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Approximate discrete variable optimization of plate structures using dual methods

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Abstract. This study presents an efficient method for optimum design of plate and shell structures, when the design variables are continuous or discrete. Both sizing and shape design variables are considered. First the structural responses such as element forces are approximated in terms of some intermediate variables. By substituting these approximate relations into the original design problem, an explicit nonlinear approximate design task with high quality approximation is achieved. This problem with continuous variables, can be solved by means of numerical optimization techniques very efficiently, the results of which are then used for discrete variable optimization. Now, the approximate problem is converted into a sequence of second level approximation problems of separable form and each of which is solved by a dual strategy with discrete design variables. The approach is efficient in terms of the number of required structural analyses, as well as the overall computational cost of optimization. Examples are offered and compared with other methods to demonstrate the features of the proposed method.

Key words: approximation; continuous variable; discrete variable; optimization; plate and shell; dual method.

1. Introduction

It is well known that the optimum design of structures can be posed as a mathematical programming problem in which the objective function reflecting the weight or cost is minimized while the design constraints are satisfied. The objective and the constraints are expressed in terms of the design variables. Examples of the design variables are the radius, thickness and cross sectional dimensions. The design constraints are bounds on stresses, displacements, etc.

Mathematically, an optimization problem can be stated as follows:

$$\text{minimize } F(X) \tag{1}$$

$$\text{subject to; } g_j(X) \leq 0 \qquad j=1,\ m \tag{2}$$

$$X_i^l \leq X_i \leq X_i^u \quad i=1,\ n \tag{3}$$

$$X_i \in D_i \tag{4}$$

where $F(X)$ and $g_j(X)$ are the objective function and constraints, respectively, and X is the vector of design variables. X_i^l and X_i^u are the lower and upper bounds on the design variable X_i. m is the number of constraints and n is the number of variables. D_i is the set of discrete variables which may be different for each design variable.

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The problem given by Eqs. (1)-(3) is, in general, a non linear programming problem and there are various techniques to solve this optimization problem (Vanderplaats 1984). Most optimization algorithms require that an initial set of design variables, $\boldsymbol{X}^0$, be specified. Beginning from this starting point, the design is updated iteratively. Probably the most common form of this iterative procedure is given by

$$\boldsymbol{X}^k = \boldsymbol{X}^{k-1} + \alpha \boldsymbol{S}^k \tag{5}$$

where k is the iteration number and $\boldsymbol{S}$ is the vector of search direction. The scalar quantity α defines the distance that we wish to move in direction $\boldsymbol{S}$ to improve the design. There are a wide variety of methods for determining the search direction, $\boldsymbol{S}$, as well as for finding the value of α (Vanderplaats 1984). In numerical optimization techniques, these methods require evaluations of the objective and constraint functions as well as their gradients. As the overall iteration process is iterative, thus to reach the optimum solution, we often require hundreds of function evaluations and gradient calculations. Each function evaluation needs an analysis of the structure under consideration. Thus, dealing with large scale optimization problems, a great number of finite element analyses of the structure is required to complete the process, thereby making the process very inefficient.

In order to solve the problem efficiently, an attempt should be made to create a high quality approximation to the design problem and solve the approximate problem completely, without actually performing any finite element analyses. Because it is an approximation, it must be repeated so that at least a few detailed finite element analyses will be needed. The key to efficiency is the creation of a high quality approximation, thus reducing the number of structural analyses.

In the past, attempts have been made to reduce the computational burden by introducing some approximation concepts (Schmit and Miura 1976, Schmit and Fleury 1980 and Salajegheh 1984). The number of the design variables was reduced by linking. This idea is reasonable as in practice some of the variables are the same. The number of constraints was also reduced by considering only the critical or the near critical constraints at each iteration. Most important of all, the number of structural analyses was reduced by employing some approximate functions to represent the constraints. A first order Taylor series expansion was used to generate the approximation forms of the constraints in terms of design variables (intermediate variables) or their reciprocals. The reason for using the reciprocal variables is due to the fact that structural response quantities such as stress and displacement are approximately linear with respect to the reciprocal variables. However, as the design constraints are, in general, highly nonlinear in terms of the design variables or their reciprocals, the quality of linearization may be poor, thus the number of structural analyses required to achieve an optimum design can be increased.

A second generation approximation techniques was developed by Salajegheh and Vanderplaats (1986, 1987), and Vanderplaats and Salajegheh (1988, 1989), by which the highest quality approximation can be achieved. The implicit structural responses such as forces, displacements, frequencies, etc., appearing in the optimization problem, are first approximated. By substituting these approximate functions into the original problem, a nonlinear explicit problem is created, the solution of which, often requires less than ten analyses of the structure. This method is very robust and efficient for large structures, where the computational cost of the analysis is high. The same idea was used by Vanderplaats and Thomas (1993) to achieve the continuous optimum solution of plate structures.

