Structural Engineering and Mechanics, Vol. 3, No. 4 (1995) 383-390
DOI: http://dx.doi.org/10.12989/sem.1995.3.4.383

A 2-D four-noded finite element containing a singularity of order λ

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Abstract. A 2-D four-noded finite element which contains a λ singularity is developed. The new element is compatible with quadratic standard isoparametric elements. The element is tested on two different examples. In the first example, an edge crack problem is analyzed using two different meshes and different integration orders. The second example is a crack perpendicular to the interface problem which is solved for different material properties and in turn different singularity order λ. The results of those examples illustrate the efficiency of the proposed element.

Key words: fracture mechanics; finite element method.

1. Introduction

In a wide range of fracture mechanics applications, especially in the field of composite materials, the displacement at the crack tip varies as r^{λ}, where r is the distance from the crack tip and λ is the order of the singularity. The value of λ ($0<\lambda\leq 1$) depends on the geometry of the problem and the material properties. In case of an homogeneous material λ is equal to 0.5. Frequently occurring cases are kinked cracks (Williams 1952) and cracks perpendicular to the interface (Cook and Erdogan 1972). The use of a special crack tip element at the singular point permitted us to get a good approximation of the field variable and its derivatives near the vicinity of the singular point and to avoid the need of an extremely fine mesh.

Many singular elements have been developed throughout the literature. For cracks in homogeneous materials, the quarter point singularity element which has been introduced by Barsoum 1976, is widely used. However, this element is not suitable for non homogeneous material (unless an extremely fine mesh is used) where λ can takes values other than 0.5. Akin (1977) has generated two-dimensional singularity elements (three node singular triangle, four node singular quadrilateral, and six node triangle) from standard conforming elements. Unfortunately, those elements are not compatible with conventional finite element and in turn convergence is not guaranteed.

Tracy and Cook (1977) have developed a 3-noded triangular element which is compatible with the conventional linear element (3-noded triangular or 4-noded quadrilateral). Recently Rochdi El Abdi (1991) has proposed a degenerated triangular element for which the shape functions are derived from those of standard isoparametric elements. This 4-noded element is compatible with the six noded triangle or the 8-noded quadrilateral.

We present here an element which has a variation of the displacements of r^{λ} along the crack face and hence a variation of the derivatives of $r^{\lambda-1}$. The interpolation functions of the dis-

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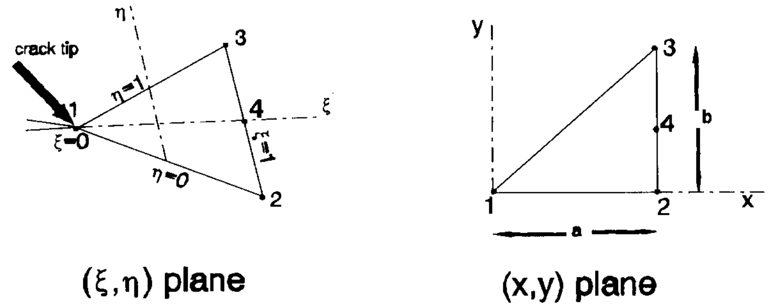


Fig. 1 4-noded element.

placements are assumed to be of lower order in ξ (where ξ is a natural coordinate) than that of the coordinates. Based on this assumption a 4-noded singular element is derived. This element can be easily connected to quadratic standard finite element, thus, convergence is guaranteed in case of an active singularity. Convergence is studied in the first example, an edge crack problem, by increasing the order of integration and by refining the mesh. A crack perpendicular to the interface problem is presented to illustrate the application to a λ singularity crack problem with different λ values. The results are compared to existing ones in the literature.

