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Saturation impulses for dynamically loaded structures with finite-deflections

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Abstract. The concept of "Saturation Impulse" for rigid, perfectly plastic structures with finite-deflections subjected to dynamic loading was put forward by Zhao, Yu and Fang (1994a). This paper extends the concept of Saturation Impulse to the analysis of structures such as simply supported circular plates, simply supported and fully clamped square plates, and cylindrical shells subjected to rectangular pressure pulses in the medium load range. Both upper and lower bounds of nondimensional saturation impulses are presented.

Key words: structures; rigid, perfectly plastic; finite-deflections; rectangular pressure pulse; saturation impulse; lower bounds; upper bounds.

1. Introduction

Over the past four decades, the dynamic plastic response and failure of structures subjected to large dynamic loading have been studied extensively (Jones 1989, Yu 1992) because of their practical applications. However, as structural configurations have become more varied, the requirements to determine their dynamic plastic behavior and failure have begun to increase dramatically (Zhao, *et al.* 1993, 1994b, Zhao 1994).

The concept of Saturation Impulse was put forward by Zhao, Yu and Fang (1994a), which concerned the dynamic plastic response of simply or fully clamped beams with finite-deflections subjected to rectangular pressure pulses in both medium and high ranges. To avoid ambiguity, we firstly define the term saturation impulse. The saturation impulse is the critical value after which the final deflection of the structure will not increase with further continuously applied load. In the example of a simply supported beam subjected to medium rectangular pressure pulse, this can be explained as follows. It has been shown by many experiments that the collaps

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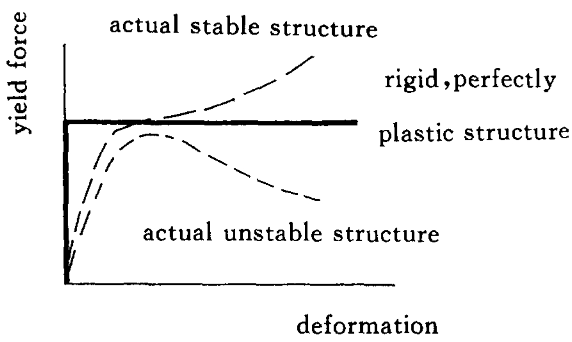

Fig. 1 Load-deformation relation for actual structures.

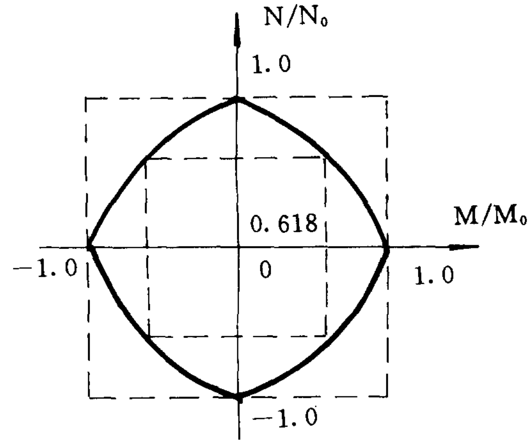

Fig. 2 Yield conditions.

loads for stable structures become larger with increase of the deflection (Yu 1989), as shown in Fig. 1, and this is why the deflection of a stable structure could not be infinite under its rigid, perfectly plastic collapse load. Once the pulse ratio is determined for a rectangular pressure pulse, an increase of the impulse means an increase in the duration of the applied load alone. When its deflection is large enough, the beam will be strengthened by the axial forces to such a extent that the continuously applied load will not produce further deflection, and then the deflection of the beam will remain constant.

This paper extends this concept to analyze other structures such as simply supported circular plates, simply supported and fully clamped square plates, and cylindrical shells subjected to rectangular pressure pulse in the medium range. The secondary effect of finite-deflections is also taken into account in each case.