Recently, the same approximate technique has been applied by Salajegheh and Vanderplaats (1993a) to the design of skeletal structures, where some or all the design variables are chosen from a prescribed set of values (discrete variables). The discrete optimum design of structures is achieved by combining the approximate techniques and branch and bound method. For practical design problems, where the design variables are linked and the number of independent design variables is chosen reasonably, the method can be used efficiently.

The same approximation concepts are used by Salajegheh and Vanderplaats (1993b) and Salajegheh (1993a) to achieve the optimum shape of two and three dimensional truss structures. In addition to the sizing variables, the coordinates of joints are also considered as design variables. In this case, the numerical results also indicate that, the optimum configuration of pin jointed structures with discrete shape variables, in conjunction with response approximation, can be achieved at little computational cost.

To further increase the efficiency of the technique for problems with great number of discrete variables, a dual strategy is used for truss type structures with sizing and shape design variables (Vanderplaats and Salajegheh 1994, Salajegheh 1993b, 1994b). The discrete variable optimization is achieved after the completion of the continuous variable optimization. The method was extended to frame structures by the same author (Salajegheh 1994a).

In the present study, the idea of force approximation presented by Salajegheh and Vanderplaats (1986, 1987) and Vanderplaats and Thomas (1993) is employed to achieve the optimum design of plate and shell structures, when the design variables are discrete. After the completion of the continuous variable optimization, dual methods are used to achieve the discrete solution. The numerical results show that both continuous and discrete solutions of plate type structures can be achieved very efficiently. This is basically due to the fact that approximating the element forces and then explicitly calculating the approximate element stresses from these yields a more accurate approximation function than direct approximation of the element stresses for plate type structures. The use of these highly accurate approximations leads to rapid convergence in the design process. Also the use of duality theory reduces the overall computational cost of discrete variable optimization.

2. Approximation concepts

The concept of choosing the functions to be approximated and the intermediate variables to be used to create a high quality approximation is fundamental to the overall efficiency and reliability of the optimization process. This is best understood by considering stress constraints for beam elements. Consider a simple rectangular beam element of width B and height H. These are the physical design variables that to be determined in an optimization problem. The maximum stress used in evaluating a stress constraint is

$$\sigma = \pm \frac{Mc}{I} \pm \frac{P}{A} \tag{6}$$

where $c=H/2$, $I=BH^3/12$ and $A=BH$ are simple functions of B and H. M is the bending moment and P is the axial force. A traditional linearization would be to create a Taylor series approximation to stress as

$$\bar{\sigma} = \sigma^0 + \sum_{i=1}^{n} \frac{\partial \sigma}{\partial X_i}(X_i - X_i^0) \tag{7}$$

where $X^T=[B, H]$. However, it is clear that the stress is highly nonlinear in the design variable B and H, so the approximation of the stress given by Eq. (7) is not accurate and a very small move limit would be necessary during the solution of the approximate problem.

Now consider how we might better approximate the stress. First A and I are considered as intermediate variables. Let the vector of the intermediate variables be as $Y^T=[A, I]$. Next, the gradients of M and P (intermediate responses) are calculated with respect to Y, M and P are approximated as

$$\bar{M}=M^0+\sum_{i=1}^{r}\frac{\partial M}{\partial y_i}(y_i-y_i^0) \tag{8}$$

$$\bar{P}=P^0+\sum_{i=1}^{r}\frac{\partial P}{\partial y_i}(y_i-y_i^0) \tag{9}$$

where r is the number of intermediate variables. When we need the value of stress, first A and I are calculated explicitly as functions of B and H. Then, the approximate member forces, $\bar{M}$ and $\bar{P}$ are evaluated. Finally, the stress and constraint are recovered in the usual fashion.

With the use of such intermediate variables and responses, we achieve two important goals. First, we allow the engineer to treat the physical dimensions B and H as design variables. Second, we retain a great deal of the non linearity of the original problem explicitly. This allows us to make very large changes in the design variables during a design iteration.

Now the same approximate strategy is applied to the optimum design of plate and shell structures. Considering a four-noded plate element with 6 degrees of freedom per node (3 translations and 3 rotations) as presented by Zienkiewicz and Taylor (1989). Thus we have 24 nodal forces in the element. As it was suggested by Vanderplaats and Thomas (1993), only the 6 components of the forces at the center of the element (element forces) are approximated. Then the approximate stresses can be calculated using these approximate forces. At the starting point of each iteration, the nodal displacements of the structure are known from the finite element analysis. The vector of element forces $\boldsymbol{P}^0$ at the starting point is determined as;

