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Periodic solutions of the Duffing equation

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Abstract. This paper presents a new linearization algorithm to find the periodic solutions of the Duffing equation, under harmonic loads. Since the Duffing equation models a single degree of freedom system with a cubic nonlinear term in the restoring force, finding its periodic solutions using classical harmonic balance (HB) approach requires numerical integration. The algorithm developed in this paper replaces the integrals appearing in the classical HB method with triangular matrices that are evaluated algebraically. The computational cost of using increased number of frequency components in the matrix-based linearization approach is much smaller than its integration-based counterpart. The algorithm is computationally efficient; it only takes a few iterations within the region of convergence. An example comparing the results of the linearization algorithm with the "exact" solutions from a 4th order Runge-Kutta method are presented. The accuracy and speed of the algorithm is compared to the classical HB method, and the limitations of the algorithm are discussed.

Keywords: duffing equation; harmonic balance method; nonlinear oscillator; linearization.

1. Introduction

One of the classical examples in nonlinear vibrations is the Duffing equation, given by

$$\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x + \lambda\omega_n^2 x^3 = F\cos(\omega t) \tag{1}$$

where λ is a parameter controlling the degree of nonlinearity, and ζ and ω_n are the damping ratio and the natural frequency of the corresponding linear system ($\lambda = 0$). Duffing equation describes the response of several physical systems under harmonic input. The solution of Eq. (1) has been studied by many researchers, and closed form solutions have been derived under certain conditions (Caughey 1971, Iwan 1969, Nayfeh and Mook 1979, Roberts and Spanos 1986). The cubic nonlinear term has thus far prohibited finding explicit solutions of the Duffing equation.

The superposition principle, which is the backbone of linear vibration theory, does not represent the behavior of nonlinear systems. However, its simplicity has led to the concept of linearization; approaching nonlinear problems by comparing their behavior to linear models satisfying certain resemblance criteria (Caughey 1963, Spanos and Iwan 1978).

This paper develops a linearization algorithm to find the periodic solutions of the Duffing

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equation, under harmonic loads. Since the algorithm developed in this paper is based on the Harmonic Balance (HB) method, Section 2 of this paper is devoted to brief discussion of the traditional HB formulation, where specific challenges in its application are identified. Section 3 develops a linearization algorithm that overcomes the difficulty of evaluating the integrals in the HB formulation by replacing them with triangular matrices that are evaluated algebraically. The algorithm is flexible; it does not limit the number of frequency components to be evaluated.

Section 4 presents a numerical example showing how the procedure modifies the natural frequency of the system for each frequency component. The results from the linearization algorithm are compared to the "exact" solutions obtained from a 4^{th} order Runge-Kutta method. The displacement of the corresponding linear system is also shown for reference. Section 5 concludes the paper by comparing the accuracy and speed of the linearization algorithm developed in this study with the classical HB method, and discussing its limitations.

2. Harmonic balance method

The harmonic balance (HB) method is a powerful tool to explore the periodic solutions of a system under periodic input (Krylov and Bogoliubov 1947). It is based on the assumption that the solution to Eq. (1) can be approximated by a truncated Fourier series

$$x(t) = x_0 + \sum_{n=1}^{N} (x_{2n-1}\cos(n\omega t) + x_{2n}\sin(n\omega t)) \tag{2}$$

where N is the number of harmonics used in the Fourier series. The first and second derivatives of $x(t)$ are

$$\dot{x}(t) = \omega + \sum_{n=1}^{N} (-nx_{2n-1}\sin(n\omega t) + nx_{2n}\cos(n\omega t)) \tag{3}$$

and

$$\ddot{x}(t) = -\omega^2 + \sum_{n=1}^{N} (n^2 x_{2n-1}\cos(n\omega t) + n^2 x_{2n}\sin(n\omega t)) \tag{4}$$

respectively.

