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The elastoplastic formulation of polygonal element method based on triangular finite meshes

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Abstract. A small strain and elastoplastic formulation of Polygonal Element Method (PEM) is developed for efficient analysis of elastoplastic solids. In this work, the polygonal elements are constructed based on traditional triangular finite meshes. The construction method of polygonal mesh can directly utilize the sophisticated triangularization algorithm and reduce the difficulty in generating polygonal elements. The Wachspress rational finite element basis function is used to construct the approximations of polygonal elements. The incremental variational form and a von Mises type model are used for non-linear elastoplastic analysis. Several small strain elastoplastic numerical examples are presented to verify the advantages and the accuracy of the numerical formulation.

Keywords: elastoplasticity; polygonal finite element method; Wachspress basis functions; meshless methods.

1. Introduction

The original idea of finite element was proposed by Courant in 1943. As an important tool for solving Partial Differential Equation (PDE) boundary value problems, Finite Element Method (FEM) has become a ubiquitous tool in scientific research and engineering practice (Bathe 1996). In two-dimensional FEM analysis, the constant strain three-node triangular and the bilinear four-node quadrangular elements are most commonly adopted by commercial software due to the simple principle and the mature meshing algorithm in preprocessing.

Recently, Polygonal Finite Element Method (PFEM) has gained much attention, and advances were made in this aspect. The shape functions on polygons can be mainly represented by the rational basis function by Wachspress (1975) and natural neighbour-based Laplace interpolation as well as Sibson interpolation (Sukumar and Moran 2001). Extensions of polygonal element methods

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based on the above methods were seen in Voronoi Cell Finite Element Method (VCFEM) (Ghost 1994), Natural Element Method (NEM) (Sukumar and Moran 1998), Natural Neighbor Galerkin Method (NNGM) (Sukumar and Moran 2001), Conforming Polygonal Finite Elements (CPFE) (Sukumar and Tabarraei 2004, Sukumar and Malsch 2006), and etc. The computational advantages of these interpolations are overwhelming. It satisfies the kronecker-delta property, meets the linear completeness requirement, and conforms the approximation between elements. It is simple and efficient in simulating mechanical property of material. It is greatly flexible in meshing of complex geometry and less sensitive to shear lock. It also has better numerical accuracy and higher rate of convergence over triangular element and quadrangular element.

Although research on polygonal element has progressed in recent years, it is at its enfant stage compared with conventional FEM and many problems need to be further studied. In this work, the polygonal elements are constructed based on traditional triangular finite meshes. The construction method of polygonal mesh can directly utilize the sophisticated triangularization algorithm and reduce the difficulty in generating polygonal element. A small strain elastoplastic formulation of Polygonal Element Method (PEM) is developed for efficient analysis of elastoplastic solids. Several small strain elastoplastic numerical examples are given to demonstrate the accuracy of the proposed method.

This paper is organized as follows: Section 2 describes the algorithm to generate polygonal mesh based on triangular finite element; Section 3 introduces the Wachspress interpolation on polygons and the mapping method from irregular polygonal element to canonical element; Section 4 provides the small strain elastoplastic formulation; Section 5 gives several numerical examples to demonstrate the accuracy and efficiency of the proposed method; Section 6 brings out the conclusion.

2. Constructing algorithm for the polygonal elements

To directly construct polygonal mesh for an arbitrarily shaped analysis domain, like that of triangular meshing or quadrangular meshing in FEM, was a most daunting task due to the complexity of meshing algorithm. As the Voronoi diagram can be constructed through some relatively mature algorithm, a commonly adopted method was to construct the polygonal element by adjusting the Voronoi diagram of the analysis domain and then use the treated polygonal mesh for analysis (Sukumar and Tabarraei 2004, Sukumar and Malsch 2006). However, some Voronoi vertices could be very close to each other, and thus the polygonal mesh could become distorted. Also, it is a rather complicated procedure when the polygonal element is to be constructed on the boundary of the analysis domain by using Voronoi diagram. Therefore, it becomes practically difficult to construct acceptable polygonal element by employing Voronoi diagram.

The following is the procedure used to construct polygon mesh in this paper.

For an arbitrary analysis domain, firstly we generate finite element mesh shown in Fig. 1 by using mature triangular meshing algorithm. With today's technology, this is realized without any difficulty.

Let the coordinate of the vertices of triangle be $(x_1, y_1), (x_2, y_2), (x_3, y_3)$ respectively and the central point of element is defined as $((x_1+x_2+x_3)/3, (y_1+y_2+y_3)/3)$. In Fig. 1, for any interior point i, all six of its surrounding triangles could be found, such as <1>-<6>. Connecting the central points of these triangles one by one, a polygonal element can be formed.