$$\boldsymbol{P}^0=\boldsymbol{DBu}^e \tag{10}$$

where $\boldsymbol{D}$ is the element material matrix in the element coordinate system, $\boldsymbol{B}$ is the strain-displacement matrix and $\boldsymbol{u}^e$ is the nodal displacement vector of the element. We are now able to approximate the element forces using a Taylor series expansion in the intermediate design variables. The intermediate design variables for plate structures are the shape design variables, the plate thickness t, and bending stiffnesses D, where $D=t^3/12$. Let the six approximate element forces in the element be as follows

$$\bar{\boldsymbol{P}}=\begin{Bmatrix}\bar{N}_x\\ \bar{N}_y\\ \bar{N}_{xy}\\ \bar{M}_x\\ \bar{M}_y\\ \bar{M}_{xy}\end{Bmatrix} \tag{11}$$

The approximate surface stresses are then calculated using

$$\bar{\sigma}=\begin{Bmatrix}\bar{\sigma}_x\\ \bar{\sigma}_y\\ \bar{\tau}_{xy}\end{Bmatrix}=\begin{Bmatrix}\dfrac{\bar{N}_x}{t}-\dfrac{\bar{M}_x z}{D}\\ \dfrac{\bar{N}_y}{t}-\dfrac{\bar{M}_y z}{D}\\ \dfrac{\bar{N}_{xy}}{t}-\dfrac{\bar{M}_{xy} z}{D}\end{Bmatrix} \tag{12}$$

where z is the fiber distance at any point.

Finally the approximate principle, maximum shear, and Von Mises stresses can be calculated. For example, the approximate Von Mises stress is

$$\sigma_{VM}=\sqrt{\bar{\sigma}_x^2+\bar{\sigma}_y^2-\bar{\sigma}_x\bar{\sigma}_y+3\bar{\tau}_{xy}^2} \tag{13}$$

This can be used to establish the approximate stress constraint as

$$\bar{g}(X)=\frac{\sigma_{VM}-\sigma_a}{\sigma_a}\leq 0 \tag{14}$$

where σ_a is the allowable stress.

2.1. Constraint deletion

In practical design problem, there are a great number of constraints involved and a large percentage of these constraints may never be critical during a design cycle, and so could be excluded from the constraint set in that cycle. This is done to decrease the cost of sensitivity analysis (gradient calculation) and to reduce the size of the approximate optimization problem. The simplest approach for constraint deletion would be to just ignore any constraint which is not within, say, 30% of being critical. In addition, if a number of constraints are active in one region of the structure, say near a stress concentration, only a small number of most critical constraints in that region is retained. In general, it is reasonable to reduce the number of retained constraints to 2-4 times the number of design variables. Any constraints not retained, which become active or violated as a result of the design changes during this cycle, will be retained on future design cycle.

3. Gradient calculation

In this approach, the gradients of element forces with respect to intermediate variables are required. First the gradient of the nodal displacements are evaluated by using equation

$$\boldsymbol{K}\boldsymbol{u}=\boldsymbol{w} \tag{15}$$

where $\boldsymbol{K}$ is the global stiffness matrix, $\boldsymbol{u}$ is the displacement vector and $\boldsymbol{w}$ is the external load vector. Differentiation of this equation with respect to the intermediate variables $\boldsymbol{Y}$ yields

$$\frac{\partial \boldsymbol{u}}{\partial Y_i}=\boldsymbol{K}^{-1}\left[\frac{\partial \boldsymbol{w}}{\partial Y_i}-\frac{\partial \boldsymbol{K}}{\partial Y_i}\boldsymbol{u}\right] \tag{16}$$

Now the gradients of the element forces with respect to $\boldsymbol{Y}$, are easily calculated from the relationship

$$P^e = DBu^e = Hu^e \tag{17}$$

as

$$\frac{\partial P^e}{\partial Y_i} = \frac{\partial H}{\partial Y_i} u^e + H \frac{\partial u^e}{\partial Y_i} \tag{18}$$

where H is a known matrix and its derivatives can be evaluated and $\partial u^e / \partial Y_i$ are the gradients of the nodal displacements associated with this element and are recovered from Eq. (16).

4. Discrete variable optimization

Having established the approximate explicit relations for the element forces, we now proceed to solve the optimization problem. By substituting these approximate equations into the original implicit problem, an approximate explicit nonlinear problem is obtained, which can be solved by numerical optimization techniques very easily, without performing the analysis of the structure. Since this is an approximation to the original problem, move limits should be imposed on the design variables and the intermediate variables to insure the quality of the approximation.