2. Simply supported and fully clamped beams

Using the approximate square yield curve in Fig. 2, the lower bound of the nondimensional saturation impulse for a rigid, perfectly plastic simply supported beam subjected to uniformly

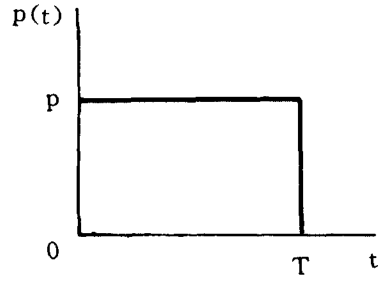

Fig. 3 Rectangular pressure pulse.

distributed medium rectangular pressure pulse (illustrated in Fig. 3) is (Zhao, *et al.* 1994a).

$$I_{low}=\frac{\pi}{\sqrt{6}}\lambda, \tag{1}$$

where $I=\frac{pT}{\sqrt{\mu H p_0}}$, $\lambda=\frac{p}{p_0}$, $p_0=\frac{2M_0}{L^2}$ is the static collapse load of the simply supported beam, μ the mass density per unit length of the beam, and L and H are half length and thickness of the beam, respectively.

As shown in Fig. 2 that an exact yield curve relating the nondimensional bending moment and membrane force lies everywhere inside a square having sides of magnitude 2, while a square with sides of length 1.236 lies everywhere inside the exact yield curve (e.g. Jones 1967). Therefore, the actual collapse pressure p_c is given by

$$0.618\ p_0 \leq p_c \leq p_0 \tag{2}$$

It is easy to show that the upper bound of the nondimensional saturation impulse is given by

$$I_{up}=\frac{1}{\sqrt{0.618}}I_{low}=1.27\ I_{low} \tag{3}$$

It should be noted that the above analysis incorporates the restriction

$$1<\lambda\leq 3. \tag{4}$$

To show the validity of the present model, here we only need to compare the upper and the lower bounds presented in this paper to the results given by Schubak, *et al.* (1989) for pulse ratio $\lambda=2$, and as shown in Fig. 4. It is evident that the point becoming horizontal in the curve is just between the lower and upper bounds given by this paper.

For a rectangular pressure pulse, the magnitude of the pulse is constant, and so the lower and the upper bounds of saturation duration of pressure pulse may be expressed respectively as

$$\tau_{low}=\frac{\pi}{\sqrt{6}} \tag{5a}$$

$$\tau_{up}=1.27\ \tau_{low} \tag{5b}$$

The lower and the upper bounds of the saturation impulse for a fully clamped beam subjected to a rectangular pressure pulse are given by

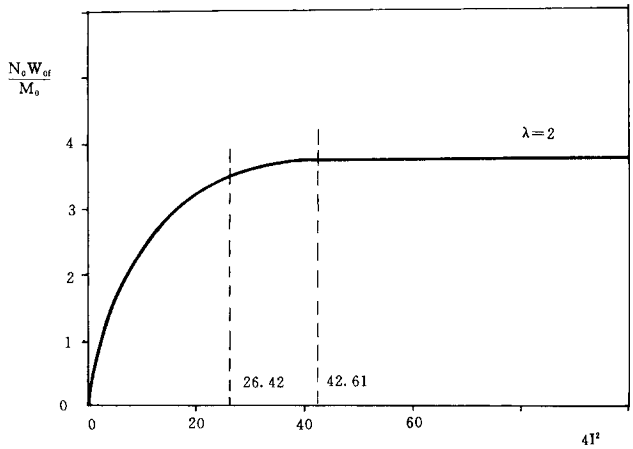

Fig. 4 Comparison for $\lambda=2$.

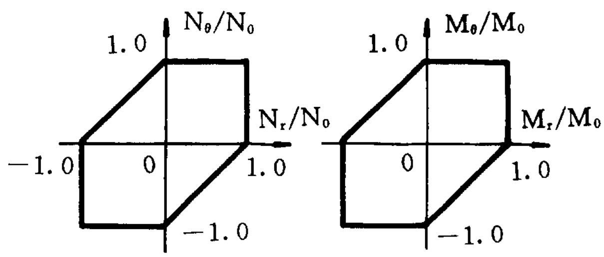

Fig. 5 Yield condition after Hodge.

$$I_{low}=\frac{\pi}{\sqrt{3}}\lambda \tag{6a}$$

$$I_{up}=1.27\ I_{low}, \tag{6b}$$

where $I=\frac{pT}{\sqrt{\mu H p_0}}$, $p_0=\frac{4M_0}{L^2}$ is the static collapse load of the fully clamped beam.

Similarly, the lower and the upper bounds of the nondimensional saturation duration for the clamped beam are

$$\tau_{low}=\frac{\pi}{\sqrt{3}} \tag{7a}$$

$$\tau_{up}=1.27\ \tau_{low} \tag{7b}$$

respectively.

