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Convergence studies on static and dynamic analysis of beams by using the U-transformation method and finite difference method

Y. Yang†

School of Civil Engineering and Transportation, South China University of Technology, Guangzhou, 510640, P. R. China

M. Cai‡ and J. K. Liu‡†

Department of Mechanics, Zhongshan University, Guangzhou 510275, P. R. China

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Abstract. The static and dynamic analyses of simply supported beams are studied by using the U-transformation method and the finite difference method. When the beam is divided into the mesh of equal elements, the mesh may be treated as a periodic structure. After an equivalent cyclic periodic system is established, the difference governing equation for such an equivalent system can be uncoupled by applying the U-transformation. Therefore, a set of single-degree-of-freedom equations is formed. These equations can be used to obtain exact analytical solutions of the deflections, bending moments, buckling loads, natural frequencies and dynamic responses of the beam subjected to particular loads or excitations. When the number of elements approaches to infinity, the exact error expression and the exact convergence rates of the difference solutions are obtained. These exact results cannot be easily derived if other methods are used instead.

Keywords: static analysis; dynamic analysis; beam; convergence rate; U-transformation; finite difference method.

1. Introduction

As an important numerical computational method, the finite difference method has broad applications in various scientific research fields, e.g., physics, mechanics, astronomy and engineering technology. The study on the convergence of difference schemes always attracts the attention of computing mathematicians and dynamicists. Dividing elements to less size and increasing the degree of the interpolation polynomial can only decrease the error but not improve the precision. So it is important to uncover the convergence of the finite difference schemes.

In 1972, Runchal discussed the convergence and accuracy of three finite difference schemes for a

† Ph.D., Corresponding author, E-mail: yiyang@scut.edu.cn
‡ Ph.D., E-mail: caiming@mail.sysu.edu.cn
‡† Professor, Ph.D., E-mail: jikeliu@hotmail.com

two-dimensional conduction and convection problem (Runchal 1972). Caldwell made an estimate of the convergence of a locally one dimensional finite difference scheme for parabolic initial-boundary value problems and error estimates for the finite scheme are presented (Caldwell 1976). Abarbanel *et al.* considered a family of spatially semi-discrete approximations, including boundary treatments, to hyperbolic and parabolic equations, and investigated the error bounds of finite difference approximations to partial differential equations (Abarbanel *et al.* 2000). Brezzi *et al.* analyzed the stability and convergence properties of the mimetic finite difference method for diffusion-type problems on polyhedral meshes (Brezzi *et al.* 2005). Borzi *et al.* studied the finite difference multigrid solution of an optimal control problem associated with an elliptic equation, and sharp convergence factor estimates of two-grid method for the optimality system are obtained by means of local Fourier analysis (Borzi *et al.* 2003). Tetsuro Yamamoto and Akio Yamamoto used an acceleration technique to investigate the convergence of various finite difference schemes (Tetsuro 2002, Akio 2005).

All the previous works only discussed the convergence of the finite difference schemes and made error estimates. The exact finite difference solution and convergence rate cannot be obtained by using the previous methods.

The U-transformation method is an analytical method for the exact analysis of structures with periodicity or nearly periodic properties, such as static and dynamic analysis of continuous beams, plane trusses, stiffened plates, mass spring systems, and cyclic bi-periodic structures (Cai *et al.* 2002). Recently, Liu *et al.* applied the U-transformation method to study the static and dynamic problems of a simply supported rectangular plate by using the two-dimensional finite difference method, and the convergence rates of explicit difference solutions were discussed (Liu *et al.* 2003, 2006).

In the present paper, the application of the U-transformation technique is extended to the convergence study of finite difference method. Without lost of generality, a beam with two simply supported ends and the central difference formula are considered. The beam subdivided by a uniform mesh may be regarded as a periodic structure. By considering an equivalent structure and adopting the U-transformation technique, a set of independent equations with only one degree of freedom can be derived from the governing equations. Therefore, the exact explicit expressions for the deflections, bending moments, buckling loads, natural frequencies and dynamic responses can be easily obtained. The convergence rates can then be determined simply from the explicit solutions. Numerical results are given to verify the exact closed-form explicit solutions obtained by the proposed technique in the present paper.