Let $\bar{g}_j$ represent the approximated form of the jth constraint and J_r indicate the set of retained constraints. Then the general form of the approximate problem, in each design cycle, can be expressed as

$$\text{minimize} \quad F(X) \tag{19a}$$

$$\text{subject to;} \quad \bar{g}_j(X) \leq 0, \quad j \in J_r \tag{19b}$$

$$X_i^l \leq X_i \leq X_i^u \quad i = 1, n \tag{19c}$$

$$Y_i^l \leq Y_i \leq Y_i^u \quad i = 1, r \tag{19d}$$

$$X_i \in D_i \tag{19e}$$

In this study, for all of the numerical examples a move limit of 80% to 100% on the initial design is used at the start of the optimization process. At the end of each approximate problem solution, if any of the constraints is more violated than the previous iteration, the move limits on each of the design variables are reduced by 50%. Move limits are never reduced less than 10%.

Ignoring Eq. (19e) the approximate explicit problem given by Eqs. (19a)-(19d) is solved. When the optimum solution is obtained, the solution is taken as the starting point for the next iteration and the process is repeated, until the problem converges. The final result is a continuous solution.

However, there are many occasions in design of structures, where the design variables must be selected from a list of discrete values. The design variables such as cross sectional areas of members, the thicknesses of plates and shells fall into this category.

The most common way of achieving a design with discrete design variables is to round-off the optimum values of the design variables, obtained from the continuous solution, to the nearest acceptable discrete values. It can be observed that such a solution may neither be optimum nor feasible.

In this study, a dual strategy is used to achieve the discrete solution. The approximate nonlinear problem given by Eqs. (19) is again approximated, using conservative approximations (Fleury and Braibant 1986). In fact, a second level approximation is used to convert the first level approximation, into a problem of separable form. Now, the second level approximate problem can be mathematically stated as

$$\text{minimize} \quad \bar{F}(X) \tag{20a}$$

$$\text{subject to;} \quad \tilde{g}_j(X) \leq 0, \qquad j \in J_r \tag{20b}$$

$$X_i^l \leq X_i \leq X_i^u \qquad i=1,\ n \tag{20c}$$

$$X_i \in D_i \tag{20d}$$

where

$$\bar{F}(X) = F(X^0) + \sum_{i=1}^{n} \frac{\partial F(X^0)}{\partial X_i}(X_i - X_i^0)$$

$$\text{if } X_i^0 \frac{\partial F(X^0)}{\partial X_i} \geq 0 \tag{21}$$

or

$$\bar{F}(X) = F(X^0) + \sum_{i=1}^{n} \frac{\partial F(X^0)}{\partial X_i}\left(\frac{1}{X_i} - \frac{1}{X_i^0}\right)(X_i^0)^2$$

$$\text{if} \quad X_i^0 \frac{\partial F(X^0)}{\partial X_i} < 0 \tag{22}$$

also

$$\tilde{g}(X) = \bar{g}(X^0) + \sum_{i=1}^{n} \frac{\partial \bar{g}(X^0)}{\partial X_i}(X_i - X_i^0)$$

$$\text{if} \quad X_i^0 \frac{\partial \bar{g}(X^0)}{\partial X_i} \geq 0 \tag{23}$$

or

$$\tilde{g}(X) = \bar{g}(X^0) + \sum_{i=1}^{n} \frac{\partial \bar{g}(X^0)}{\partial X_i}\left(\frac{1}{X_i} - \frac{1}{X_i^0}\right)(X_i^0)^2$$

$$\text{if} \quad X_i^0 \frac{\partial \bar{g}(X^0)}{\partial X_i} < 0 \tag{24}$$

It can be seen that Eqs. (21) and (23) are a direct linear approximation and Eqs. (22) and (24) are reciprocal approximations. The bar symbol (−) denotes first level approximation and tilde (~) represents second level approximation. In addition, if X_i^0 or X_i are near zero or if X_i may cross zero (in case of shape optimization), a direct approximation is always used.

It can be seen that both $\bar{F}(X)$ and $\tilde{g}(X)$ are separable functions, that is

$$\bar{F}(X) = \sum_{i=1}^{n} \bar{f}_i(X_i) \tag{25}$$

$$\tilde{g}_j(X) = \sum_{j \in J_r} \tilde{g}_{ji}(X_i) \tag{26}$$

The Lagrangian function can now be written as follows:

$$L(X,\ \lambda) = \sum_{i=1}^{n} \bar{f}_i(X_i) + \sum_{j \in J_r} \lambda_j \sum_{i=1}^{n} \tilde{g}_{ji}(X_i) \tag{27}$$

Using duality theory, $L(X, \lambda)$ is minimized with respect to X and then maximized with respect to λ, subject to non-negativity constraints on the dual variables. Also, using the property that the minimum of a separable function is the sum of the minimums of the individual parts, we can state the dual problem as

$$\text{maximize } L(\lambda)=\sum_{i=1}^{n} \min_{X_i}\left[\bar{f}_i(X_i)+\sum_{j\in J_r} \lambda_j \tilde{g}_{ji}(X_i)\right] \tag{28a}$$

$$\text{subject to; } \lambda_j \geq 0 \qquad j\in J_r \tag{28b}$$

It can be seen that a number of one dimensional minimization is carried out with respect to primal variables X_i, the results of which is a continuous solution. Now, the next lower and upper discrete values to X_i, are found and whichever minimizes the one dimensional function will be the discrete solution for X_i. The process of minimizing the one dimensional functions of X_i and maximizing the unconstrained function given by Eq. (28) with respect to λ is repeated, until the problem converges. For details of the one dimensional minimization and the maximization of dual problem, the paper by the same author (Salajegheh 1994a) can be consulted.