3. Simply supported circular plate

If the limited interaction yield surface proposed by Hodge in 1960 (illustrated in Fig. 5) is used, the final deflection at the center of the circular plate subjected to a uniformly distributed medium rectangular pressure pulse is given by

$$\frac{W_m}{H}=\frac{1}{2}\left[\sqrt{1+2\lambda\left(1-\cos\frac{2I}{\lambda}\right)(\lambda-1)}-1\right], \tag{8}$$

where $I=\frac{pT}{\sqrt{\mu H p_0}}$ is the nondimensional impulse, $\lambda=\frac{p}{p_0}$.

Similar to Zhao, *et al.* (1994a), the lower bound of the nondimensional saturation impulse for a simply supported circular plate under medium load is given by

$$I_{low}=\frac{\pi}{3}\lambda \tag{9}$$

In the same manner as the beams, the upper bound of the nondimensional saturation impulse is taken as

$$I_{up}=1.27\ I_{low} \tag{10}$$

It should be noted that the restriction for the pulse ratio is

$$1<\lambda\leq 2. \tag{11}$$

For a rectangular pressure pulse in medium range, the lower and upper bounds of corresponding nondimensional saturation duration of pulse are

$$\tau_{low}=\frac{\pi}{2} \tag{12a}$$

$$\tau_{up}=1.27\ \tau_{low} \tag{12b}$$

respectively.

4. Simply supported and fully clamped square plates

Consider a rigid, perfectly plastic square plate of width $2L$ which is either simply supported or fully clamped around the outer boundary. If the plate is subjected to a medium rectangular pressure pulse, the maximum transverse displacement at the centre is (Jones 1971).

$$\frac{W_m}{H}=\frac{1}{2}\left[\sqrt{1+2\lambda\left(1-\cos\frac{2I}{\lambda}\right)(\lambda-1)}-1\right], \tag{13}$$

for a simply supported square plate, and

$$\frac{W_m}{H}=\sqrt{1+2\lambda\left(1-\cos\frac{\sqrt{2}I}{\lambda}\right)(\lambda-1)}-1 \tag{14}$$

for a fully clamped square plate.

Similarly, the nondimensional saturation impulse for a simply supported square plate is

$$I_{low}=\frac{\pi}{2}\lambda \tag{15a}$$

$$I_{up}=1.27\ I_{low} \tag{15b}$$

The restribution for the pulse ratio is $1<\lambda\leq 2$. Obviously, the corresponding lower and upper bounds of the nondimensional saturation duration for a rectangular pressure pulse are

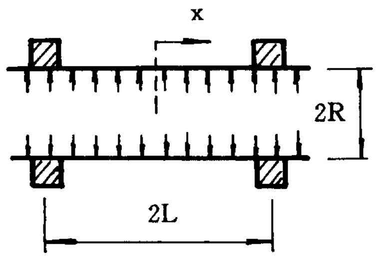

Fig. 6 Illustration for a circular cylindrical shell.

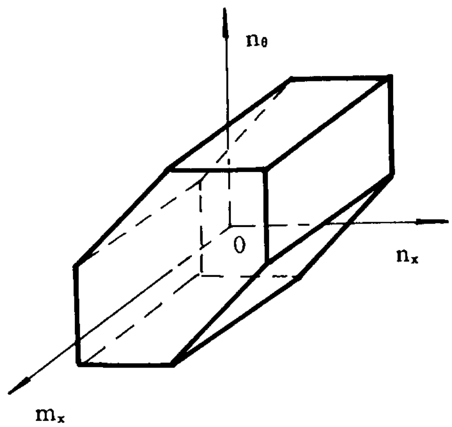

Fig. 7 Yield condition after Hodge and Shield.

$$\tau_{low}=\frac{\pi}{2} \tag{16a}$$

$$\tau_{up}=1.27\ \tau_{low} \tag{16b}$$

respectively.

The nondimensional saturation impulse for a fully clamped square plate is

$$I_{low}=\frac{\pi}{\sqrt{2}}\lambda. \tag{17a}$$

$$I_{up}=1.27\ I_{low} \tag{17b}$$

And

$$\tau_{low}=\frac{\pi}{\sqrt{2}} \tag{18a}$$

$$\tau_{up}=1.27\ \tau_{low} \tag{18b}$$

are the corresponding lower and upper bounds for fully clamped square plates subjected to rectangular pressure pulse in the medium range.