The cubic term in Eq. (1) can be approximated as the following truncated Fourier series (Liu, Thomas *et al.* 2006)

$$x^3(t) \approx r_0 + \sum_{n=1}^{N} (r_{2n-1}\cos(n\omega t) + r_{2n}\sin(n\omega t)) \tag{5}$$

where

$$r_0 = \frac{1}{2\pi}\int_0^{2\pi}\left(x_0 + \sum_{n=1}^{N} (x_{2n-1}\cos(nt) + x_{2n}\sin(nt))\right)^3 dt \tag{6}$$

$$r_{2n-1} = \frac{1}{\pi}\int_0^{2\pi}\left(x_0 + \sum_{n=1}^{N} (x_{2k-1}\cos(kt) + x_{2k}\sin(kt))\right)^3 \cos(nt)dt \tag{7}$$

$$r_{2n} = \frac{1}{\pi}\int_0^{2\pi}\left(x_0 + \sum_{n=1}^{N} (x_{2k-1}\cos(kt) + x_{2k}\sin(kt))\right)^3 \sin(nt)dt \tag{8}$$

When Eqs. (2)-(5) are substituted into the differential equation, and the terms associated with each frequency are balanced, a system of algebraic equations is obtained. The Fourier coefficients of the assumed solution can then be found by solving the resulting system of equations.

The solution of many weakly nonlinear systems under multiple-frequency input can be effectively approximated by adding the solutions for independent frequencies, as the application of superposition principle is not likely to introduce large errors when the nonlinearity is sufficiently low (Tezcan and Spanos 2006). On the other hand, a truncated Fourier series assumption generally fails to describe the response of systems with a strong nonlinearity. Since contribution from other harmonics must be taken into account, classical HB approach will yield a large system of coupled equations, making the solution prohibitive (Urabe and Reiter 1964).

3. Derivation of the linearization algorithm

This section develops a new algorithm to find the periodic solutions of Duffing equation. Since Duffing oscillator is known to be stable around the origin, zero initial conditions are assumed in the derivation.

Starting with the main idea of equivalent linearization, if there exists a k_{nl} satisfying

$$k_{nl}x(t) = \lambda\omega_n^2 x^3(t) \tag{9}$$

the nonlinear system in Eq. (1) is equivalent to the following linear system

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + (\omega_n^2 + k_{nl})x(t) = g(t) \tag{10}$$

For $\lambda > 0$, k_{nl} can be thought as the additional stiffness introduced by the nonlinearity.

To illustrate the basic idea used in the linearization procedure developed in this paper, Fig. 1 shows two block diagram representations of the Duffing system. The function H is the complex frequency response function of the linear time invariant (LTI) system with damping ratio ζ and

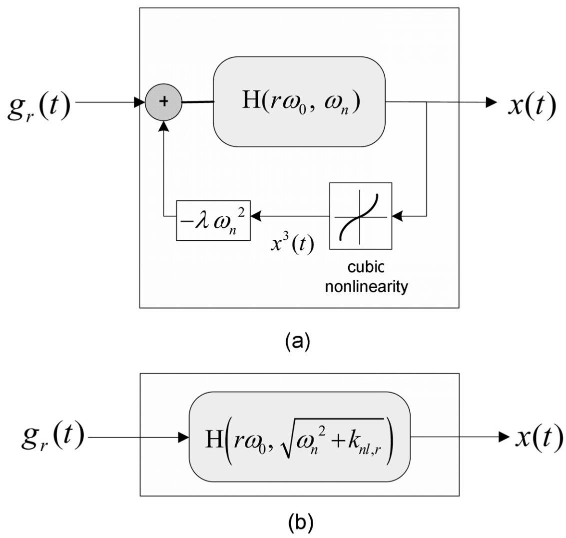

Fig. 1 Two block diagram representations of the Duffing system

natural frequency ω_n, given by

$$H(\omega, \omega_n) = \frac{1}{-\omega^2 + \omega_n^2 + 2i\zeta\omega_n\omega} \tag{11}$$

Note that Fig. 1(a) corresponds to the feedback form

$$\ddot{x}(t) + 2\zeta\omega_n\dot{x}(t) + \omega_n^2 x(t) = g(t) - \lambda\omega_n^2 x^3(t) \tag{12}$$

while Fig. 1(b) represents the linearized form to be obtained through the iterative algorithm developed in this paper.

