frac{\partial\phi_i^s}{\partial\xi} \tag{51}$$

Utilizing Eq. (48), the following equation can be derived from Eq. (50) as

$$\frac{\mathrm{d}^2\Omega_{im}(\tau)}{\mathrm{d}\tau^2} + \omega_{im}^2\Omega_{im}(\tau) = f_{im}(\tau) \tag{52}$$

where

$$f_{im}(\tau) = \frac{1}{N_{im}}\int_s^1 \xi F_i(\xi, \tau)R_{im}(\xi)\mathrm{d}\xi \tag{53}$$

The solution of Eq. (48) is

$$\Omega_{im}(\tau) = \Omega_{im}(0)\cos\omega_{im}\tau + \frac{\dot{\Omega}_{im}(0)}{\omega_{im}}\sin\omega_{im}\tau + \frac{1}{\omega_{im}}\int_0^\tau f_{im}(p)\sin\omega_{im}(\tau - p)\mathrm{d}p \tag{54}$$

where $\Omega_{im}(0)$ and $\dot{\Omega}_{im}(0)$ are unknown constants. By means of Eqs. (44) and (45a), $\Omega_{im}(0)$ and $\dot{\Omega}_{im}(0)$ can be determined as

$$\begin{aligned}\Omega_{im}(0) &= \frac{1}{N_{im}}\int_s^1 \xi[\chi_{0i}(\xi) - v_i^s(\xi, 0)]R_{im}(\xi)\mathrm{d}\xi \\ \dot{\Omega}_{im}(0) &= \frac{1}{N_{im}}\int_s^1 \xi[\chi_{1i}(\xi) - \dot{v}_i^s(\xi, 0)]R_{im}(\xi)\mathrm{d}\xi\end{aligned} \tag{55}$$

Thus exact solutions of the displacement and electric potential are finally obtained as

$$\begin{aligned}v(\xi, \eta, \tau) &= \sum_{i=0}^{\infty}\left[\sum_{m=1}^{\infty} R_{im}(\xi)\Omega_{im}(\tau) + A_{1i}(\tau)\xi^{-1}\right]\cos(\alpha_i\eta) \\ \phi(\xi, \eta, \tau) &= \sum_{i=0}^{\infty}\left[\sum_{m=1}^{\infty} T_{im}(\xi)\Omega_{im}(\tau) + A_{2i}(\tau)I_0(\beta_i\xi) + B_{2i}(\tau)K_0(\beta_i\xi)\right]\sin(\alpha_i\eta)\end{aligned} \tag{56}$$

4. Numerical results and discussions

4.1 Natural frequency

We will first exam the eigenroots of the eigenequation (46). Specially, we should note here that the eigenroots of Eq. (46) are just the non-dimensional natural frequencies of the finite piezoelectric hollow cylinder which is traction free and electrically shorted at both two ends and two cylindrical surfaces. In the numerical calculations, the physical constants are taken from as (Paul and Sarma 1977)

Table 1 Comparison of the first five nondimensional natural frequencies for s = 1/6 and l = 5/3

Torsional vibration mode m	Present method					Paul and Sarma (1977)				
	Order of terms in trigonometric series i					Order of terms in trigonometric series i				
	1	2	3	4	5	1	2	3	4	5
1	3.1434	6.2854*	9.4268*	12.5681	15.7095	3.1416	11.9958	13.9013	12.2191	14.6376
2	10.6915	11.9980	13.9043	16.2001	18.7427	10.6907	18.1633	19.4744	16.1970	18.7393
3	17.3298	18.1656	19.4784	21.1792	23.1827	17.3290	24.8005	25.7763	21.1739	23.1768
4	24.1968	24.8026	25.7802	27.0890	28.6835	24.1962	31.7187	32.4874	27.0832	28.6763
5	31.2490	31.7205	32.4909	33.5393	34.8405	31.2485	38.8034	39.4342	33.5338	34.8331

Table 2 First eight nondimensional natural frequencies of the finite piezoelectric hollow cylinder with $s = a/b = 0.25$ for different shape parameter $l = L/b$