2. Mathematical formulation

Fig. 1 shows the 4-noded element in the (ξ, η) and the (x, y) planes. The crack tip is located at node 1. Two different shape functions are assumed for the interpolation of the coordinates and the displacements, i.e.;

$$[X]=\sum N'_i\ [x_i] \tag{1}$$

$$[U]=\sum N_i\ [u_i] \tag{2}$$

where $[X]$ and $[U]$ are the coordinate and displacement vector, respectively, i.e. $[X]^T=[x\ y]$ and $[U]^T=[u_x\ u_y]$. $[u_i]$ represents the nodal displacements and $[x_i]$ represents the nodal coordinates. Introducing a singularity λ along the ξ direction, the relation between ξ and r, where r is another local parameter measuring the distance from the crack tip with its origin ($r=0$) at the crack tip and $r=a$ at the face 2-3, is given by:

$$\xi=\left(\frac{r}{a}\right)^{\lambda} \tag{3}$$

Now the shape function N' and N can be introduced as:

$$\begin{aligned}
N'_1&=1-\xi^{1/\lambda}\\
N'_2&=\xi^{1/\lambda}(1-\eta)(1-2\eta)\\
N'_3&=\xi^{1/\lambda}\eta(2\eta-1)\\
N'_4&=4\xi^{1/\lambda}\eta(1-\eta)\\
N_1&=1-\xi\\
N_2&=\xi\,(1-\eta)(1-2\eta)
\end{aligned}$$

$$N_3 = \xi\eta(2\eta - 1)$$
$$N_4 = 4\xi\eta(1-\eta) \tag{4}$$

The Jacobian matrix $[J]$ is evaluated as:

$$[J] = \begin{bmatrix} \sum N'_{i,\xi}\, x_i & \sum N'_{i,\xi}\, y_i \\ \sum N'_{i,\eta}\, x_i & \sum N'_{i,\eta}\, y_i \end{bmatrix}$$

$$[J] = \begin{bmatrix} \dfrac{\partial x}{\partial \xi} & \dfrac{\partial y}{\partial \xi} \\ \dfrac{\partial x}{\partial \eta} & \dfrac{\partial y}{\partial \eta} \end{bmatrix} = \begin{bmatrix} \sum \dfrac{\partial N'}{\partial \xi} x_i & \sum \dfrac{\partial N'}{\partial \xi} y_i \\ \sum \dfrac{\partial N'}{\partial \eta} x_i & \sum \dfrac{\partial N'}{\partial \eta} y_i \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} \tag{5}$$

The inverse of $[J]$ is obtained as:

$$[J]^{-1} = \frac{1}{J} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} \tag{6}$$

where J is the determinant of the Jacobian matrix.

The global strain displacement matrix $[B]$ can be calculated as follows:

$$[B] = [J]^{-1}[DN] \tag{7}$$

where the matrix $[DN]$ contains the derivatives of the shape functions N with respect to ξ and η.

The element stiffness matrix is finally obtained as:

$$[K] = \int_A [B]^T[D][B]t\,dA \tag{8}$$

Eq. (8) should be numerically integrated. The following quadrature rule is used (Robert, *et al.* 1982);

$$\int_A \phi\, dA = \frac{1}{2} \sum w_i J \phi_i \tag{9}$$

where w_i is the weighting factor at the point where ϕ is evaluated. However, to be able to use the integration rules of Eq. (8) the function ϕ should be written in terms of the area coordinates (L_1, L_2, L_3) in stead of ξ and η. Thus, The matrix $[B]$ in Eq. (7) is transformed to the area coordinates using the following relations:

$$\xi = 1 - L_1$$
$$\eta = \frac{L_3}{1-L_1} \tag{10}$$

Fig. 2 shows the different integration options.

It can be easily shown that the elements has a stress singularity at node 1 of order $\lambda - 1$. It has continuity along sides 1-2 and 1-3 with the similar singular elements while along side 2-3 with 8-noded quadratic standard isoparametric finite elements.

3. Numerical results

The new 4-noded element has been introduced in a finite element computer program CALM