Thus, the overall process is composed of two level approximations. In the first level, all the functions that are computationally expensive to evaluate, are approximated. These functions - include element forces, displacements, etc. Then, by substituting these approximate functions into the original problem, an approximate design problem will be achieved. In the second level approximation, the approximate problem obtained in the first level, is again approximated to achieve a convex and separable problem. This problem is now can be solved by dual methods for continuous and discrete variables.

The main characteristic of a discrete optimization problem is that in general, the discrete design space is disjoint and non convex. Thus, the optimum solution may be a local optimum. There is no guarantee that the final solution is a global optimum. To assure that the solution is a local optimum, there are various methods to check the optimality. One such method is that the Kuhn-Tucker conditions (Vanderplaats 1984) should be satisfied at the optimum point. However, in discrete design problems, any of the constraints may not be active at the optimum point, as the constraints may not pass through the discrete points. So, even local optimality of the solution cannot be assured, unless a great amount of search is performed. In the present study, different starting points have been used and the results are also compared with those of branch and bound method and it is observed that satisfactory results are obtained.

The only methods that are capable of finding the global optimum are simulated annealing and genetic algorithms (see the review paper by Arora, *et al.* 1994). However, these methods are not efficient as there are a great number of function evaluations involved.

5. Continuous-discrete optimization process

The overall optimization procedure is summarized as follows:

(1) Perform an analysis of the structure with the proposed design.

(2) Evaluate all the constraints and retain only the critical or near critical constraints.

(3) Calculate the gradients of the structural responses (forces, etc.) with respect to the intermediate variables for the responses included in the objective function and the retained constraints.

(4) Using these gradients, construct the approximation of the responses, and evaluate the appro-

ximate stresses. Also create the approximate explicit nonlinear problem.
(5) Solve the approximate continuous variable optimization problem with the imposed move limits.
(6) Check for convergence to the continuous optimum; if satisfied, proceed to discrete variable optimization. Otherwise update the problem and repeat from Step (1)
(7) Solve the current approximate problem with discrete variables by dual methods with move limits.
(8) Check for convergence. If converged terminate. Otherwise update the discrete problem and repeat Steps similar to (1), (2), (3), (4) and (7).

The efficiency of the method is based on the number of required analyses of the structure during the optimization process. The method has been applied to a number of problems for discrete sizing and shape variable optimization and the numerical results indicate that the approach is very efficient as compared to the results obtained in a previous study using branch and bound for the discrete solution (Salajegheh 1994c). Computationally, the numerical results show that the duality theory approach is typically one to two orders of magnitude more efficient than the branch and bound approach for problems of more than twenty discrete variables. Thus, it is concluded that duality theory is normally the preferred approach.

6. Examples

Three examples are offered here to demonstrate the efficiency of the method. The DOT (VMA 1992) program is used to solve the continuous variable optimization and to maximize the dual problem. For comparison of the results, the problems are chosen from Vanderplaats and Thomas (1993), Salajegheh (1994c) and Moore and Vanderplaats (1990).

6.1. Cantilever plate

This example consists of finding the minimum mass of the 20 element cantilever plate shown in Fig. 1. The plate is loaded with a 450.0 in-lb moment at the tip and has material properties of Young's modulus $E=10.0\ E6$ psi, Poisson's ratio $\nu=0.3$, and weight density $\rho=0.298$ lb/in^3. The 20 element thicknesses are the design variables. The initial element thicknesses are all 1.0 inch. There is a displacement constraint on the tip of 0.5 inches and bending stress constraints on the elements of 30000 psi. The discrete set for all the design variables are considered as

$$t_i \in \{0.30,\ 0.31,\ 0.32,\ 0.33, \cdots\}\ \text{(in)}$$

The continuous solution to this problem was obtained by Vanderplaats and Thomas (1993). This problem is statically determinate if a beam model is used and nearly statically determinate

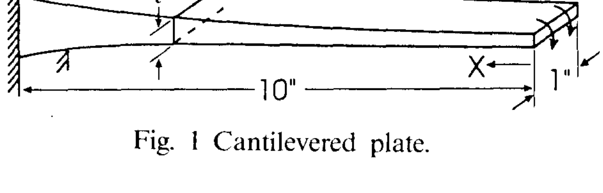

Fig. 1 Cantilevered plate.