For any given node k on the boundary in Fig. 1, its surrounding elements are (1), (2) and (3). A

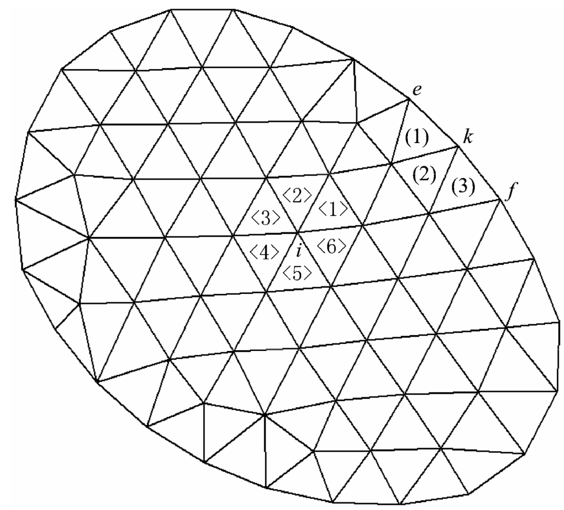

Fig. 1 Triangular finite element mesh for an arbitrary analysis domain

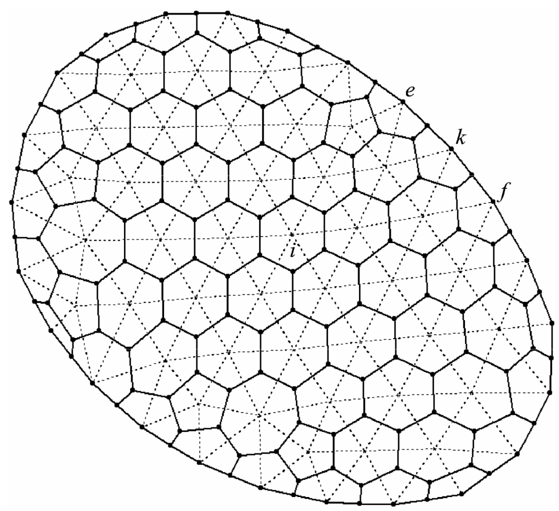

Fig. 2 Construction of polygonal mesh for an arbitrary analysis domain

polygonal element with vertices surrounding k is formed when the centers of the associate elements and the mid points of the associate sides (k-e and k-f) are connected.

In similar approach, polygonal mesh shown in Fig. 2 could be constructed on the basis of triangular mesh as is shown in Fig. 1. The black dots in Fig. 2 are nodes of polygonal element. The number of the polygonal elements equals to the number of the nodes of triangular mesh in Fig. 1. This simply states that fewer polygonal elements in Fig. 2 are generated than triangular elements.

3. Wachspress interpolation

Rational basis function was firstly developed by Wachspress (1975) to build shape function on irregular polygons. The Wachspress interpolation based on perspective geometry satisfies the kronecker property and precisely linearity and continuity on boundary side. However, due to its complicated construction process, it has received limited attention long time after.

The past several years has seen the rising interests on improving the Wachspress function which helps to improve the usability of Wachspress interpolation (Dasgupta 2003, Dikshit and Ojha 1991, Dikshit and Ojha 2002, Elisabeth and Gautam 2004, Floater 2003, Laydi and Aoubiza 1995, Meyer and Lee 2002, Powar and Rana 1991). In this paper, a generalized barycentric co-ordinate for n-gons by Meyer and co-workers (Meyer and Lee 2002) was used to calculate the shape function of polygonal element according to the following.

$$\Phi_i(\mathbf{x}) = \frac{w_i(\mathbf{x})}{\sum_{j=1}^{n} w_j(\mathbf{x})}, \quad w_i(\mathbf{x}) = \frac{A(p_{i-1}, p_i, p_{i+1})}{A(p_{i-1}, p_i, p)A(p_i, p_{i+1}, p)} = \frac{\cot\gamma_i + \cot\delta_i}{\|\mathbf{x}-\mathbf{x}_i\|^2}, \tag{1}$$

In the above equation, $A(a, b, c)$ is the signed area of triangle $[a, b, c]$, and γ_i and δ_i are show in Fig. 3(a).

Meyer and co-workers derived the coordinate expression for $\cot\delta_i$ and $\cot\gamma_i$ where (a_1, a_2),

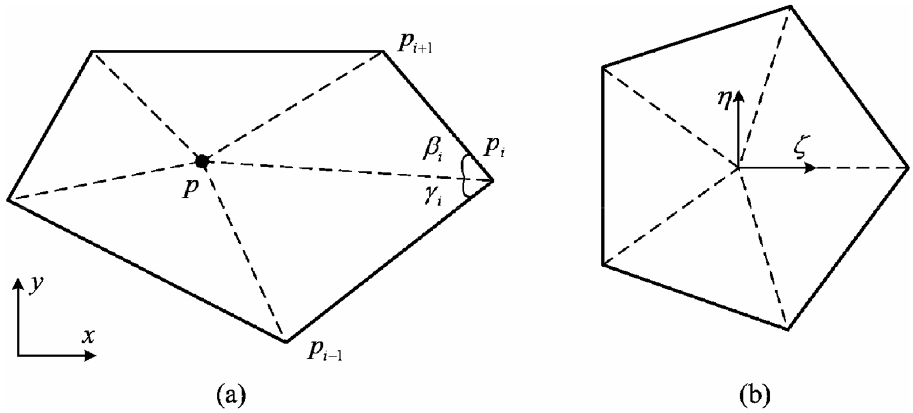

Fig. 3 The calculation of Wachspress shape function for an arbitrary polygonal element