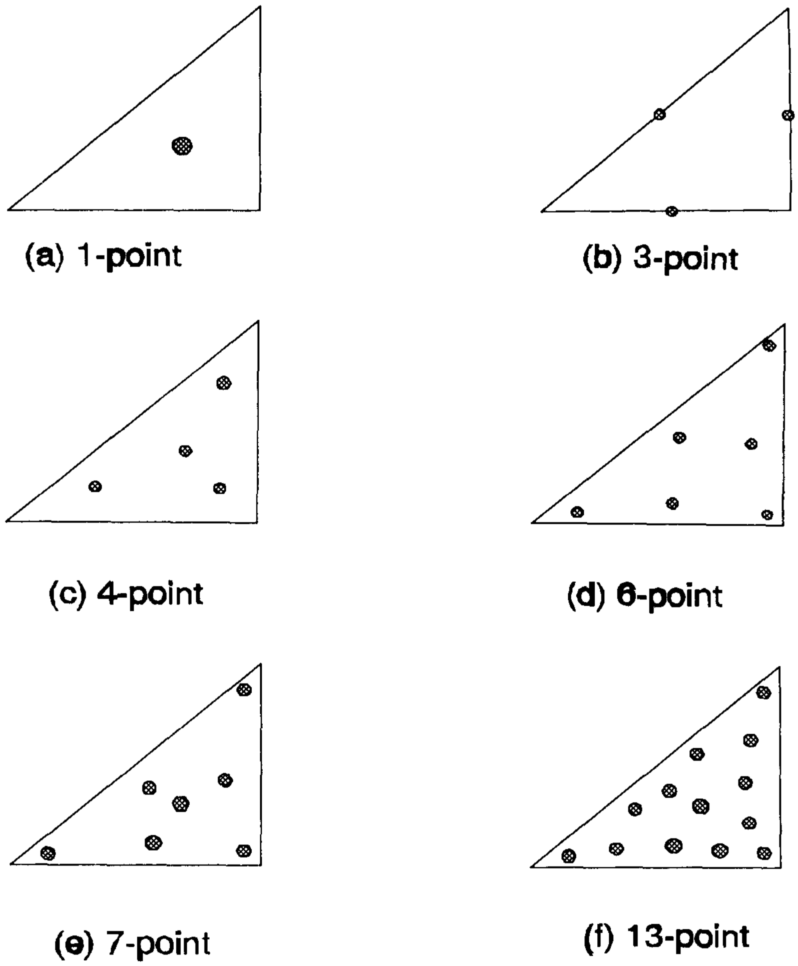


Fig. 2 Integration rules.

(Geyskens, *et al.* 1991). Two examples are studied below to illustrate the effectiveness of the proposed elements. The first example is the edge crack sample assuming plane strain condition. The problem configuration is shown in Fig. 3. The crack length to the plate width ratio is 0.3. The new singular element is used to approximate the displacement and the stress field in the first row around the crack tip with a λ value of 0.5, while 8-noded quadratic standard isoparametric elements are used elsewhere. Two rather coarse meshes, which are shown in Figs. 4(a) and 4(b), are used for this problem. Due to symmetry only half of the problem has to be analyzed.

The crack tip element size to the crack length is 1 : 6 in mesh 1 and 1 : 12 in mesh 2. The stress intensity factor is extrapolated to the crack tip using the crack face displacement. For plane strain problem the following expression could be used:

$$K_i = \frac{E\sqrt{2\pi}}{8(1-\nu^2)\sqrt{r}} u_i \tag{11}$$

where u_i is the opening displacement of the point i on the crack face. E and ν are the Young's modulus and the Poisson ratio, respectively.

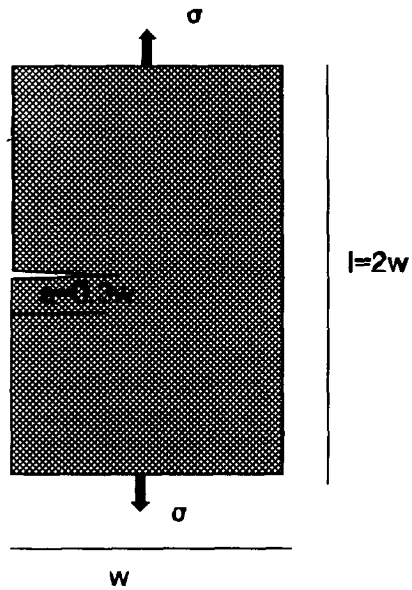
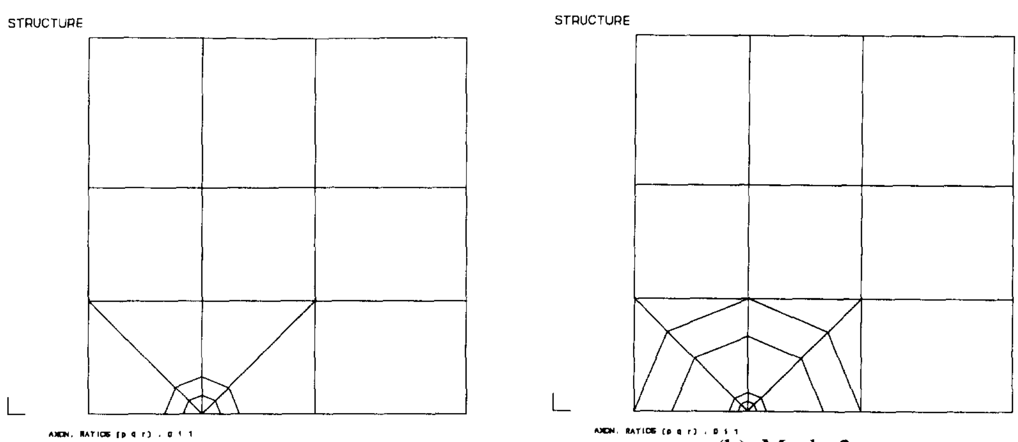