Shape parameter $l = L/b$	Torsional vibration mode m	Order of terms in trigonometric series i				
		1	2	3	4	5
$l = 0.5$	1	10.4737	20.9451	31.4168	41.8886	52.3604
	2	14.8275	23.4285	33.1245	43.1842	53.4027
	3	20.8608	27.6468	36.2318	45.6117	55.3842
	4	27.9240	33.3034	40.7145	49.2488	58.4165
	5	35.4803	39.8546	46.2293	53.8988	62.3878
	6	43.2875	46.9411	52.4636	59.3344	67.1408
	7	51.2349	54.3576	59.1937	65.3625	72.5243
	8	59.2673	61.9871	66.2699	71.8349	78.4091
$l = 1.0$	1	5.2378	10.4737	15.7094	20.9451	26.1809
	2	11.7285	14.8275	18.8936	23.4285	28.2073
	3	18.7829	20.8608	23.9240	27.6468	31.7984
	4	26.4068	27.9240	30.2825	33.3034	36.8238
	5	34.2987	35.4803	37.3658	39.8546	42.8412
	6	42.3242	43.2875	44.8464	46.9411	49.5030
	7	50.4236	51.2349	52.5589	54.3576	56.5854
	8	58.5673	59.2673	60.4156	61.9871	63.9502
$l = 2.0$	1	2.6193	5.2378	7.8559	10.4737	13.0915
	2	10.8153	11.7285	13.1094	14.8275	16.7797
	3	18.2260	18.7829	19.6757	20.8608	22.2916
	4	26.0135	26.4068	27.0495	27.9240	29.0093
	5	33.9967	34.2987	34.7960	35.4803	36.3411
	6	42.0799	42.3242	42.7283	43.2875	43.9960
	7	50.2186	50.4236	50.7632	51.2349	51.8349
	8	58.3909	58.5673	58.8600	59.2673	59.7868
$l = 5.0$	1	1.0478	2.0955	3.1430	4.1905	5.2378
	2	10.5452	10.7004	10.9540	11.2995	11.7285
	3	18.0669	18.1580	18.3088	18.5177	18.7829
	4	25.9023	25.9659	26.0716	26.2188	26.4068
	5	33.9117	33.9603	34.0412	34.1541	34.2987
	6	42.0112	42.0505	42.1158	42.2071	42.3242
	7	50.1611	50.1940	50.2488	50.3253	50.4236
	8	58.3415	58.3697	58.4168	58.4827	58.5673
$l = 10.0$	1	0.5239	1.0478	1.5716	2.0955	2.6193
	2	10.5061	10.5452	10.6101	10.7004	10.8153
	3	18.0441	18.0669	18.1049	18.1580	18.2260
	4	25.8864	25.9023	25.9288	25.9659	26.0135
	5	33.8995	33.9117	33.9320	33.9603	33.9967
	6	42.0014	42.0112	42.0276	42.0505	42.0799
	7	50.1529	50.1611	50.1748	50.1940	50.2186
	8	58.3344	58.3415	58.3532	58.3697	58.3909

$$c_1 = \frac{c_{66}}{c_{44}} = 1.400552\,, \quad \varepsilon_1 = \frac{\varepsilon_{11}}{\varepsilon_{33}} = 0.955642\,, \quad e_1 = \frac{e_{14}}{\sqrt{c_{44}\varepsilon_{33}}} = \sqrt{0.002933} \tag{57}$$

For the sake of comparison, the normalization method for the roots (natural frequencies) adopted in Paul and Sarma (1977) is reused in this section. The comparison of the first five nondimensional natural frequencies with those by Paul and Sarma (1977) for the finite piezoelectric hollow cylinder with the geometric parameters $s = a/b = 1/6$ and $l = L/b = 5/3$ are listed in Table 1. From Table 1, we find that the obtained results are very close to those gained by Paul and Sarma (1977). The correctness of the present calculation is then verified. Furthermore, by inspecting Table 1, we learn that the results followed by "*" are just those omitted by Paul and Sarma (1977).

Table 2 show the first eight nondimensional natural frequencies for the finite piezoelectric hollow cylinder with $s = a/b = 0.25$ for different shape parameter $l = L/b$. Clearly, the natural frequencies decrease with the increase of l. It is physical reasonable that the torsional stiffness of the finite hollow cylinder decrease with the increase of l. We also notice that the natural frequencies of the first mode are very sensitive to l.