Note that the right hand side of Eq. (12) can be thought of the input excitation to the LTI system with the complex frequency response function given in Eq. (11). To recast Eq. (12) in matrix form, $g(t)$ and $x(t)$ must be expressed as vectors of the same size, where each row corresponds to a discrete frequency.

Consider a case where the periodic solutions can be calculated with sufficient accuracy, using Nf discrete frequencies, where Nf is chosen to include the input frequencies, as well as the frequencies where nontrivial solutions are expected. Then the input $g(t)$ can be rewritten to include Nf components as follows

$$g(t) = \sum_{r=1}^{Nf} (e_r \cos(r\omega_0 t) + f_r \sin(r\omega_0 t)) \tag{13}$$

Note that except under special conditions, Nf will exceed the number of frequencies necessary to define $g(t)$, and it will be necessary to add frequency components with $e_r = f_r = 0$ to the input signal. Although this process might seem an unnecessary burden, it leads to an algorithm that is more accurate and significantly faster than the classical HB formulation, as will be discussed in the conclusion part.

The periodic solutions of Eq. (10) when subjected to $g(t)$ take the form

$$x(t) = \sum_{r=1}^{Nf} (a_r \cos(r\omega_0 t) + b_r \sin(r\omega_0 t)) \tag{14}$$

where

$$a_r = \frac{e_r(k_{nl} + \omega_n^2 - (r\omega_0)^2) - f_r(2\omega_n(r\omega_0)\zeta)}{(k_{nl} + \omega_n^2 - (r\omega_0)^2)^2 + (2\omega_n(r\omega_0)\zeta)^2} \tag{15}$$

and

$$b_r = \frac{f_r(k_{nl} + \omega_n^2 - (r\omega_0)^2) + e_r(2\omega_n(r\omega_0)\zeta)}{(k_{nl} + \omega_n^2 - (r\omega_0)^2)^2 + (2\omega_n(r\omega_0)\zeta)^2} \tag{16}$$

respectively. For a linear system, the sine and cosine coefficients are obtained by substituting $k_{nl} = 0$ in Eqs. (15) and (16), respectively.

For harmonic input and response, the orthogonality of sine and cosine functions can be utilized to define the k_{nl} in Eq. (9). Substituting $x(t)$ and $g(t)$ in Eq. (9), multiplying both sides with $\omega_0/\pi x_r(t)$ and integrating the resulting expression from $t = 0$ to $t = 2\pi/\omega_0$ yields a value of k_{nl} that satisfies Eq. (9). The resulting k_{nl} value is frequency dependent, since its value changes with the response component x_r. This paper uses the notation $k_{nl,r}$, to represent the k_{nl} corresponding to the r^{th} frequency component, which is given by the equation

$$k_{nl,r}(a_r^2+b_r^2) = \frac{3}{4}\lambda\omega_n^2(a_rI_{rc}+b_rI_{rs}) \tag{17}$$

where

$$I_{rc} = \frac{4\omega_0}{3\pi}\int_0^{\frac{2\pi}{\omega_0}} x^3\cos(r\omega_0 t)dt \tag{18}$$

and

$$I_{rs} = \frac{4\omega_0}{3\pi}\int_0^{\frac{2\pi}{\omega_0}} x^3\sin(r\omega_0 t)dt \tag{19}$$

The term $k_{nl,r}$ in Eq. (17) can be simplified as

$$k_{nl,r} = c_r(a_rI_{rc}+b_rI_{rs}) \tag{20}$$

where c_r is a frequency dependent constant defined as

$$c_r = \frac{3\lambda\omega_n^2}{4(a_r^2+b_r^2)} \tag{21}$$

I_{rc} and I_{rs} functions in Eq. (18) and Eq. (19) can be written as

$$I_{rc} = \{a\}[C_{r2}]\{a\}^T+\{b\}[C_{r1}]\{b\}^T \tag{22}$$

and

$$I_{rs} = \{a\}[S_{r1}]\{a\}^T+\{b\}[S_{r2}]\{b\}^T \tag{23}$$

respectively. In Eq. (22) and Eq. (23), $\{a\}$ and $\{b\}$ are row vectors containing the cosine and sine coefficients of $x(t)$, respectively. If the number of frequency components being computed is Nf, the $\{a\}$ and $\{b\}$ will be vectors of size $1\times Nf$. The matrices $[C_{r1}]$, $[C_{r2}]$, $[S_{r1}]$, $[S_{r2}]$ are frequency dependent upper triangular matrices of size $Nf\times Nf$. Through algebraic manipulation, the elements of these four matrices are found to be