(b_1, b_2) and (x, y) denote the co-ordinates of the vertices of triangle $[p_i, p_{i+1}, p]$ and come to the following

$$\cot\delta_i = \frac{(\mathbf{p}_{i+1}-\mathbf{p}_i)\cdot(\mathbf{p}-\mathbf{p}_i)}{|(\mathbf{p}_{i+1}-\mathbf{p}_i)\times(\mathbf{p}-\mathbf{p}_i)|} = \frac{(b_1-a_1)(x-a_1)+(b_2-a_2)(y-a_2)}{(b_1-a_1)(y-a_2)-(x-a_1)(b_2-a_2)} \tag{2}$$

and $\cot\gamma_i$ can be calculated in the same manner.

Expression (1) and (2) are only applicable to calculate Wachspress shape functions for convex polygons. Whereas the polygonal element obtained by the proposed method are of arbitrary geometry and convex requirement might be violated. Thus the isoparametric map by Sukumar and Tabarraei (2004) is utilized in this paper. For instance, an arbitrary polygonal element shown in Fig. 3(a) was mapped to canonical element shown in Fig. 3(b). The Wachspress shape functions was evaluated first in canonical element and then the interpolation was obtained for physical element. The isoparametric mapping between Fig. 3(a) and Fig. 3(b) are determined by

$$x = \sum_{i=1}^{n}\Phi_i(\zeta, \eta)x_I, \quad y = \sum_{i=1}^{n}\Phi_i(\zeta, \eta)y_i \tag{3}$$

In the above equation, n is the number of vertices of polygons and x_i, y_i denotes the coordinates of vertices.

The rational function based on Wachspress interpolation on polygon is expressed as

$$\mathbf{u}^h(\mathbf{x}) = \sum_{i=1}^{n}\Phi_i(\zeta, \eta)\mathbf{u}_i = \Phi\cdot a \tag{4}$$

Noting that except for the above Wachspress interpolation, the Point Interpolation (Liu and Gu 2001), the Radial Point Interpolation (Liu *et al.* 2002), and other meshless approximations (Lu *et al.* 2006, Most and Bucher 2005) can also be employed to construct the approximation of the polygonal element in Fig. 3(a) and Fig. 3(b) and are capability of developing different types of PFEM.

4. Small strain elastoplastic formulation

Consider a two-dimensional problem with small displacements on the domain Ω bounded by Γ. The elastoplastic problem can be described in incremental variational form

$$\begin{aligned}&\iint_{\Omega} \delta(\Delta\boldsymbol{\varepsilon})^{T\,\tau}\mathbf{D}_{ep}\Delta\boldsymbol{\varepsilon} d\boldsymbol{\Omega} - \iint_{\Omega} \delta(\Delta\mathbf{u})^{T}\Delta\mathbf{F} d\boldsymbol{\Omega} - \int_{\Gamma_t} \delta(\Delta\mathbf{u})^{T}\Delta\mathbf{T} d\boldsymbol{\Gamma} \\ &= -\iint_{\Omega} \delta(\Delta\boldsymbol{\varepsilon})^{T\,t}\boldsymbol{\sigma} d\boldsymbol{\Omega} + \iint_{\Omega} \delta(\Delta\mathbf{u})^{T\,t}\mathbf{F} d\boldsymbol{\Omega} + \int_{\Gamma_t} \delta(\Delta\mathbf{u})^{T\,t}\mathbf{T} d\boldsymbol{\Omega}\end{aligned} \tag{5}$$

Where ${}^t\boldsymbol{\sigma}$ is the stress tensor for the time t; ${}^t\mathbf{F}$ is the body force vector for the time t; ${}^t\mathbf{T}$ is the traction boundary for the time t; $\Delta\boldsymbol{\varepsilon}$ and $\Delta\mathbf{u}$ are the incremental strain and displacement for the time t.