Table 1 Results for cantilever plate

Design variable	Continuous optimum	Discrete optimum (B & B)*	Discrete optimum (dual)
1	0.430	0.43	0.43
2	0.425	0.43	0.43
3	0.420	0.42	0.42
4	0.413	0.42	0.42
5	0.406	0.40	0.41
6	0.398	0.40	0.40
7	0.392	0.40	0.40
8	0.387	0.38	0.39
9	0.379	0.38	0.38
10	0.369	0.37	0.37
11	0.360	0.36	0.36
12	0.350	0.35	0.36
13	0.350	0.35	0.35
14	0.328	0.33	0.33
15	0.315	0.31	0.31
16	0.300	0.30	0.30
17	0.300	0.30	0.30
18	0.300	0.30	0.30
19	0.300	0.30	0.30
20	0.300	0.30	0.30
Mass (lb)	1.076	1.077	1.078

*Branch & Bound

with the plate model. Therefore, approximations of the displacement with respect to the reciprocals of the design variables and element forces and the corresponding element stresses with respect to the design variables are nearly exact. The continuous optimum solution is achieved in nearly one design cycle. In fact, two iterations are performed, the results of both are almost similar. The discrete solution is also obtained in one cycle. Again two iterations are carried out, the results of which are the same. Thus 4 analyses of the structure are required to complete both the continuous and discrete solutions, 2 of which are used only to check the feasibility of the solutions and the assurance of the convergence. The results for this problem are presented in Table 1. The results are close to those of branch and bound method (Salajegheh 1994c), however the computer time required to complete the discrete solution by dual approach is about 1/15 of the time needed by branch and bound.

6.2. Cantilever shell

This example consists of finding the minimum volume of the cantilever shell shown in Fig. 2. The shell has a distributed load of 7848 lb applied to the free end and material properties of $E=2.9E7$ psi and $\nu=0.3$. The half model of the structure shown in Fig. 3 was used with symmetric boundary conditions for the optimization. This model has 12 thin plate elements along the 90 degree arc and 48 elements along the length of the structure (the same as Vanderplaats and Thomas 1993). Initially the shell thickness is 1.0 inch and the radius of each end is 2.5 inches. The design variables are considered as the thickness and radii of the fixed and free

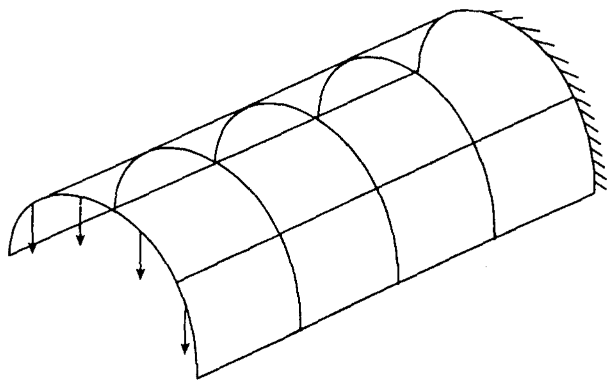

Fig. 2 Cantilevered shell.

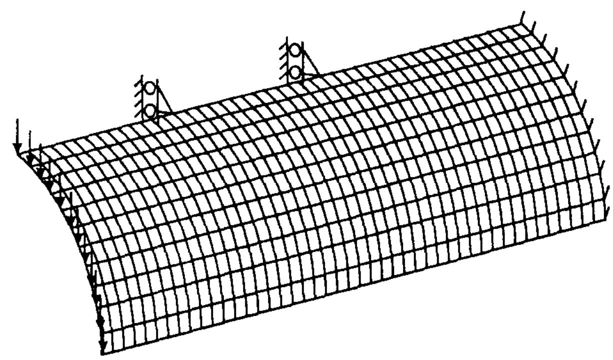

Fig. 3 Cantilevered shell model.

ends; $X=[t, r_1, r_2]$. r_1 and r_2 are the radii of the fixed and free ends, respectively. In fact, in this problem both sizing and shape design variables are considered. The Von Mises stress on the top and bottom of each element is considered to be less than 10000 psi. The initial structure has constraints that are violated by 142.2%.

The discrete values for the design variables are taken as;

$$t \in \{0.5,\ 0.75,\ 1.0,\ 1.5,\ 2.0,\ 2.5, \cdots\} \text{ (in)}$$

$$r_1,\ r_2 \in \{0.1,\ 0.2,\ 0.3,\ 0.4,\ 0.5, \cdots\} \text{ (in)}$$

The continuous optimum design is completed with 4 analyses of the structure and one extra analysis of the shell is required to obtain the discrete solution. The results are given in Table 2. The results of branch and bound and dual methods are similar and the computer time for both approaches is nearly the same. This is because there are only three design variables involved in this problem.