$$\begin{aligned} i\ge j \quad & C_{r1}(i,j) = \sum_{k=1}^{Nf} a_k(2(\delta_{j-i-|k-r|}+\delta_{j-i-k-r})+(\delta_{j-i}-2)(\delta_{j+i-|k-r|}+\delta_{j+i-k-r})); \\ i<j \quad & C_{r1}(i,j) = 0 \end{aligned} \tag{24}$$

$$\begin{aligned} i\ge j \quad & S_{r1}(i,j) = \sum_{k=1}^{Nf} b_k(2(\delta_{j-i-|k-r|}-\delta_{j-i-k-r})-(\delta_{j-i}-2)(\delta_{j+i-|k-r|}-\delta_{j+i-k-r})); \\ i<j \quad & S_{r1}(i,j) = 0 \end{aligned} \tag{25}$$

$$\begin{aligned} i\ge j \quad & C_{r2}(i,j) = C_{r1}(i,j)+4\delta_{j-i+k-r}\sum_{k=1}^{Nf}\left(\frac{1}{3}a_i\delta_{j-i}\delta_{j-k}+a_k(2-\delta_{j-i}-\delta_{j-k})\right); \\ i<j \quad & C_{r2}(i,j) = 0 \end{aligned} \tag{26}$$

$$\begin{aligned} i\ge j \quad & S_{r2}(i,j) = S_{r1}(i,j)+4\delta_{j-i+k-r}\sum_{k=1}^{Nf}\left(\frac{1}{3}b_i\delta_{j-i}\delta_{j-k}+b_k(2-\delta_{j-i}-\delta_{j-k})\right); \\ i<j \quad & S_{r2}(i,j) = 0 \end{aligned} \tag{27}$$

In Equations (24)-(27), $\delta(x)$ is the discrete unit impulse function defined as

$$\delta_x = \begin{cases} 1 & x = 0 \\ 0 & x \neq 0 \end{cases} \tag{28}$$

Through algebraic manipulation and simplification, $k_{nl,r}$ in Eq. (20) becomes

$$k_{nl,r} = \frac{3\lambda\omega_n^2}{4(a_r^2 + b_r^2)}(\{a\}(a_r[C_{r2}] + b_r[S_{r1}])\{a\}^T + \{b\}(a_r[C_{r1}] + b_r[S_{r2}])\{b\}^T) \tag{29}$$

Note that, the Eq. (29) involves the sine and cosine coefficients of the response, which are unknown. An iterative scheme can be used to calculate the $k_{nl,r}$ values, where the response of the linear system ($\lambda = 0$) is used as the initial estimate of the nonlinear response. Using Eqs. (24) to (29), the frequency dependent stiffness terms ($k_{nl,r}$) are calculated, and the natural frequency of the corresponding system is updated as

$$\omega_{n,r} = \sqrt{\omega_n^2 + k_{nl,r}} \tag{30}$$

The updated system, then, enables computation of an improved response estimate. The algorithm is repeated until the solution converges.

Fig. 2 depicts the idea used in the linearization algorithm described in this paper. When the algorithm converges, the r^{th} component of the response of the nonlinear system is given by the response of the linear system with damping ζ and natural frequency $\sqrt{\omega_n^2 + k_{nl,r}}$. Note that although the final $x_r(t)$ appears to be a direct mapping from the r^{th} component of the input, the $k_{nl,r}$ term accounts for the contribution from other frequencies as well.