Fig. 3 Edge crack sample.



(a) Mesh 1. (b) Mesh 2

Fig. 4 Edge crack problem.

The results are given in Table 1. The stress intensity factor is calculated in dimensionless form, i.e., $K/\sigma(\pi a)^{1/2}$. The analytical solution of this problem is 1.6629 (Rook and Cartwright 1976).

The percentage differences of the computed values from the reference value are given in parenthetic. By refining the finite element mesh from 1 to 2, convergence is achieved. The SIF value computed by one-integration point is less accurate than the other integration schemes. In general, the results are in good agreement with the reference value.

The second example is a crack perpendicular to the interface problem taken from Lin and Mar (1976). The problem configuration is shown in Fig. 5. The crack length (2a) is 2 inch, lying in panel 1 (material 1), and perpendicular to the interface of the two panels. Due to symme-

Table 1 Edge crack problem - convergence study

Int. Scheme	$K/\sigma(\pi a)^{1/2}$ Mesh 1	Mesh 2
1	1.626(2.2%)	1.635 (1.7%)
3	1.655(0.4%)	1.6229(0.03%)
4	1.656(0.4%)	1.6637(0.05%)
6	1.655(0.4%)	1.6630(0.006%)
7	1.654(0.5%)	1.6620(0.05%)
13	1.656(0.4%)	1.6625(0.02%)

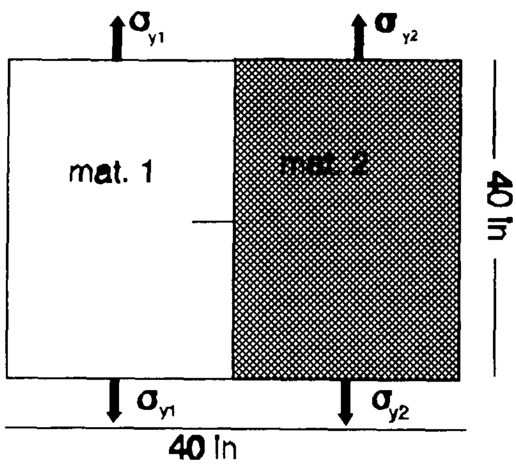


Fig. 5 Crack perpendicular to the interface problem.

try, only one half of the model is analyzed. The finite element mesh is shown in Fig. 6. The crack tip at the interface will be denoted as tip 2, and the other tip in material 1 will be denoted as tip 1. This problem has been solved analytically in Cook and Erdogan (1972) for an infinite plate with pressure loading at the crack surfaces. However, in the present analysis, the stresses σ_{y1} and σ_{y2} in both materials are prescribed according to $\sigma_{y2}=E_2/E_1\ \sigma_{y1}$ to ensure constant strain ε_y in plane stress condition.

The stress intensity factors are computed using the following expression:

$$K_i = \lim_{r\to 0} 2\sqrt{2}\mu^* \lambda \frac{u_{yi}(r,\ \pi)}{r^\lambda} \tag{12}$$

where

$$\mu^* = \mu_1 m \frac{(1-2\lambda)(m+k_2)+(1+2\lambda)(1+mk_1)}{(m+k_2)(1+mk_1)\sin\pi\lambda}$$

μ_1, μ_2=the shear moduli for material 1 and 2, respectively; $m=\mu_1/\mu_2$; $k_i=3-4\nu_i$ for plane strain condition and $(3-\nu_i)/(1+\nu_i)$ for plane stress condition; $u_{yi}(r,\pi)$=the opening displacement at a position i on the crack face.