5. Cylindrical shells

Consider an infinitely long circular cylindrical shell reinforced by equally spaced reinforcing rings (Fig. 6) and subjected to a uniformly distributed radial rectangular pressure pulse. For simplicity, the linearized yield surface for a cylindrical shell after Hodge and Shield (illustrated in Fig. 7) is used. From the Appendix we know that the permanent transverse displacement at the midspan is

$$\left(\frac{W_0}{H}\right)_{max}=\frac{1}{4}\left[\sqrt{1+2\lambda\left(1-\cos\frac{\sqrt{6}I}{\lambda}\right)(\lambda-1)}-1\right], \tag{19}$$

where $I=\frac{pT}{\sqrt{\mu H p_0}}$ is the nondimensional impulse, and the meaning of other symbols are explained in the Appendix.

Similarly, the lower and upper bounds of the nondimensional saturation impulse is

$$I_{low}=\frac{\pi}{\sqrt{6}}\lambda \tag{20a}$$

$$I_{up}=\frac{1}{\sqrt{0.618}}I_{low}=1.27\ I_{low} \tag{20b}$$

It should be noted that the following inequalities must be met

$$1<\lambda\leq 3. \tag{21}$$

The corresponding lower and upper bounds of the saturation duration of the cylindrical shell subjected to a rectangular pressure pulse in the medium range are

$$\tau_{low}=\frac{\pi}{\sqrt{6}} \tag{22a}$$

$$\tau_{up}=1.27\ \tau_{low} \tag{22b}$$

respectively.

6. Conclusions

This paper extends the concept of Saturation Impulse to the analyses of structures such as simply supported circular plates, simply and fully clamped square plates and cylindrical shells subjected to rectangular pressure pulses in the medium range. Since approximate yield surfaces are used, both lower and upper bounds of the saturation impulses for such structures are presented. In case of rectangular pressure pulses in the medium range, once the magnitude of the pulse is given, the saturation impulse is, therefore, equivalent to saturation duration of the pulse. It should be noted that this paper only deals with one special pulse shape, namely rectangular pressure pulses, and that the phenomenon of saturation impulse may exist for other kinds of pulse shape in the medium range; in these cases the saturation impulses are not equivalent to the saturation durations of the pulse.

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Notations

H	thickness of beam or plate
I	$pT/\sqrt{\mu H p_0}$, nondimensional pressure pulse
L	half span of the beam and cylindrical shell, or half width of a square plate
m_x	M_x/M_0
M_x	axial bending moment
M_0	$\sigma_0 H^2/4$
$n_{x,\theta}$	$N_{x,\theta}/N_0$, nondimensional axial forces
$N_{x,\theta}$	axial and circumferential bending moments
N_0	$\sigma_0 H$
p	magnitude of the rectangular pressure pulse
p_0	collapse load
t_f	duration of response
T	pulse duration
u	axial displacement
w	transverse deflection
λ	p/p_0, pulse ratio
μ	mass density
σ_0	uniaxial yield stress
τ	$\sqrt{\frac{p_0}{\mu H}}\,T$, nondimensional duration of pressure pulse
$(\ )'$	$\frac{\partial}{\partial x}(\ )$
$(\dot{\ })$	$\frac{\partial}{\partial t}(\ )$

Subscripts:

low	lower bound
up	upper bound

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Appendix

Dynamic plastic behavior of cylindrical shells to medium rectangular pressure pulse

This problem is a axisymmetrical one, and the equations of motion for a cylindrical shell illustrated in Fig. 6 are (Jones 1970)

$$n'_x+\frac{(n_x-n_\theta)}{R}w'+\frac{pw'}{N_0}-\mu\ddot{w}\frac{w'}{N_0}-\frac{\mu\ddot{u}}{N_0}=0 \tag{A1a}$$

$$m''_x+\frac{4}{H}n_x w''+\frac{p}{M_0}-\frac{4n_\theta}{HR}-\frac{\mu\ddot{w}}{M_0}+\frac{\mu\ddot{u}w'}{M_0}=0, \tag{A1b}$$

where $m_x=\frac{M_x}{M_0}$, $n_{n,\theta}=\frac{N_{n,\theta}}{N_0}$, M_x is the bending moment, and N_x and N_θ are axial and circumferential membrane forces, respectively. w is the transverse displacement, μ denotes the mass density of the shell per unit length. $(\ )'=\frac{\partial}{\partial x}(\ )$, $(\dot{\ })=\frac{\partial}{\partial t}(\ )$. $M_0=\frac{\sigma_0 H^2}{4}$, $N_0=\sigma_0 H$, and σ_0 is the tensile yield stress.