The incremental form of Eq. (4) is

$$\Delta\mathbf{u}^h(\mathbf{x}) = \Phi\cdot\Delta\mathbf{a} \tag{6}$$

The discretized system can be obtained by substituting Eq. (6) into Eq. (5)

$$\mathbf{K}_{ep}\cdot\Delta\mathbf{a} = \Delta\mathbf{Q} \tag{7}$$

where

$$\mathbf{K}_{ep} = \sum_e \mathbf{K}_{ep}^e, \quad \Delta\mathbf{a} = \sum_e \Delta\mathbf{a}^e, \quad \Delta\mathbf{Q} = \sum_e {}^{t+\Delta t}\mathbf{Q}_l^e - \sum_e {}^{t+\Delta t}\mathbf{Q}_i^e$$

and

$$\mathbf{K}_{ep}^e = \int_{\Omega_e} \mathbf{B}^T\cdot\mathbf{D}_{ep}\cdot\mathbf{B} d\Omega \tag{7a}$$

$${}^{t+\Delta t}\mathbf{Q}_l^e = \int_{\Omega_e} \mathbf{\Phi}^T\cdot{}^{t+\Delta t}\mathbf{F} d\Omega + \int_{\Gamma_t} \mathbf{\Phi}^T\cdot{}^{t+\Delta t}\mathbf{T} d\Gamma \tag{7b}$$

$${}^t\mathbf{Q}_i^e = \int_{\Omega_e} \mathbf{B}^T\cdot{}^t\boldsymbol{\sigma} d\Omega \tag{7c}$$

$\mathbf{B}$ is the strain matrix; ${}^{t+\Delta t}\mathbf{Q}_l^e$ is the external load vector; ${}^t\mathbf{Q}_i^e$ is the internal load vector; $\mathbf{D}_{ep} = \mathbf{D}_e - \mathbf{D}_p$ is the elastoplastic matrix.

The linear isotropic hardening law and the von Mises yield criterion are employed in this work. The von Mises yield criterion is written as follows

$$f(\boldsymbol{\sigma}, \bar{\boldsymbol{\varepsilon}}^p) = \sqrt{\frac{3}{2}S_{ij}S_{ij}} - \sigma_Y(\bar{\boldsymbol{\varepsilon}}^p) \tag{8}$$

where S_{ij} denotes the stress deviator tensor and σ_Y denotes the yield stress; $\bar{\boldsymbol{\varepsilon}}^p$ indicates effective plastic strain.

Using the associated flow rule and the consistency condition, we get

(a) the plastic matrix $\mathbf{D}_p$ for plane stress case as

$$\mathbf{D}_p = \frac{E}{Q}(1-\mu^2)\begin{bmatrix}(S_x+\mu S_y)^2 & (S_x+\mu S_y)(\mu S_x+S_y) & (S_x+\mu S_y)(1-\mu)\tau_{xy} \\ & (\mu S_x+S_y)^2 & (\mu S_x+S_y)(1-\mu)\tau_{xy} \\ sym. & & (1-\mu)^2\tau_{xy}^2\end{bmatrix} \tag{9}$$

where, $Q = S_x^2 + S_y^2 + 2\mu S_x S_y + 2(1-\mu)\tau_{xy}^2 + \frac{4(1-\mu^2)\sigma_i^2 H'}{9E}$; $\sigma_i = \sqrt{\frac{3}{2}S_{ij}S_{ij}}$; E is the Young's modulus; μ is the Poisson's ratio; H' is the tangent modulus.

(b) the plastic matrix $\mathbf{D}_p$ for plane strain case as

$$\mathbf{D}_p = \frac{9G^2}{(H'+3G)\sigma_i^2}\begin{bmatrix} S_x^2 & S_x S_y & S_x \tau_{xy} \\ & S_y^2 & S_y \tau_{xy} \\ sym. & & \tau_{xy}^2 \end{bmatrix} \tag{10}$$

where $G = E/2(1+\mu)$ is the shear modulus.

Assume that at the time t, the stress state is in an elastic state and enters into an plastic state in the time $t + \Delta t$. In such a case, there exists a scaling factor r which can be decided by

$$f({}^t\boldsymbol{\sigma} + r\Delta\boldsymbol{\sigma}^e, {}^t\bar{\boldsymbol{\varepsilon}}^p) = 0 \tag{11}$$

where the trial stress increment $\Delta\boldsymbol{\sigma}^e = \mathbf{D}_e \Delta\boldsymbol{\varepsilon}^e$. Hence, the stress increment can be obtained as

$$\Delta\boldsymbol{\sigma} = r\Delta\boldsymbol{\sigma}^e + \int_{{}^m\boldsymbol{\varepsilon}+r\Delta\boldsymbol{\varepsilon}}^{{}^{t+\Delta t}\boldsymbol{\varepsilon}} \mathbf{D}_{ep} d\boldsymbol{\varepsilon} = 0 \tag{12}$$

The explicit method or implicit method can be used for the evaluation of the above integrations. For ideal plasticity materials, we needn't to compute the integrations of Eq. (12).