2. Simply supported beam

Consider a beam with two simply supported ends. The beam is divided into n equal elements and then the beam may be regarded as a periodic structure with n substructures as shown in Fig. 1. L denotes the length of the beam and a is the length of an element, $F(x)$ is the load function, and j denotes the nodal number. The boundary conditions can be expressed as $w_1 = w_{n+1} = 0$ and $M_1 = M_{n+1} = 0$, where w_j and M_j denote the deflection and bending moment of the j-th node, respectively. The equivalent system with cyclic periodicity must satisfy these two restrained conditions.

Only the structures with cyclic periodicity may be analyzed by using the U-transformation

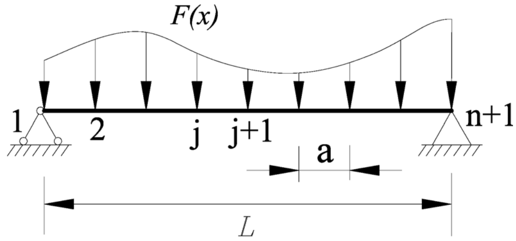

Fig. 1 Simply supported beam with n elements

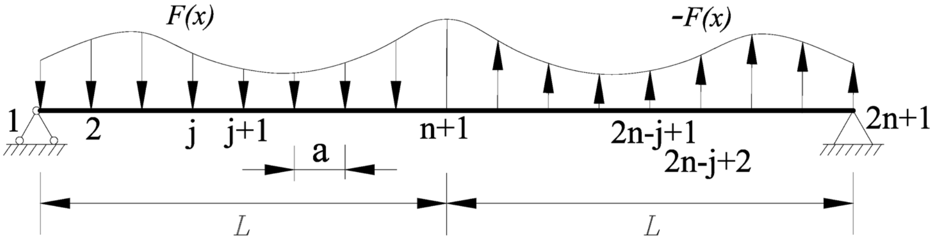

Fig. 2 Equivalent system with $2n$ elements

method. Firstly, it is necessary to extend the original beam by its symmetrical image and apply anti-symmetric loading on the corresponding extended part about the original loading as shown in Fig. 2. The two ends can be imaginarily put together and treated as one point in mathematics. A structure with such characteristics is a kind of cyclic periodic structure. It is possible to use the U-transformation method to analyze the equivalent system as shown in Fig. 2 instead of the original beam as shown in Fig. 1. The boundary conditions at both extreme ends for the original beam are satisfied automatically in its equivalent system.

3. Deflection

Consider the cyclic periodic system as shown in Fig. 2. The deflection equation of the j-th node can be expressed as

$$\mathrm{EI}\frac{\mathrm{d}^4 w_j}{\mathrm{d}x^4} = F_j, \quad j = 1,2,\ldots,2n \tag{1}$$

where F_j denotes the loading for the node j, and EI is the flexural rigidity. The loading F_j must satisfy the anti-symmetric condition, i.e.

$$F_j = -F_{2n-j+2}, \quad j = 2,\ldots,n; \quad F_1 = F_{n+1} = 0 \tag{2}$$

Substituting the central difference scheme for $\mathrm{d}^4 w_j/\mathrm{d}x^4$ into the governing Eq. (1) results in

$$\frac{\mathrm{EI}}{a^4}(w_{j-2} - 4w_{j-1} + 6w_j - 4w_{j+1} + w_{j+2}) = F_j, \quad j = 1,2,\ldots,2n \tag{3}$$

in which $a = L/n$.

Every displacement of the cyclic periodic structures may be expressed as a series of cyclic modes U_s, and the coefficients q_s denote the mode co-ordinate (Liu *et al*. 2003). Now the U-transformation method is used to uncouple the difference Eq. (3), i.e., let

$$w_j = \frac{1}{\sqrt{2n}} \sum_{m=1}^{2n} e^{i(j-1)m\psi} q_m, \quad j = 1, 2, \ldots, 2n \tag{4a}$$

or

$$q_m = \frac{1}{\sqrt{2n}} \sum_{j=1}^{2n} e^{-i(j-1)m\psi} w_j, \quad m = 1, 2, \ldots, 2n \tag{4b}$$

in which q_m is the generalized displacement, $i = \sqrt{-1}$, and $\psi = \pi/n$.