6.3. Pressure loaded plate

This example consists of finding the minimum weight design of a 100×100 inch clamped plate subject to a uniform pressure load of 100 psi (Moore and Vanderplaats 1990). The plate is made of steel with material properties of $E=29.0E6$ psi, $\nu=0.208$, and $\rho=0.283$ lb/in^3. The

Table 2 Results for cantilever shell

Design variable	Continuous optimum	Discrete solution (B & B)*	Discrete solution (dual)
t	0.72	0.75	0.75
r_1	5.15	5.20	5.20
r_2	1.62	1.60	1.60
Weight (in^3)	81.0	84.8	84.8

*Branch & Bound

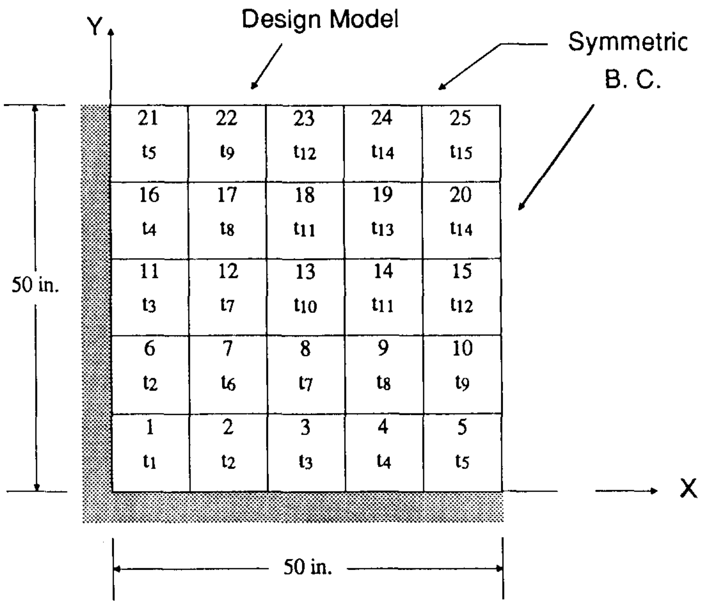

Fig. 4 Pressure loaded plate.

initial thickness of the plate is 2.0 in. and the Von Mises stress on each element is considered to be less than 10000 psi.

The analysis and design model consist of a 25 finite element quarter model of the plate with the appropriate symmetry boundary conditions and is shown in Fig. 4. The lower and upper bounds on the plate thicknesses are 0.01 and 10.0 in., respectively. In this problem, the 25 plate thicknesses are linked so that they are controlled by 15 independent design variables in the configuration shown in Fig. 4. The design model was also used by Moore and Vanderplaats (1990).

All the design variables are allowed to take the following discrete values;

$$t_i \in \{0.01,\ 0.02,\ 0.5,\ 1.0,\ 1.5,\ 2.0,\ 2.5, \cdots\}\ \text{(in)}$$

The continuous optimization is completed in 11 structural analyses, and 2 additional analyses are required for discrete optimization, however, the last analysis is only used to assure the convergence. The results are presented in Table 3. In this problem, there is a significant difference between the time required by branch and bound and the present method. The computer time measured in clock time for both approaches and the time required for dual method is about

Table 3 Results for pressure loaded plate

Design variable	Continuous solution	Discrete solution (B & B)*	Discrete solution (dual)
1	0.01	0.01	0.01
2	0.01	0.01	0.01
3	2.50	2.50	2.50
4	3.65	4.00	3.50
5	4.75	5.00	5.00
6	0.02	0.02	0.02
7	0.01	0.01	0.01
8	0.02	0.02	0.02
9	0.02	0.02	0.02
10	2.15	2.00	2.00
11	0.01	0.01	0.01
12	4.44	4.50	4.50
13	4.33	4.50	4.50
14	4.92	5.00	5.00
15	4.41	4.50	4.50
mass (lb)	1474.0	1514.0	1490.4

*Branch & Bound

1/20 of that needed for branch and bound. In addition, the optimum weight obtained by the present approach is less than that obtained by Salajegheh (1994c) without violating any of the constraints. At the optimum point, a final analysis is carried out and the constraints are evaluated. The feasibility of the solution is checked through the values of the constraints.

In the examples under investigation, the numerical results indicate that for some variables, the optimum values are the same as round-off solution. This is basically due to the fact that in these cases the optimum solution is not far away from the continuous solution. The other reason is that the available discrete values are widely spaced and thus some of the design variables are forced to choose the nearest discrete values.