For the purpose of numerical integration of Eq. (7), the quadrature rules on a triangle are used by sub-dividing the polygonal element V_0 in Fig. 3(b) into sub-triangles. The polygonal element in Fig. 3(b) is a canonical element in a local coordinate system (ζ, η) with (0,0) as its center, and hence the canonical element V_0 can be easily divided into n sub-triangles as shown in Fig. 3(b). The numerical integration for a function $G(x,y)$ over V_i (a n-gons) is written as

$$\int_{V_0} G(x,y)dxdy = \int_{V_0} G_1(\zeta, \eta)|\mathbf{J}|d\zeta d\eta = \sum_{j=1}^{n} G_2(\mathbf{L}_j) \cdot A_j \cdot w_j \tag{13}$$

where, A_j is the jth sub-triangle in Fig. 3(b) and w_j is the weight of the quadrature point. In the present work, three Gaussian points are used in each sub-triangle.

Newton-Raphson method is used for solving the nonlinear Eq. (7).

5. Numerical examples

The present Polygonal Finite Element Method (PFEM) is coded in standard C++. Cases are run in order to examine the PFEM in two dimensional elastoplastic analysis.

5.1 Thick cylinder

Consider an axially restrained thick cylinder of inside radius $a = 10\,m$, outside radius $b = 20\,m$ and thickness $t = 1\ m$ as shown in Fig. 4(a). A distributed pressure $p = 12\ KPa$ is applied to the inner boundary of cylinder. The material is ideal elastoplastic with Young's modulus $E = 85570$ kPa

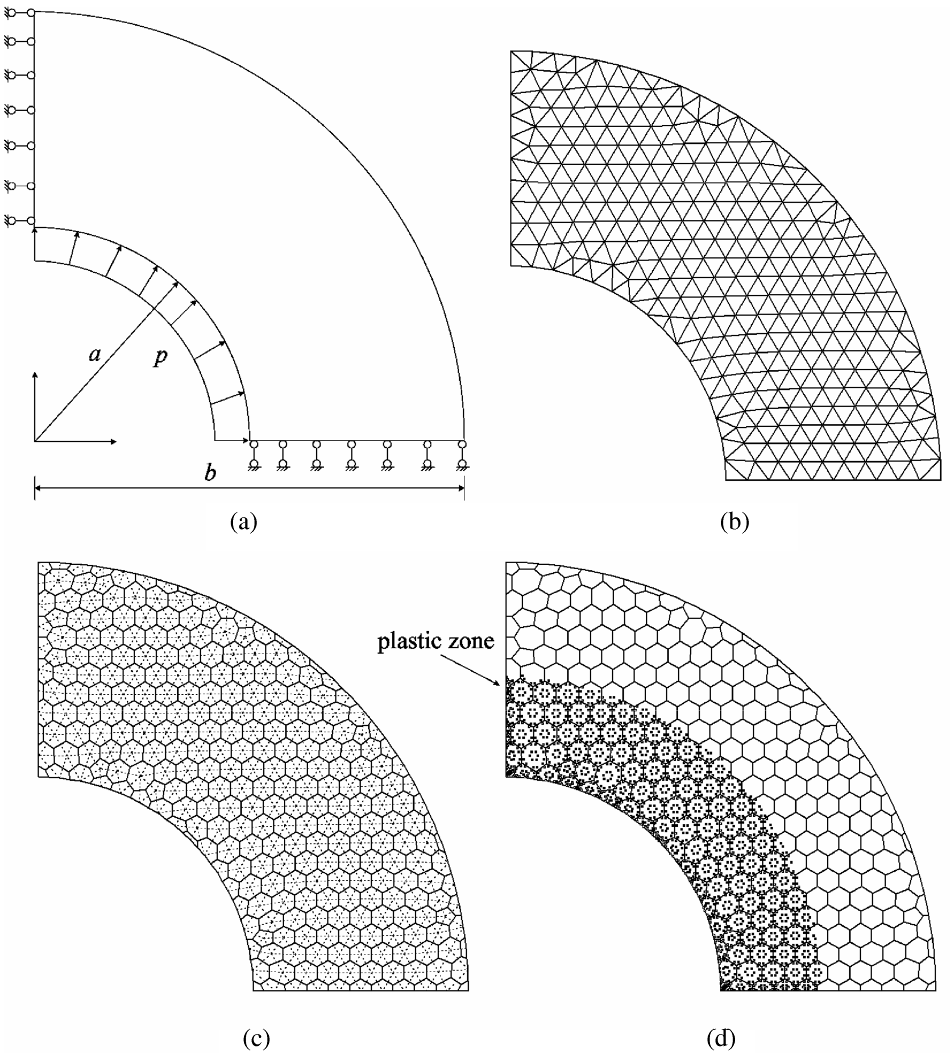

Fig. 4 Thick cylinder: (a) model, (b) triangular meshes, (c) polygonal meshes, (d) plastic zone

and Poisson's ratio μ = 0.3. The von Mises yield criterion is adopted and the initial yield stress σ_s = $10\sqrt{3}kPa$. The problem is solved for the plane stress case. The solution to this problem is given as follows (Chen 2003)