Applying the U-transformation (4) to Eq. (3) results in

$$\frac{EI}{a^4}(6 - 8\cos m\psi + 2\cos 2m\psi) q_m = f_m, \quad m = 1, 2, \ldots, 2n \tag{5}$$

where

$$f_m = \frac{1}{\sqrt{2n}} \sum_{j=1}^{2n} e^{-i(j-1)m\psi} F_j, \quad m = 1, 2, \ldots, 2n \tag{6}$$

Consider the case of a beam subjected to an uniform load with magnitude p_0, i.e., $F(x) = p_0$, then

$$F_j = -F_{n+j} = p_0, \quad j = 2, \ldots, n; \quad F_1 = F_{n+1} = 0 \tag{7}$$

Substituting Eq. (7) into Eq. (6) yields

$$f_m = \frac{-2ip_0}{\sqrt{2n}} \frac{\sin m\psi}{1 - \cos m\psi}, \quad m = 1, 3, \ldots, 2n-1$$

$$f_m = 0, \quad m = 2, 4, \ldots, 2n \tag{8}$$

Inserting Eq. (8) in Eq. (5), the generalized displacement q_m becomes

$$q_m = \frac{-2ia^4 p_0}{\sqrt{2n}EI} \frac{\sin m\psi}{(1 - \cos m\psi)(6 - 8\cos m\psi + 2\cos 2m\psi)}, \quad m = 1, 3, \ldots, 2n-1$$

$$q_m = 0, \quad m = 2, 4, \ldots, 2n \tag{9}$$

Now every nodal displacement can be obtained from the U-transformation (4) with Eq. (9), as

$$w_j = \frac{-ia^4 p_0}{nEI} \sum_{m=1,3}^{2n-1} \frac{\sin m\psi}{(1 - \cos m\psi)(6 - 8\cos m\psi + 2\cos 2m\psi)}, \quad j = 1, 2, \ldots, 2n \tag{10}$$

The maximum deflection occurs at the center of the beam. If n is even, substituting $j = n/2 + 1$ into Eq. (10) yields

$$w_{\max} = \frac{2a^4 p_0}{nEI} \sum_{m=1,3}^{n-1} \frac{\sin\frac{m\pi}{2} \sin m\psi}{(1 - \cos m\psi)(6 - 8\cos m\psi + 2\cos 2m\psi)} \tag{11}$$

Expanding the right side of Eq. (11) into power series of ψ results in

$$w_{\max} = \frac{5}{384}\frac{p_0 L^4}{\mathrm{EI}}\left(1 + \frac{4}{5}n^{-2}\right) + \mathrm{O}(n^{-4}) \tag{12}$$

The first term on the right hand side of Eq. (12) represents the limiting solution, which is in agreement with the analytical solution. Then the second term represents the main error of the finite difference solution. When n approaches to infinity, the deflection converges from the above analytical solution at an asymptotic rate of n^{-2}. And meanwhile the precise coefficient of the error term $p_0L^4/96\mathrm{EI}$ is determined. Comparing the finite difference solution (12) with the exact finite element solution found by the U-transformation and the finite element method (Chan *et al*. 1998), it can be found that the former converges slower than the latter whose convergence rate is n^{-4}.

4. Bending moment

The governing equation of the bending moment can be expressed as

$$\frac{\mathrm{d}^2 M_j}{\mathrm{d}x^2} = -P_j = \begin{cases} -p_0,\ j = 2, 3, \ldots, n \\ p_0,\ j = n+2, \ldots, 2n \\ 0,\ j = 1, n+1 \end{cases} \tag{13}$$

Substituting the difference scheme for $\mathrm{d}^2M_j/\mathrm{d}x^2$ into the governing Eq. (13) results in

$$M_{j+1} - 2M_j + M_{j-1} = F'_j\ , \quad j = 1, 2, \ldots, 2n \tag{14}$$

where

$$F'_j = \begin{cases} -p_0a^2,\ j = 2, 3, \ldots, n \\ p_0a^2,\ j = n+2, \ldots, 2n \\ 0,\ j = 1, n+1 \end{cases} \tag{15}$$