4.2 Transient response

As an illustrative example, the transient responses of a finite piezoelectric hollow cylinder with electrically shorted and free-free ends subjected to a time dependent electric potential at the external surface will be performed. The physical constants are presented in Eq. (57) and the boundary conditions at the two cylindrical surfaces are employed as

$$\phi_a(\eta, \tau) = 0.0\,, \quad \phi_b(\eta, \tau) = \sin\left(\frac{\pi}{l}\eta\right)(1 - e^{-2\tau}) \tag{58a}$$

$$p_a(\eta, \tau) = 0\,, \quad p_b(\eta, \tau) = 0 \tag{58b}$$

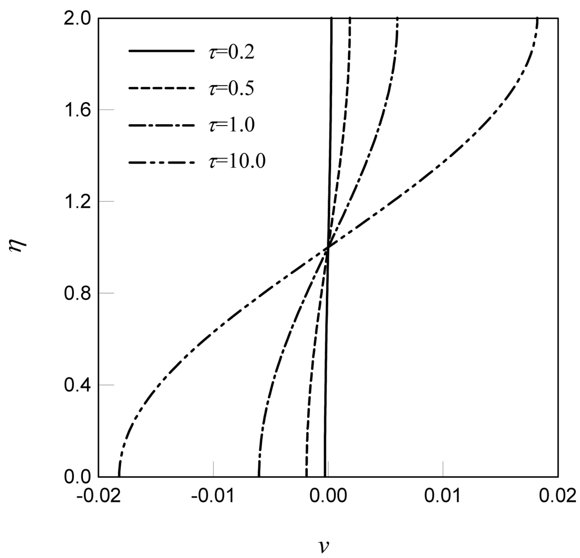

Fig. 2 Distributions of displacement along the axial direction at the internal surface ($\xi = 0.2$)

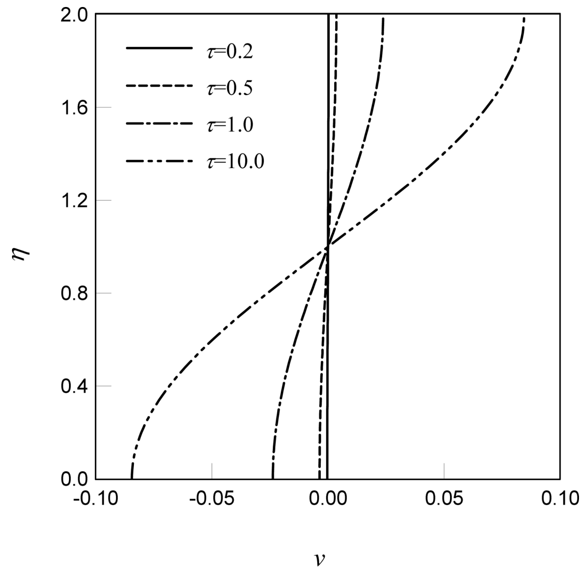

Fig. 3 Distributions of displacement along the axial direction at the external surface ($\xi = 1.0$)

In the following calculation, the geometric parameters of the finite hollow cylinder are taken as $s = a/b = 0.2$, $l = L/b = 2.0$. Also, the first 40 terms in Eq. (49) are employed.

Figs. 2 and 3 show the distributions of displacement along the axial direction at the internal surface ($\xi = 0.2$) and external surface ($\xi = 1.0$) at different times, respectively. We find that the distributions of displacement is anti-symmetric with the middle plane ($\eta = 1.0$). From Figs. 3 and 4, we also find that at the each time, the maximum amplitude of the displacement appears at the two ends.

The Distributions of electric potential along the axial direction at the middle surface ($\xi = 0.6$) are