In plastic zone

$$\sigma_r = -p + \frac{2}{\sqrt{3}}\sigma_s \ln\frac{r}{a}, \quad \sigma_\theta = -p + \frac{2}{\sqrt{3}}\sigma_s\left(1 + \ln\frac{r}{a}\right) \tag{14}$$

In elastic zone

$$\sigma_r = -\frac{\sigma_s}{\sqrt{3}}\frac{r_s^2}{b^2}\left(\frac{b^2}{r^2} - 1\right), \quad \sigma_\theta = \frac{\sigma_s}{\sqrt{3}}\frac{r_s^2}{b^2}\left(\frac{b^2}{r^2} + 1\right) \tag{15}$$

where σ_r is the radial stress; σ_θ is the shear stress; r_s is the plastic radius and can be determined by

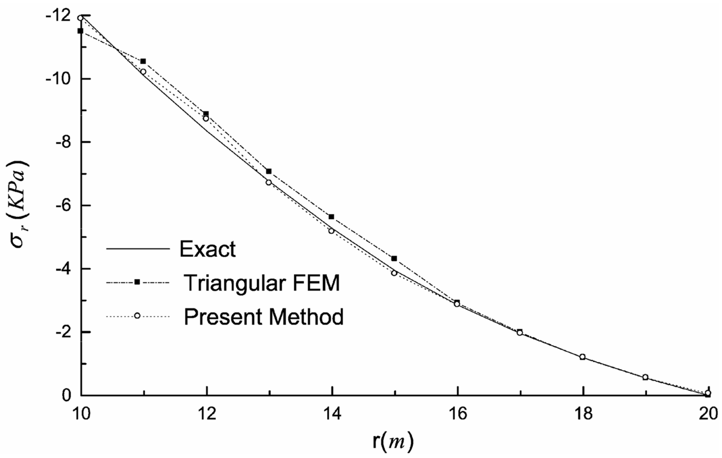

Fig. 5 Comparison of the radial stress for cylinder problem

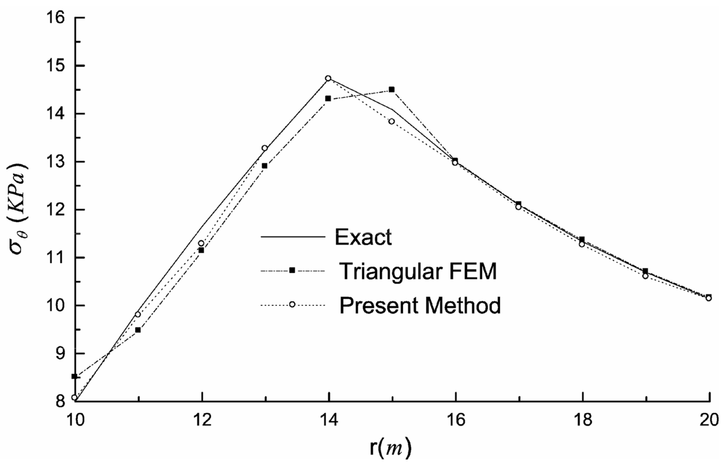

Fig. 6 Comparison of shear stress for cylinder problem

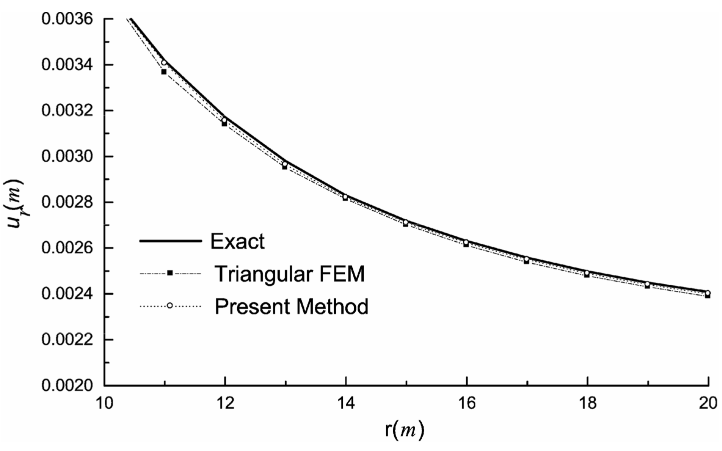

Fig. 7 Comparison of displacement u_r for cylinder problem

$$p = \frac{2\sigma_s}{\sqrt{3}}\left[\ln\frac{r_s}{a} + \frac{1}{2}\left(1 - \frac{r_s^2}{b^2}\right)\right] \tag{16}$$

The traditional triangular elements and the respective polygonal elements produced by the algorithm of Section 2 are shown in Fig. 4(b) and Fig. 4(c).

The plastic zone (yield integral points) computed by PFEM is shown in Fig. 4(d). The comparisons of the radial stress and the shear stress are shown in Fig. 5 and Fig. 6. The comparison of displacement u_r is shown in Fig. 7. We can find that the results obtained by PFEM are in good agreement with the analytical solutions.