Applying the U-transformation

$$M_j = \frac{1}{\sqrt{2n}}\sum_{m=1}^{2n} \mathrm{e}^{\mathrm{i}(j-1)m\psi} q_m\,, \quad j = 1, 2, \ldots, 2n \tag{16}$$

in Eq. (14) results in

$$2(\cos m\psi - 1)q_m = f_m = \begin{cases} \dfrac{2\mathrm{i}p_0a^3}{\sqrt{2n}}\dfrac{\sin m\psi}{1-\cos m\psi},\ m = 1, 3, \ldots, 2n-1 \\ 0,\ m = 2, 4, \ldots, 2n \end{cases} \tag{17}$$

From Eq. (17) the generalized moment q_m can be expressed as

$$q_m = \frac{\mathrm{i}p_0a^2}{\sqrt{2n}}\frac{\sin m\psi}{(\cos m\psi - 1)(1-\cos m\psi)}, \quad m = 1, 3, \ldots, 2n-1;\ q_m = 0, m = 2, 4, \ldots, 2n \tag{18}$$

Now every nodal bending moment can be obtained from the U-transformation (16) with Eq. (18), i.e.

$$M_j = -\frac{\mathrm{i}p_0a^2}{2n}\sum_{m=1,3}^{2n-1}\frac{e^{\mathrm{i}(j-1)m\psi}\sin m\psi}{(1-\cos m\psi)^2}, \quad j = 1,2,\ldots,2n \tag{19}$$

The maximum moment occurs at the center of the beam. If n is even, substituting $j = n/2+1$ into Eq. (19) yields

$$M_{\max} = \frac{p_0a^2}{n}\sum_{m=1,3}^{n-1}\frac{\sin\frac{m\pi}{2}\sin m\psi}{(1-\cos m\psi)^2} = \frac{p_0L^2}{8}\left(1+(-1)^{n/2}\frac{\pi}{15}n^{-3}\right)+\mathrm{O}(n^{-4}) \tag{20}$$

When n approaches to infinity, the moment converges to the analytical solution at an asymptotic rate of n^{-3}, i.e., $\lim_{n\to\infty} M_{\max} = p_0L^2/8$. And at the same time the precise coefficient of the error term $(-1)^{n/2}p_0\pi L^2/120$ is derived. The explicit finite element solution for $M_{\max}$ (Chan *et al.* 1998) converges to the analytical solution at an asymptotic rate of n^{-2}. So the difference solution for $M_{\max}$ converges faster than the finite element one.

5. Buckling load

Consider now the simply supported beam subjected to the axial pressure with magnitude P_x at two ends as shown in Fig. 3. Similar to that described above, an equivalent system with cyclic periodicity may be produced.

The buckling equation can be expressed as

$$\mathrm{EI}\frac{\mathrm{d}^4w_j}{\mathrm{d}x^4}+P_x\frac{\mathrm{d}^2w_j}{\mathrm{d}x^2} = 0, \quad j = 1,2,\ldots,2n \tag{21}$$

Substituting the difference schemes for $\mathrm{d}^4w_j/\mathrm{d}x^4$, $\mathrm{d}^2w_j/\mathrm{d}x^2$ and the U-transformation (4) into the buckling Eq. (21) results in

$$\frac{\mathrm{EI}}{a^4}(6-8\cos m\psi+2\cos 2m\psi)q_m+\frac{P_x}{a^2}(2\cos m\psi-2)q_m = 0, \quad m = 1,2,\ldots,2n \tag{22}$$

From Eq. (22) the critical load can be expressed as

$$P_x = -\frac{\mathrm{EI}}{a^2}\frac{(6-8\cos m\psi+2\cos 2m\psi)}{2\cos m\psi-2}, \quad m = 1,2,\ldots,2n \tag{23}$$

Expanding the right side of Eq. (23) into power series of ψ results in

$$P_x = \frac{\mathrm{EI}\pi^2m^2}{L^2}\left[1-\frac{m^2\pi^2}{12}n^{-2}+\mathrm{O}(n^{-4})\right], \quad m = 1,2,\ldots,2n-1 \tag{24}$$

The critical load P_{cr}, i.e., the minimum buckling load P_x can be obtained, by substituting $m = 1$ into Eq. (24), as