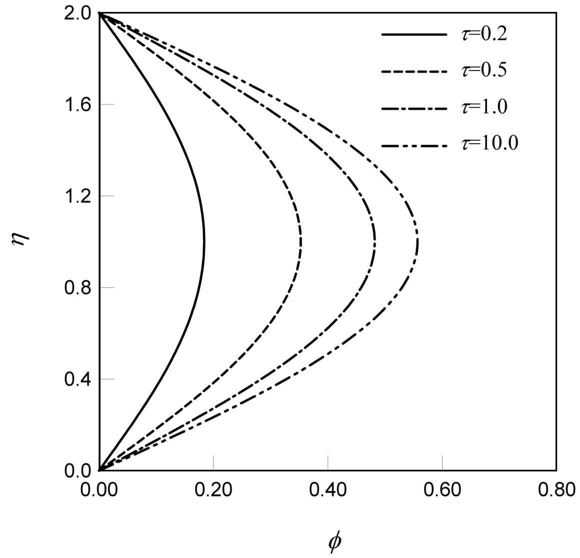

Fig. 4 Distributions of electric potential along the axial direction at the middle surface ($\xi = 0.6$)

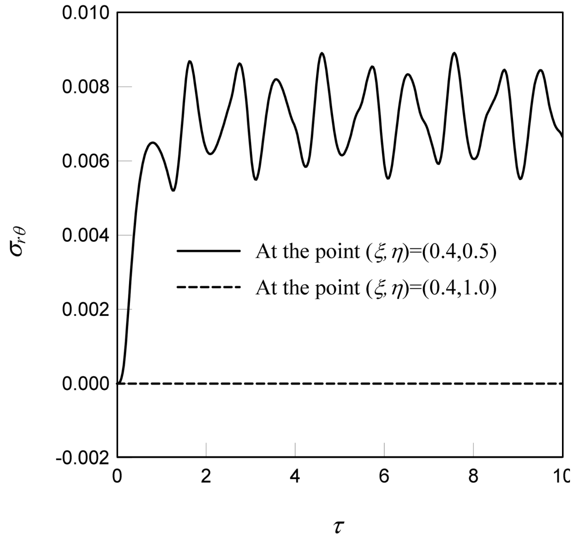

Fig. 5 Dynamic responses of shearing stress $\sigma_{r\theta}$ at two prescribed points

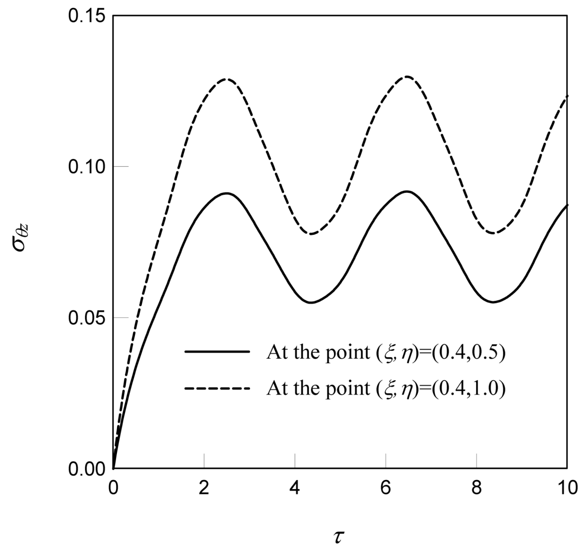

Fig. 6 Dynamic responses of shearing stress $\sigma_{\theta z}$ at two prescribed points

illustrated in Fig. 4. We notice that the distributions of electric potential is symmetric with the middle plane ($\eta = 1.0$) and the maximum amplitude appears at the middle plane. Also, with the time processing, maximum amplitude increases gradually.

Figs. 5 and 6 depict the dynamic responses of shearing stresses $\sigma_{r\theta}$ and $\sigma_{\theta z}$ at the points $(\xi, \eta) = (0.4, 0.5)$ and $(0.4, 1.0)$ respectively. By the numerical tests, we observe that the dynamic responses of $\tau_{r\theta}$ always keep zero at the middle plane ($\eta = 1.0$). The same phenomena can also be found for electric displacement D_z. Such numerical results can be easily verified by substituting the obtained series solution (56) into Eq. (8).