It is assumed that

$$\dot{u}=\ddot{u}'=\ddot{u}=0. \tag{A2}$$

The response of the cylindrical shell under a rectangular pressure pulse in the medium range is divided into two stages

(1) first stage $t\in[0,\ \tau]$;
(2) second stage $t\in[\tau,\ t_f]$.

For simplicity, the linearized yield surface for a cylindrical shell after Hodge and Shield (shown in Fig. 7) is used. If it is assumed that the shape of the displacement field under dynamic loading in the medium range which produces finite-deflections is the same as that developed for the corresponding static collapse load, then

$$w(x,\ t)=W_0(t)\ (1-x/L)\ \text{when}\ t\in[0,\ \tau] \tag{A3a}$$

and

$$w(x,\ t)=W_1(t)\ (1-x/L)\ \text{when}\ t\in[\tau,\ t_f] \tag{A3b}$$

Substituting (A3a) and (A2) into (A1a,b) and neglecting higher-order terms containing w' gives

$$m'_x=\frac{4}{H}\frac{W_0}{L}-\frac{p(t)}{M_0}x+\frac{4}{HR}x+\frac{\mu}{M_0}\ddot{W}_0\left(x-\frac{x^2}{2L}\right) \tag{A4}$$

It should be noted that $n_\theta=n_x=1$ has been used in the derivation of (A4).

Integrating (A4) and noticing $m_x=1$ at the midspan between two adjacent reinforcing rings yields

$$m_x = \frac{4}{H}\frac{W_0}{L}x - \frac{p(t)}{M_0}\frac{x^2}{2} + \frac{4}{HR}\frac{x^2}{2} + \frac{\mu}{M_0}\frac{x^2}{2}\left(1 - \frac{x}{3L}\right) + 1 \tag{A5}$$

By using the boundary condition $m_x = -1$ at $x = L$, we obtain the differential equation of $W_0(t)$ in the first stage as follows

$$\ddot{W}_0 + \omega^2 W_0 = \frac{3}{2\mu}\left[p(t) - p_0\right], \tag{A6}$$

where

$$\omega^2 = \frac{3N_0}{\mu L^2}$$

and

$$p_0 = \frac{N_0}{R} + \frac{4M_0}{L^2}$$

p_0 is the static collapse load of the cylindrical shell.

By using $w = \dot{w} = 0$ at $t = 0$, the solution of Eq. (A6) may be written in the following form

$$w(x,\ t) = \frac{H}{4}(\lambda - 1)(1 - \cos\omega t)(1 - x/L) \tag{A7}$$

The corresponding differential equation of $W_1(t)$ is

$$\ddot{W}_1 + \omega^2 W_1 = -\frac{3}{2\mu}p_0 \tag{A8}$$

By using the requirements of the continuity conditions for displacements and velocity at $t = \tau$, then a dimensionless deflection $\frac{w(x,t)}{H}$ is

$$\frac{w(x,t)}{H} = \frac{1}{4}\{\cos\omega t + \lambda(\cos\omega\tau - 1)\cos\omega t + \lambda\sin\omega\tau\sin\omega t - 1\}\left(1 - \frac{x}{L}\right) \tag{A9}$$

The response time t_f can be obtained by using $w(x,\ t_f) = 0$, thus

$$t_f = \frac{1}{\omega}\tan^{-1}\frac{\lambda\sin\omega\tau}{1 + \lambda(\cos\omega\tau - 1)} \tag{A10}$$

The permanent transverse displacement at the midspan is

$$\left(\frac{W_0}{H}\right)_{max} = \frac{1}{4}\left[\sqrt{1 + 2\lambda\left(1 - \cos\frac{\sqrt{6}I}{\lambda}\right)(\lambda - 1)} - 1\right], \tag{A11}$$

where $I = \frac{pT}{\sqrt{\mu H p_0}}$ is the nondimensional impulse.

It should be noted that the pulse ratio λ must satisfy the following inequalities

$$1 < \lambda \leq 3. \tag{A12}$$

It may be shown that these solutions are both statically and kinematically admissible.