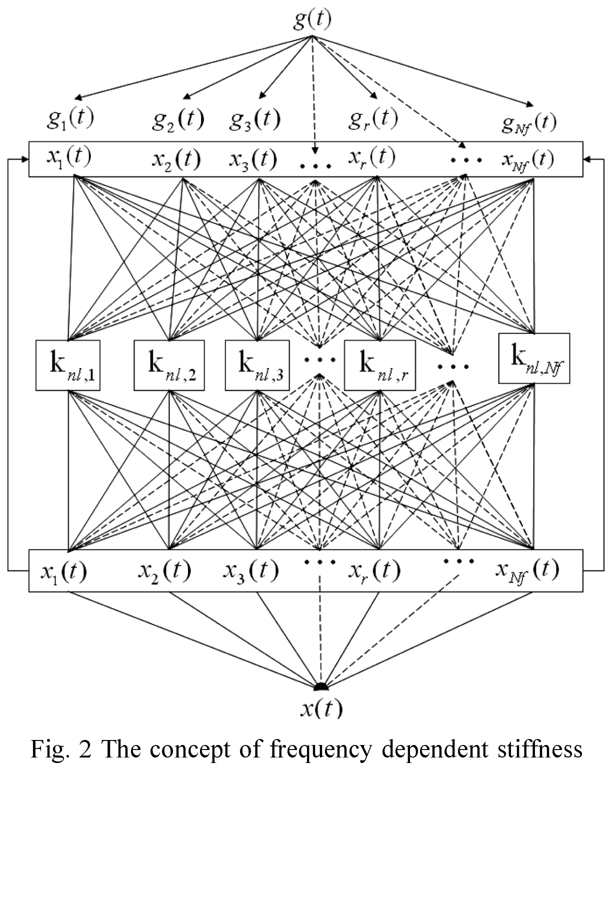

Fig. 2 The concept of frequency dependent stiffness

The matrix-based linearization procedure can be summarized as follows

Step 1: Find e_r and f_r satisfying Eq. (13)

Step 2: Initialize the a_r and b_r values by assuming linear response, using Eqs. (15) and (16).

Step 3: For all r, calculate $[C_{r1}]$ and $[S_{r1}]$ matrices using Eqs. (24) and (25), respectively.

Step 4: For all r, calculate $[C_{r2}]$ and $[S_{r2}]$ matrices using Eqs. (26) and (27), respectively.

Step 5: For all r, calculate the $k_{nl,r}$ values using Eq. (29), and update the natural frequency using $\omega_{n,r} = \sqrt{\omega_n^2 + k_{nl,r}}$.

Step 6: Calculate a_r and b_r values using the updated natural frequency in Eqs. (15) and (16).

Step 7: If the solution has not converged, go to step 3.

Step 8: Calculate the response using

$$x(t) = \sum_{r=1}^{Nf} x_r(t) = \sum_{r=1}^{Nf} (a_r \cos(r\omega_0 t) + b_r \text{cossin}(r\omega_0 t)) \tag{31}$$

It is very important to note that, although the method developed in this paper expresses the response of the Duffing equation as a sum of components representing each frequency, this summation should not be confused with linear superposition, as the components are not independent of each other. That is, while $x_r(t)$ represents the r^{th} frequency component in the solution, the effects from other frequencies are introduced through the $[C_{r1}]$, $[C_{r2}]$, $[S_{r1}]$ and $[S_{r2}]$ matrices. Hence, $x_r(t)$ does not model the response of the system when subjected to a harmonic force of frequency $r\omega_0$.

4. Application

In this section, we demonstrate how the linearization procedure scales the response amplitudes by using modified natural frequencies for each component. The system and load parameters are deliberately chosen such that the nonlinear response can be approximated by adjusting two natural frequency terms that match the frequency content of the input ($Nf = 2$). Primary and secondary resonances are eliminated by defining the natural frequency away from the frequency content of the applied load. Consider the system

$$\ddot{x} + 0.5\dot{x} + 1.6\pi^2(1 + \lambda x^2)x = g_1(t) \qquad x(0) = 0;\ \dot{x}(0) = 0 \tag{32}$$

Let $\lambda = 5$ and $g_1(t) = 50\cos(t) + 10\sin(t) + 20\cos(2t) + 30\sin(2t)$.