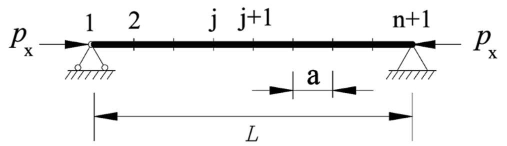

Fig. 3 Simply supported beam subjected to the axial pressure at two ends

Table 1 Buckling load P_{cr}

n	2	4	8	16	32	∞
P_{cr}	0.79438	0.94860	0.98715	0.99679	0.99920	1.00000
Multiplier			$\mathrm{EI}\pi^2/L^2$			

$$P_{cr} = \frac{\mathrm{EI}\pi^2}{L^2}\left[1 - \frac{\pi^2}{12}n^{-2} + \mathrm{O}(n^{-4})\right] \tag{25}$$

The first term on the right hand side of Eq. (25) represents the limiting solution that is in agreement with the analytical solution. Then the second term represents the main error of the buckling load found by finite difference method. When n approaches to infinity, the buckling load converges from below the exact analytical solution at an asymptotic rate of n^{-2}. Some numerical results of Eq. (25) are given in Table 1. The convergence rate of the exact finite element solution is n^{-4}, so the finite element solution for P_{cr} converges faster than the finite difference one.

6. Natural frequency

Consider the cyclic periodic system as shown in Fig. 2. The dynamic equations for all substructures are of the same form, i.e.

$$\mathrm{EI}\frac{\mathrm{d}^4 w_j}{\mathrm{d}x^4} + \rho\frac{\mathrm{d}^2 w_j}{\mathrm{d}t^2} = F_j, \quad j = 1, 2, \ldots, 2n \tag{26}$$

where ρ denotes the mass per unit length. The loading functions F_j must satisfy the anti-symmetric condition expressed as Eq. (2).

Substituting the difference scheme for $\mathrm{d}^4 w_j/\mathrm{d}x^4$ into the governing Eq. (26) results in

$$\frac{\mathrm{EI}}{a^4}(w_{j-2} - 4w_{j-1} + 6w_j - 4w_{j+1} + w_{j+2}) + \rho\frac{\mathrm{d}^2 w_j}{\mathrm{d}t^2} = F_j, \quad j = 1, 2, \ldots, 2n \tag{27}$$

Applying the U-transformation to Eq. (27), i.e., substituting

$$w_j = \frac{1}{\sqrt{2n}}\sum_{m=1}^{2n} \mathrm{e}^{\mathrm{i}(j-1)m\psi} q_m, \quad j = 1, 2, \ldots, 2n \tag{28}$$

with $\psi = \pi/n$ into Eq. (27) results in

$$\ddot{q}_m + \frac{\mathrm{EI}}{\rho a^4}(6 - 8\cos m\psi + 2\cos 2m\psi) q_m = \frac{f_m}{\rho}, \quad m = 1, 2, \ldots, 2n \tag{29}$$

where f_m has been defined by Eq. (6).

Inserting $q_m = Q_m(x) \cdot \mathrm{e}^{\mathrm{i}\omega t}$ and $f_m = 0$ into Eq. (29) yields

$$\left[\frac{\mathrm{EI}}{\rho a^4}(6 - 8\cos m\psi + 2\cos 2m\psi) - \tilde{\omega}_m^2\right] Q_m = 0, \quad m = 1, 2, \ldots, 2n \tag{30}$$

where $\tilde{\omega}_m$ denotes the natural frequency found by finite difference method. The natural frequencies can be obtained from Eq. (30) as

Table 2 Natural frequencies $\tilde{\omega}_m^2$

m \ n	2	4	8	16	32	∞
1st	64.0000 (-33.4091)[a]	87.84531 (-9.56378)	94.93423 (-2.47486)	96.78499 (-0.62410)	97.25273 (-0.15636)	97.40909
2nd		1024.000 (-534.545)	1405.525 (-153.020)	1518.948 (-39.598)	1548.560 (-9.986)	1558.545
3rd		2984.155 (-4905.982)	6243.611 (-1646.526)	7445.543 (-444.594)	7776.805 (-113.3327)	7890.136
4th			16384.00 (-8552.73)	22488.40 (-2448.33)	24303.16 (-633.56)	24936.73
5th			31323.15 (-29557.53)	51778.11 (-9102.57)	58479.45 (-2401.23)	60880.68
Multiplier			$EI/\rho L^4$			

[a]The numbers in the round bracket denote the error.