7. Conclusions

A two level approximation method is presented to achieve the continuous-discrete variable optimization of plate type structures. To reduce the number of detailed finite element analyses of the structure, the element forces are approximated in terms of some intermediate variables. In order to reduce the cost of discrete optimization, a dual strategy is used after the completion of the continuous variable optimization. In the process of optimization, only the critical constraints are retained to reduce the cost of gradient calculation. It is shown that the combination of force approximation and dual methods in conjunction with the idea of constraint deletion form the basis of an efficient optimization method. In particular, for plate type structures, with great number of degrees of freedom and large number of constraints and design variables, the optimum design is achieved very efficiently. The results are compared with those of branch and bound and the computational time is reduced substantially.

References

Arora, J.S., Huang, M.W. and Hsieh, C.C. (1994), "Methods for optimization of nonlinear problems with discrete variables: A review", *Struct. Optim.*, **8**(2/3), 69-85.

DOT Users Manual (1992), VMA Engineering, Goleta, CA.

Fleury, C. and Braibant, V. (1986), "Structural optimization: A new dual method using mixed variables", *Int. J. Num. Meth. Engng*, **23**(3), 409-429.

Moore, G.J. and Vanderplaats, G.N. (1990), "Improved approximations for static stress constraints in shape optimal design of shell structures", *Proc. AIAA/ASME/ASCE/ASC 31st Structures, Structural Dynamics and Materials Conf.*, (held in Long beach, CA), Washington, D.C.: AIAA., 161-170.

Salajegheh, E. (1984), "Optimum design of double layer grids", *Proc. 3rd Int. Conference of Space Structures*, held at the Univ. of Surrey, England, Nooshin, H. (Editor), Elsevier Applied Publishers, London, 661-668.

Salajegheh, E. and Vanderplaats, G.N. (1986, 1987), "An efficient approximation method for structural synthesis with reference to space structures", *Int. J. of Space Structures*, **2**, 165-175.

Salajegheh, E. and Vanderplaats, G.N. (1993a), "Efficient optimum design of structures with discrete design variables", *Int. J. Space Structures*, **8**(3), 199-208.

Salajegheh, E. and Vanderplaats, G.N. (1993b), "Optimum design of trusses with discrete sizing and shape variables", *Struct. Optim.* **6**(2), 79-85.

Salajegheh, E. (1993a), "Efficient optimum design of structures", *Proc. 4th Int. Conference on Space Structures*, held at the Univ. of Surrey, England, Parke, G.A.P. and Howard, C.M. (Editors), Thomas Telford Services Ltd., London, 1735-1745.

Salajegheh, E. (1993b), "Structural optimization technique involving discrete design variables", *Proc. 2nd Asian-Pacific Conference on Computational Mechanics*, held at the Univ. of New South Wales, Australia, Valliappan, S., Pulmano, V.A. and Tin-Loi, F. (Editors), Balkema, Rotterdam, 975-980.

Salajegheh, E. (1994a), "Approximate discrete variable optimization of frame structures with dual methods", *Proc. 17th Energy-Sources Technology Conference & Exhibition (ETCE)*, held at the University of Texas, U.S.A., ASME 1994, PD, **61**, Tribology Symposium, 81-89.

Salajegheh, E. (1994b), "A two level approximation technique for structural optimization", In press in the Journal of Engineering, Islamic Republic of Iran.

Salajegheh, E. (1994c), "Optimum design of plate and shell structures with discrete design variables", *Proc. of the Second International Conference on Computational Structures Technology*, Topping, B.H.V. and Papadrakakis, M. (Editors), Advances in Structural Optimization, Civil-Comp Ltd., Edinburgh, Scotland, 113-117.

Schmit, L.A. and Miura, H. (1976), "Approximation concepts for efficient structural synthesis", NASA CR-2552.

Schmit, L.A. and Fleury, C. (1980), "Discrete-continuous variables structural synthesis using dual methods", *AIAA J.*, **18**, 1515-1524.

Vanderplaats, G.N. (1984), *Numerical optimization techniques for engineering design: with applications*, McGraw-Hill, Inc., New York.

Vanderplaats, G.N. and Salajegheh, E. (1988), "An efficient approximation technique for frequency constraints in frame optimization", *Int. J. Num. Mech. Engng.*, **26**, 1057-1069.

Vanderplaats, G.N. and Salajegheh, E. (1989), "A new approximation method for stress constraints in structural synthesis", *AIAA J.*, **27**, 352-358.

Vanderplaats, G.N. and Thomas, H.L. (1993), "An improved approximation for stress constraints in plate structures", *Struct. Optim.* **6**(4), 1-7.

Vanderplaats, G.N. and Salajegheh, E. (1994), "An approximation method for structural synthesis with discrete sizing and shape variables, using duality theory", *AIAA J.*, to appear.

Zienkiewicz, O.C. and Taylor, R.L. (1989), *The finite element method*, 4th Edn, McGraw-Hill Book Company, UK.