The load and the response of the linear system ($\lambda = 0$) are shown in Fig. 3.

The $k_{nl,r}$ and $\omega_{n,r}$ values resulting from the first five iterations of the linearization algorithm are presented in Table 1. As the table shows, convergence was achieved within a few iterations.

The time domain representations and the power spectra of the linear system response ($\lambda = 0$), the "exact" response as calculated from the 4th order Runge-Kutta algorithm, and the result of the linearization procedure described in this paper are presented in Fig. 4. The top row corresponds to the linear system ($\lambda = 0$), and has been provided as a tool to measure the deviation of the Duffing system from its linear counterpart. The second row represents the "exact solution calculated using a fourth order Runge-Kutta (RK4) algorithm. Comparison of the "exact" solution with the linear solution shows that, for this specific example, the effect of the nonlinearity manifests itself as an amplitude reduction, while the phase remains unaffected. Also, the relative strength of the frequency components is similar. The third row shows the "linearized" solution, at the end of the five iterations. Note that the "linearized" solution and the "exact" solution almost overlap after about 5

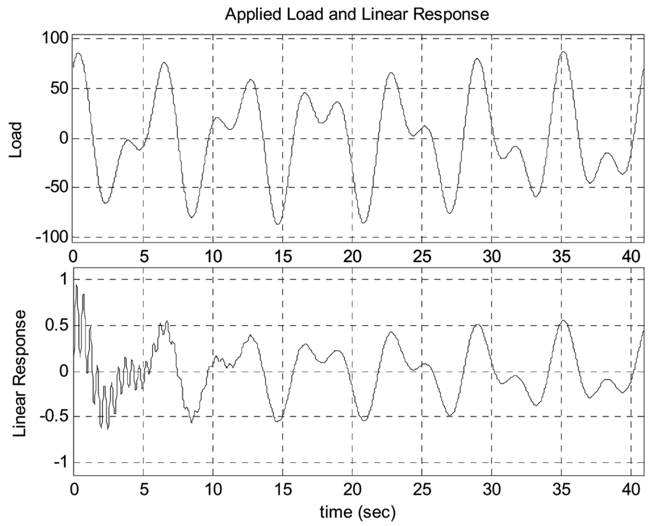

Fig. 3 Applied load and linear response for the example application

Table 1 $k_{nl,r}$ and $\omega_{n,r}$ values from the first five iterations

	Iteration 1	Iteration 2	Iteration 3	Iteration 4	Iteration 5
$k_{nl,1}$	127.1190	49.5494	59.0641	57.1040	57.4799
$k_{nl,2}$	70.4833	104.4850	98.2742	99.4863	99.2512
$\omega_{n,1}$	16.8829	14.4036	14.7302	14.6635	14.6763
$\omega_{n,2}$	15.1128	16.1987	16.0059	16.0437	16.0364

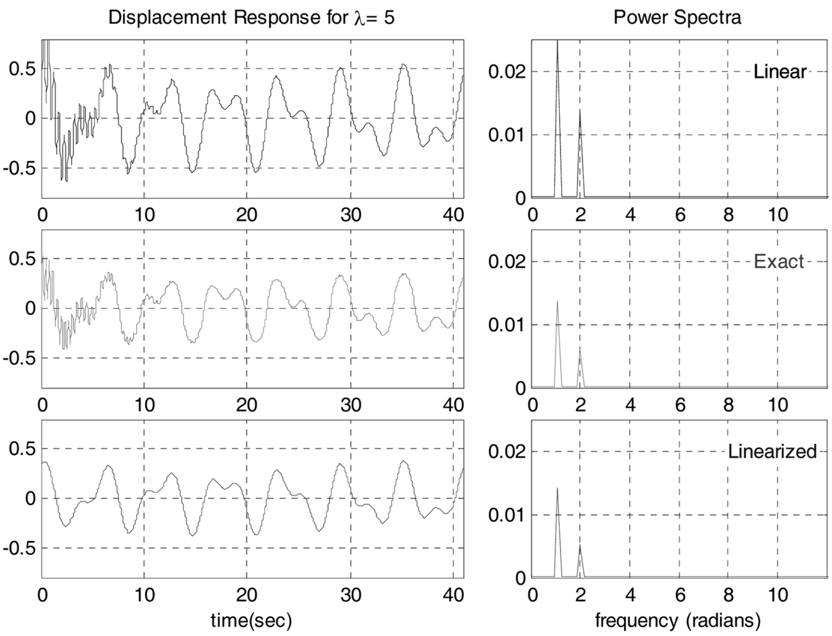

Fig. 4 Displacement response and power spectra for the example application (after 5 iterations)