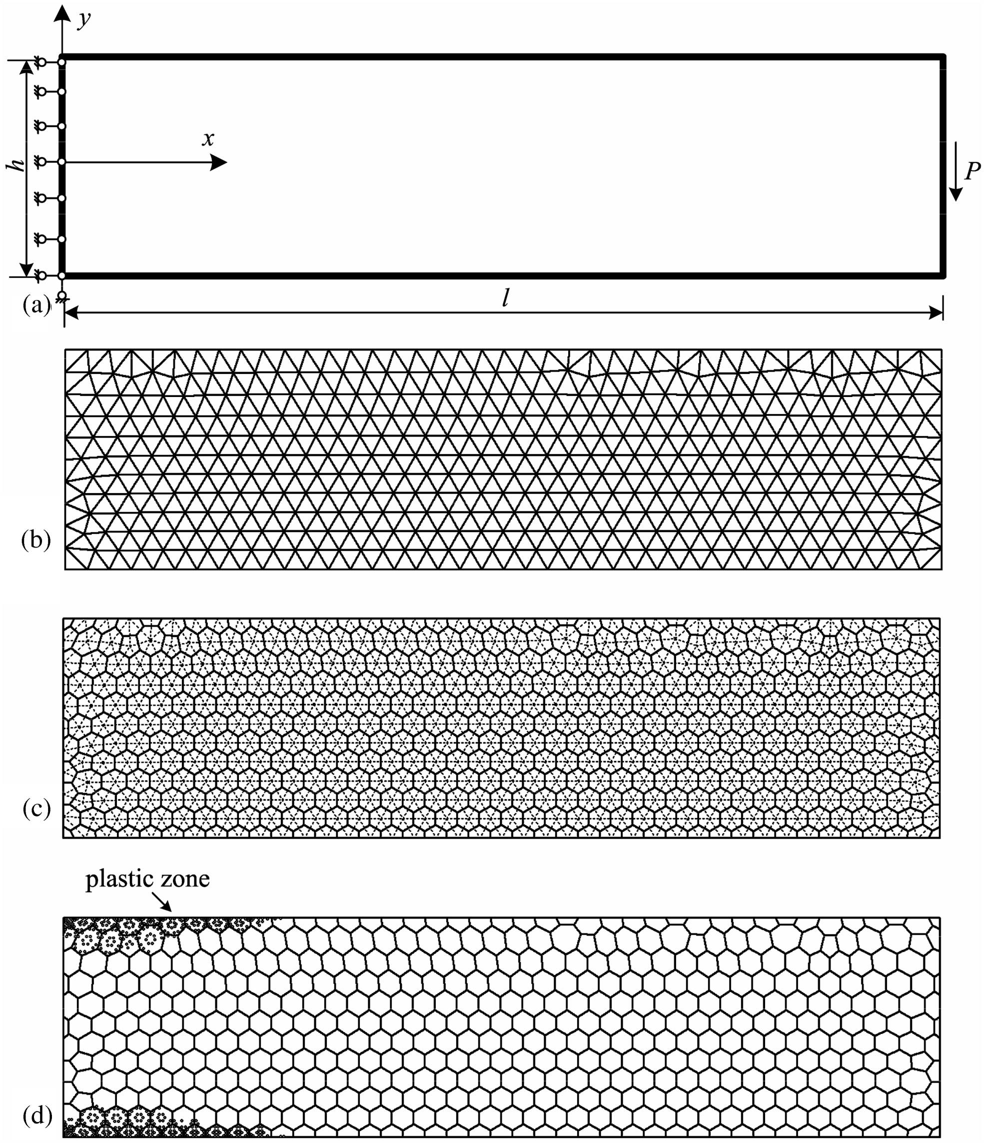

Fig. 8 Cantilever beam:(a) model, (b) triangular meshes, (c) polygonal meshes, (d) plastic zone

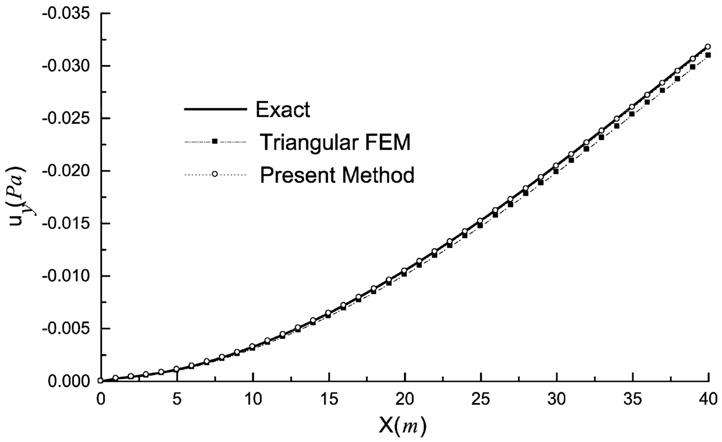

Fig. 9 Comparison of displacement u_y at y = 0 for Cantilever beam

5.2 Cantilever beam

The second example considered here is a rectangular cantilever beam of length l = 40 m and height h = 10 m as shown in Fig. 8(a). The beam is fixed at one end and subjected to concentrated load P = 11 kN at the free end. The material is ideal elastoplastic and the Von Mises yield criterion is adopted. The material parameters are E = 86670 kPa, μ = 0.3 and $\sigma_s = 10\sqrt{3}kPa$. Plane strain condition is assumed in this case.