The present singular elements are used in the first row around tip 2 with the appropriate λ value (depending on the material properties of both materials (Cook and Erdogan 1972), and around tip 1 with λ equal to 0.5. 8-noded quadratic elements are used in the second row around

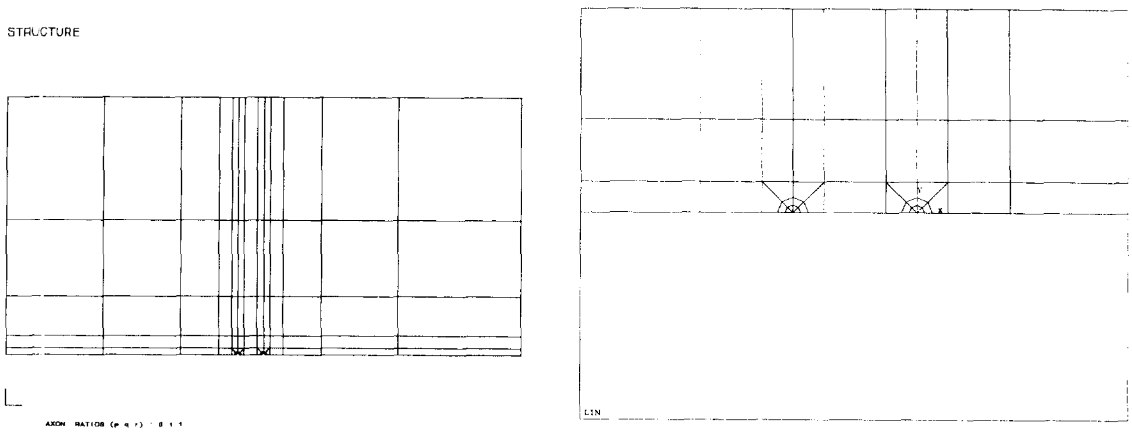


Fig. 6 (a) Finite element mesh; (b) Details near the crack tips.

Table 2 Stress intensity factors at tip 1

μ_1/μ_2	λ	$K_1/(\sigma_{y1}\, a^{1/2})$ Present Elem. (234 d.o.f)	Cook and Erdogan 1972	Lin and Mar 1976 (244 d.o.f.)
0.5	0.5	0.8343	0.871	0.833
0.0433	0.5	0.8458	0.879	0.855
1.0	0.5	0.9816	1.00	0.995
23.08	0.5	1.370	1.353	1.371
138.46	0.5	1.5450	1.509	1.529

Table 3 Stress intensity factors at tip 2

μ_1/μ_2	λ	$K_2/(\sigma_{y1}\, a^{1-\lambda})$ Present Elem. (234 d.o.f)	Cook and Erdogan 1972	Lin and Mar 1976 (244 d.o.f.)
0.0072	0.7335	4.866	4.922	4.978
0.0433	0.711	4.113	4.176	4.241
1.0	0.5	0.9816	1.00	0.995
23.08	0.1758	0.097	0.074	0.095
138.46	0.0749	0.0197	0.0079	0.0196

the crack tips, then they are reduced to linear elements. The crack tip element size to the crack length ratio is 1 : 16. The total numbers of degrees of freedom used in the current model are 234. The results are presented in Table 2 and 3. The current results are compared to that of Cook and Erdogan (1972) and Lin and Mar (1976). The values between parenthetic are the numbers of degrees of freedom used by the present model and by Lin and Mar (1976). In Lin and Mar (1976), a hybrid singular element was used to model the singularity near the crack tips of this problem.

From Tables 2 and 3, it could be shown that although a coarse mesh has been used for this problem, the current results are in good agreement with the two references values.

4. Conclusions

A 2-D singular finite element was developed. The element is compatible with the quadratic standard isoparametric elements. The interpolation functions of the displacements were assumed to be in lower order than that of the coordinates. The element has been successfully used in computing the stress intensity factors for problems of an edge crack and a crack perpendicular to the interface. These case studies have indicated that convergence can be reached in case of an active λ singularity problem.

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