seconds, and the power spectra of the two are almost the same.

5. Conclusions

This paper introduces a linearization algorithm to find the periodic solutions of Duffing's equation around the origin. The new algorithm replaces the integrals appearing in the HB method with triangular matrices. An example application of the proposed method has been presented. The example shows how the method scales the response amplitudes by using modified natural frequencies for each component when the natural frequency of the system is away from the frequency content of the applied load.

The triangular matrices used in the algorithm lead to an increased efficiency when compared to calculating the integrals that appear in the classical HB formulation. Five iterations of the example presented above, required only 0.05 seconds of CPU time, whereas the classical HB method took 4.25 sec. The difference becomes more critical with increasing number of frequencies, since the number of terms in the cube of the response function for Nf frequencies is $(8Nf^3 + 12Nf^2 + 4Nf)/6$, whereas the number of operations in the matrix-based linearization increases only linearly with Nf, due to the triangular form of the matrices used.

The mean squared error, as measured from the steady-state part of the Runge-Kutta solution, is smaller for the matrix based linearization than the classical HB method, although the two responses are virtually indistinguishable when plotted on the same graph. The accuracy of the classical HB method, relative to the matrix-based linearization, decreases with increased number of iterations. We believe this is direct result of the increased round-off errors, that accumulate faster in the classical HB method, rather than a difference caused by the formulation of the two methods.

It should be noted that the linearization algorithm developed in this paper, although being computationally efficient and flexible, is subject to the limitations inherent in all linearization algorithms and it only gives the stable periodic solutions of the Duffing equation. Therefore, the result obtained from this algorithm should not be treated as the total solution of the Duffing equation, which can exhibit several nonlinear phenomena that cannot be captured through linearization.

References

Caughey, T.K. (1963), "Equivalent linearization techniques", *J. Acoust. Soc. Am.*, **35**(11), 1706-1711.

Caughey, T.K. (1971), "Nonlinear theory of random vibrations", *Adv. Appl. Mech.*, **11**, 209-253.

Iwan, W.D. (1969), "On defining equivalent systems for certain ordinary nonlinear differential equations", *Int. J. Nonlinear Mech.*, **4**, 325-334.

Jordan, D.W. and P. Smith (1999), "Nonlinear ordinary differential equations: An introduction to dynamical systems", Oxford University Press.

Krylov, N.N. and N.N. Bogoliubov (1947), *Introduction to Nonlinear Mechanics*. Princeton University, Princeton.

Liu, L., J.P. Thomas, *et al.* (2006), "A comparison of classical and high dimensional harmonic balance approaches for a Duffing oscillator", *J. Comput. Phys.*, **215**(1), 298-320.

Nayfeh, A.H. and D.T. Mook (1979), *Nonlinear Oscillations*, John Wiley, New York.

Roberts, J. and P.D. Spanos (1986), *Random Vibration and Statistical Linearization*, Wiley, New York.

Spanos, P.D. and W.D. Iwan (1978), "On the existence and uniqueness of solutions generated by equivalent linearization", *Int. J. Nonlinear Mech.*, **13**(2), 71-78.

Tezcan, J. and P.D. Spanos (2006), "A numerical approach for nonlinear system response spectrum determination via wavelets", *Proceedings of the 5th International Conference on Computational Stochastic Mechanics*, Rodos, Greece, Mill-Press International.

Urabe, M. and A. Reiter (1964). *Numerical Computation of Nonlinear Forced Oscillations by Galerkin's Procedure*. Mathematics Research Center, University of Wisconsin.