The traditional triangular elements and the respective polygonal elements produced by the algorithm of Section 2 are shown in Fig. 8(b) and Fig. 8(c).

The plastic zone (yield integral points) obtained by PFEM is shown in Fig.8(d). The comparisons of the displacement u_y are shown in Fig. 9.

6. Conclusion

In this paper, a method for polygonal mesh generation based on traditional triangular finite meshes was adopted. This method can overcome the difficulty in generating polygon mesh and can serves as a good basis for further research and for the application of polygonal finite element method. The rational finite element basis function proposed by Wachspress and revised by Meyer and co-workers, was used to construct the approximations of polygonal element. An incremental form of small strain elastoplastic formulation of Polygonal Element Method (PEM) is developed for efficient analysis of two-dimensional solids. Several numerical examples are given to demonstrate the efficiency and accuracy of the proposed method.

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References

Bathe, K.J. (1996), *Finite Element Procedures*, Prentice-Hall, Englewood Cliffs, NJ.

Chen, W.F. (2003), *Elasticity and Plasticity*, China Architecture & Building Press.

Dasgupta, G. (2003), "Integration within polygonal finite elements", *J. Aerospace Eng.*, **16**(1), 9-18.

Dikshit, H.P. and Ojha, A. (1991), "Dimensions of spaces of Wachspress type C1 rational finite element", *Comput. Math. Appl.*, **22**(3), 23-26.

Dikshit, H.P. and Ojha, A. (2002), "On C1-Continuity of Wachspress quadrilateral patches", *Comput. Aided Geom. D.*, **19**, 207-222.

Elisabeth, A.M. and Gautam, D. (2004), "Interpolations for temperature distributions: a method for all non-concave polygons", *Int. J. Numer. Meth. Eng.*, **41**(1), 2165-2188.

Floater, M.S. (2003), "Mean value coordinates", *Comput. Aided Geom. D.*, **20**, 93-99.

Ghost, S. and Mallett, R.L. (1994), "Voronoi cell finite elements", *Comput. Struct.*, **50**(1), 33-46.

Laydi, M.R. and Aoubiza, B. (1995), "Wachspress rational finite element of arbitrary degree", *Comptes Rendus de L Academie Des Sciences Serie I-Mathematique*, **320**(11), 1391-1394.

Liu, G.R. and Gu, Y.T. (2001), "A local point interpolation method for stress analysisof two-dimensional solids", *Struct. Eng. Mech.*, **11**(2), 221-236.

Liu, G.R., Yan, L., Wang, J.G. and Gu, Y.T. (2002), "Point interpolation method based on local residualformulation using radial basis functions", *Struct. Eng. Mech.*, **14**(6), 713-732.

Lu, X.Z., Jiang, J.J. and Ye, L.P. (2006), "A composite crack model for concrete based on meshless method", *Struct. Eng. Mech.*, **23**(3), 217-232.

Meyer, M., Lee, H., Barr, A.H. and Desbrun, M. (2002), "Generalized barycentric coordinates on irregular polygons", *J. Graphics Tools*, **7**(1), 13-22.

Most, T. and Bucher, C. (2005), "A Moving Least Squares weighting function for the Element-free Galerkin Method which almost fulfills essential boundary conditions", *Struct. Eng. Mech.*, **21**(3), 315-332.

Powar, P.L. and Rana, S.S. (1991), "A counterexample of the construction of C1 rational finite element due to Watchspress", *Comput. Math. Appl.*, **22**(3), 17-22.

Sukumar, N., Moran, B. and Semenov, Y. (2001), "Natural neighbour galerkin method", *Int. J. Numer. Meth. Eng.*, **50**, 1-27.

Sukumar, N., Moran, B. and Belytschko, T. (1998), "The natural element method in solid mechanics", *Int. J. Numer. Meth. Eng.*, **43**, 839-887.

Sukumar, N. and Tabarraei, A. (2004), "Conforming polygonal finite elements", *Int. J. Numer. Meth. Eng.*, **61**, 2045-2066.

Sukumar, N. and Malsch, E.A. (2006), "Recent advances in the construction polygonal finite interpolants", *Arch. Comput. Meth. Eng.*, **13**(1), 129-163.

Wachspress, E.L. (1975), *A Rational Finite Element Basis*, New York: Academic Press, Inc.