$$\tilde{\omega}_m^2 = \frac{EI}{\rho a^4}(6 - 8\cos m\psi + 2\cos 2m\psi), \quad m = 1, 2, \ldots, 2n \tag{31}$$

Expanding the right side of Eq. (31) into power series of ψ results in

$$\tilde{\omega}_m^2 = \omega_m^2\left[1 - \frac{m^2\pi^2}{6}n^{-2} + O(n^{-4})\right], \quad m = 1, 2, \ldots, 2n \tag{32}$$

where ω_m denotes the analytical solution for the m-th natural frequency of the simply supported beam, i.e.

$$\omega_m^2 = m^4\pi^4\frac{EI}{\rho L^4}, \quad m = 1, 2, \ldots, 2n \tag{33}$$

When n approaches to infinity, the natural frequencies determined by the finite difference method converge from below the exact solutions at an asymptotic rate of n^{-2}. The first five natural frequencies are given in Table 2. The convergence rate of the natural frequencies found by the U-transformation and the finite element method is n^{-4} (Chan *et al.* 1998), and the finite element solutions for ω_m converge faster than the finite difference ones.

7. Dynamic response

Let us consider a concentrated load of magnitude $p(t)$ acting at the midpoint of a simply supported beam. If the number of the substructures is even, the loading function may be expressed as

$$F_{n/2+1} = -F_{n+n/2+1} = \frac{p(t)}{a} \tag{34}$$

with other nodal loading being equal to zero.

Substituting Eq. (34) into Eq. (6) yields

$$f_m = \frac{-2\mathrm{i}p(t)}{a\sqrt{2n}}\sin\frac{m\pi}{2}, \quad m = 1,2,\ldots,2n \tag{35}$$

and then inserting Eq. (35) in Eq. (29) results in

$$\ddot{q}_m + \frac{\mathrm{EI}}{\rho a^4}(6 - 8\cos m\psi + 2\cos 2m\psi)q_m = \frac{-2\mathrm{i}p(t)}{\rho a\sqrt{2n}}\sin\frac{m\pi}{2}, \quad m = 1,2,\ldots,2n \tag{36}$$

The solution for q_m of Eq. (36) with zero initial condition can be expressed as a Duhamel integral

$$q_m = \frac{-2\mathrm{i}}{\rho a\sqrt{2n}\,\tilde{\omega}_m}\sin\frac{m\pi}{2}\int_0^t p(\tau)\sin\tilde{\omega}_m(t-\tau)\,\mathrm{d}\tau, \quad m = 1,2,\ldots,2n \tag{37}$$

where $\tilde{\omega}_m$ has been defined by Eq. (32). Now the response function of the deflection can be obtained from the U-transformation (4) with Eq. (37), i.e.

$$w_j = \frac{1}{\sqrt{2n}}\sum_{m=1,3}^{2n-1} \mathrm{e}^{\mathrm{i}(j-1)m\psi} q_m, \quad j = 1,2,\ldots,2n \tag{38}$$

The response function of the deflection at the center of the beam may be found by substituting $j = n/2 + 1$ into Eq. (38), i.e.

$$\tilde{w}_c(t) = \frac{2}{\rho L}\sum_{m=1,3}^{n-1}\frac{1}{\tilde{\omega}_m}\int_0^t p(\tau)\sin\tilde{\omega}_m(t-\tau)\,\mathrm{d}\tau \tag{39}$$

The analytical solution by using mode method is given as

$$w_c(t) = \frac{2}{\rho L}\sum_{m=1,3}^{n-1}\frac{1}{\omega_m}\int_0^t p(\tau)\sin\omega_m(t-\tau)\,\mathrm{d}\tau \tag{40}$$

The finite difference solution shown in Eq. (39) converges but does not converge uniformly to the analytical solution when the number of elements approaches to infinity. The convergence rate is dependent on the characteristic of the loading function.

8. Conclusions

In the present work, the application of the U-transformation has been extended to convergence studies of the static and dynamic analysis of the finite difference method. Explicit finite difference solutions for deflections, bending moments, buckling loads, frequencies and dynamic response have been obtained. It has been shown that when the number of elements approaches to infinity, the results are the same as the exact analytical solutions. The convergence rates of the solutions are precisely determined. The proposed method in the present paper is also applicable to the static and dynamic analysis of two and three-dimensional systems.

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