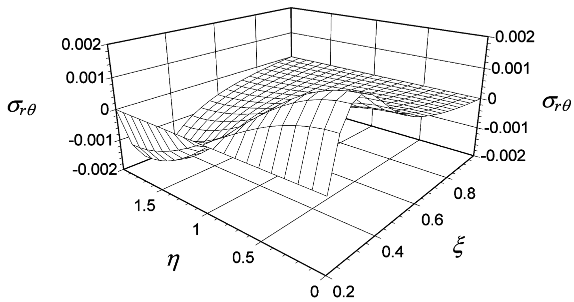

Fig. 7 2D distribution of shearing stress $\sigma_{r\theta}$ at the time $\tau = 0.2$

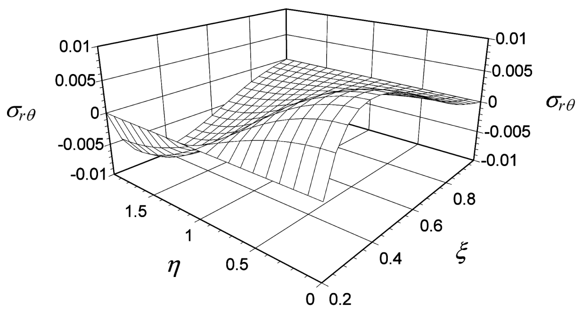

Fig. 8 2D distribution of shearing stress $\sigma_{r\theta}$ at the time $\tau = 20.0$

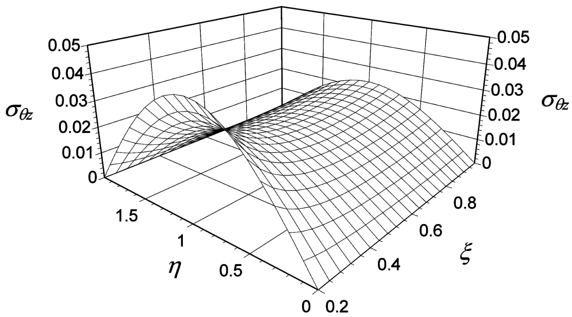

Fig. 9 2D distribution of shearing stress $\sigma_{\theta z}$ at the time $\tau = 0.2$

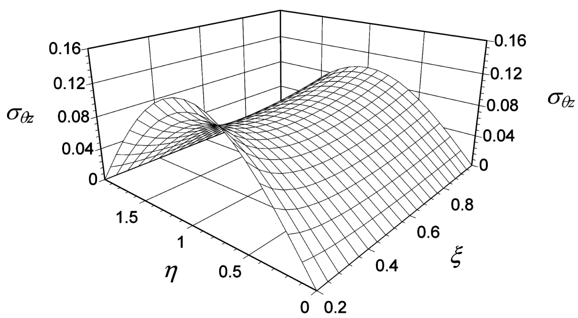

Fig. 10 2D distribution of shearing stress $\sigma_{\theta z}$ at the time $\tau = 20.0$

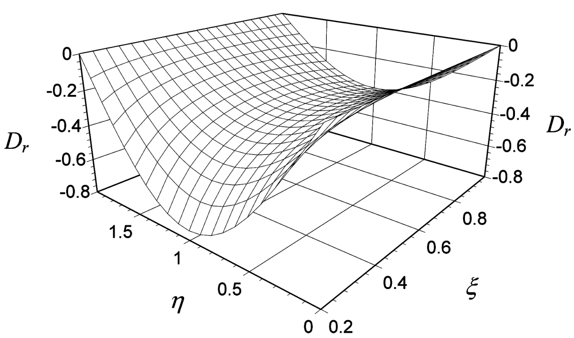

Fig. 11 2D distribution of radial electric displacement D_r at the time $\tau = 0.2$

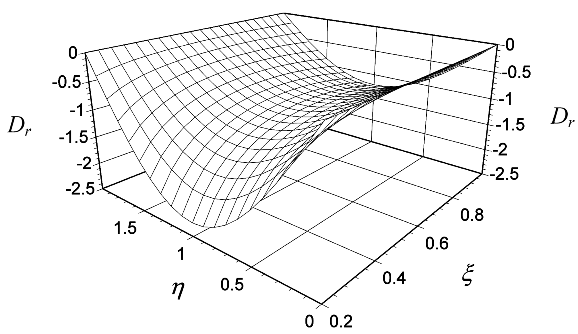

Fig. 12 2D distribution of radial electric displacement D_r at the time $\tau = 20.0$

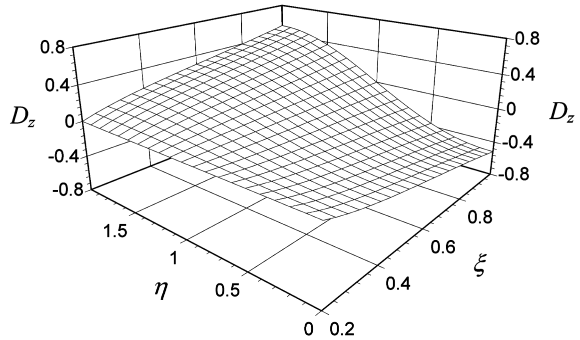

Fig. 13 2D distribution of axial electric displacement D_z at the time $\tau = 0.2$

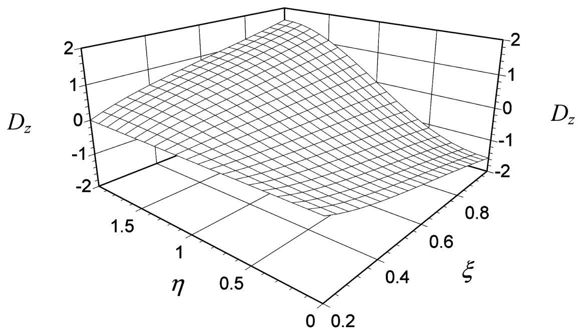

Fig. 14 2D distribution of axial electric displacement D_z at the time $\tau = 20.0$

The 2D distributions of the shearing stresses $\sigma_{r\theta}$, $\sigma_{\theta z}$ and the electric displacements D_r and D_z at the initial time $\tau = 0.2$ and the long time $\tau = 20.0$ are shown in Figs. 7-14. The surfaces display clearly that distribution forms of the mechanical and electric fields. It is noticed that the distribution forms of $\sigma_{\theta z}$ and D_r are like a saddle. The distribution forms of $\sigma_{\theta z}$ is convex while that of D_r is concave. We also find that the maximum amplitudes of the field distributions at the time $\tau = 20.0$ are always larger than those at $\tau = 0.2$.

5. Conclusions

An exact dynamic solution is developed for torsional vibration of a finite piezoelectric hollow cylinder with free-free ends. The hollow cylinder is made of crystal class 622 and polarized in axial direction. The excitation can be dynamic shearing stress or time dependent electric potential applied on the internal and external surfaces.

The obtained solutions of the displacement and electric potential are expressed as a sum of two infinite series. One series contains Bessel functions and the other contains trigonometric functions. Numerical tests show the validity of the present solution. The potential application of the present solution will be found in exact analysis of dynamic behavior of piezoelectric torsional actuators.

Acknowledgements

Foundation of China (Nos. 10872179 and 10725210), the Natural Science Foundation of Zhejiang province (No. Y7080298) and Zijin Plan of Zhejiang University.

References

Lin, S.Y. (1996), "Torsional vibration of coaxially segmented, tangentially polarized piezoelectric ceramic tubes", *J. Acoust. Soc. Am.*, **99**, 3476-3480.

Mitra, A.K. and Mukherji, P. (1972), "On the torsional vibration of a finite circular cylinder of non-homogeneous material by a particular type of twist on one of its plane ends", *Pure Appl. Geophy.*, **95**, 75-79.

Paul, H.S. and Sarma, K.V. (1977), "Torsional wave propagation wave propagation in a finite piezoelectric cylindrical shell", *Proc. Indian Nat. Sci. Acad.*, **43A**, 169-181.

Singh, B.M., Rokne, J. and Dhaliwal, R.S. (2006), "Torsional vibration of functionally graded finite cylinders", *Meccanica*, **41**, 459-470.

Srinivasamoorthy, V.R. and Anandam, C. (1980), "Torsional wave propagation in an infinite piezoelectric cylinder (622) crystal class", *J. Acoust. Soc. Am.*, **67**, 2034-2035.

Timoshenko, S., Young, D.H. and Weaver, W. (1974), *Vibration Problems in Engineering* (4th edition), John Wiley & Sons, New York.

Wang, X., Xia, X.H. and Hao, W.H. (2003), "An elastodynamic solution of finite long orthotropic hollow cylinder under torsion impact", *J. Sound Vib.*, **267**, 67-86.

Xie, S. and Liu, K. (1998), "Transient torsional wave propagation in a transversely isotropic tube", *Arch. Appl. Mech.*, **68**, 589-596.