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A combined finite element-Riccati transfer matrix method for free vibration of structures

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Abstract. A combination of Riccati transfer matrix method and finite element method is proposed for obtaining vibration frequencies of structures. This method reduces the propagation of round-off errors produced in the standard transfer matrix method and finds out the values of the frequency by Newton-Raphson method. By this technique, the number of nodes required in the regular finite element method is reduced and therefore a microcomputer may be used. Besides, no plotting of the value of the determinant versus assumed frequency is necessary. As the application of this method, some numerical examples are presented to demonstrate the accuracy as well as the capability of the proposed method for the vibration of structures.

Key words: finite element; transfer matrix; Riccati transformation; vibration, Newton-Raphson method

1. Introduction

In vibration analysis of structures, exact solutions for the natural frequencies are possible only for a limited set of simple structures and boundary conditions. Approximate numerical methods are therefore important for the analysis of more complex systems. The different numerical methods available include the Holzer-Myklestad type transfer matrix method. This method is successful for systems described by a single space variable such as beams and shafts.

For more complicated structures, the finite element method has proved to be powerful and versatile. However, the disadvantage of the finite element method is that for some systems large matrices are produced which require large computers to handle them. In order to reduce the size of the matrices, some substructure techniques have been proposed which consist of keeping the important degrees of freedom and suppressing the less important ones. Which degrees of freedom in the substructure are to be retained depends on judgment and on the physical system. However, this approach may lead to considerable inaccuracy if the wrong degrees of freedom are suppressed. Recently, Dokainish and others (Dokanish 1972, Ohga 1983, 1987, Degen 1985) suggested a method in which the finite element technique is combined with the transfer matrix approach (FE-TM) for obtaining frequencies of vibration of thin plates and shells. In this approach, as the size of stiffness and mass matrices was equal to the number of degrees of freedom in only one subsystem, it had the advantage of reducing the size of a matrix to much less than that obtained by the ordinary finite element method. However, the method has drawbacks: numerical instabilities occur when transfer matrix method is used for calculation of high resonant frequencies, it requires calculation at a significant number of frequencies and interpolation must

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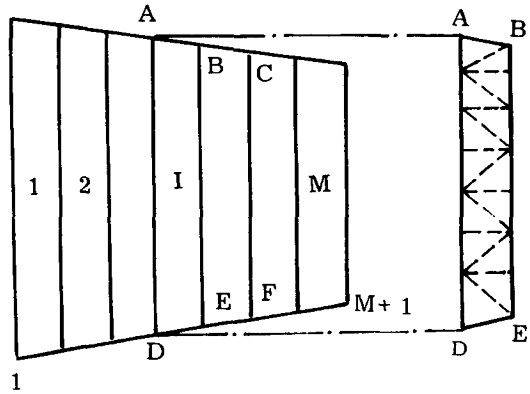

Fig. 1 Subdivision of structure into strips and finite elements.

be used if even only a few of the lowest natural frequencies are to be determined.

To overcome these drawbacks, the Riccati transformation of state vectors is proposed to use as a means of reducing the propagation of round off errors. In this method, the Riccati transfer matrix and its derivatives with respect to frequency are formulated for the right boundary. This transfer matrix relation is then used in the determination of natural frequencies via a Newton-Raphson iterative technique. The proposed method gives a quadratic convergence to a natural frequency from the trial value on either side of the true natural frequency and hence allows a greater degree of error in the selection of the trial frequency.

2. Finite element-Riccati transfer matrix method (FE-RTM)

Without losing generality, we consider the plate shown in Fig. 1. It is divided into m strips and each strip is subdivided into finite elements. Edge BE is the left section of strip $i+1$ or the right section of strip i. There are $2n$ nodes on strip i. Here n nodes on the left section AD, and n nodes on the right section BE.

2.1. Transfer matrix relation for a strip

Proceeding as in Dokanish (1972), we obtain i strip's transfer matrices

$$\left\{\begin{matrix} U \\ -F \end{matrix}\right\}_i^R = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \left\{\begin{matrix} U \\ F \end{matrix}\right\}_i^L = [A]_i \left\{\begin{matrix} U \\ F \end{matrix}\right\}_i^L \tag{1}$$

where

$$[A_{11}]_i = -[B_{12}]_i^{-1}[B_{11}]_i,\quad [A_{12}]_i = [B_{12}]_i^{-1},$$

$$[A_{21}]_i = -[B_{12}]_i + [B_{22}]_i[B_{12}]_i^{-1}[B_{11}]_i,\quad [A_{22}]_i = -[B_{22}]_i[B_{12}]_i^{-1}$$

$$[B] = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = [K]_i - \omega^2[M]_i$$

and $[K]_i$ is the final stiffness matrix of the i strip, $[M]_i$ is the mass matrix of the i strip, ω is the natural frequency of free vibrations $\{U\}_i^L$, $\{U\}_i^R$, $\{F\}_i^L$ and $\{F\}_i^R$ are the left and right displacement and force vectors of section i.

Since the problem of free vibrations is considered, we can get a relation of the form:

$$\begin{Bmatrix} U \\ -F \end{Bmatrix}_i^R = \begin{Bmatrix} U \\ F \end{Bmatrix}_{i+1}^L \tag{2}$$

Substitution in Eq. (1) leads to

$$\begin{Bmatrix} U \\ F \end{Bmatrix}_{i+1}^L = [A]_i \begin{Bmatrix} U \\ F \end{Bmatrix}_i^L \tag{3}$$

2.2. The Riccati transformation of state vectors

In order to reduce the propagation of round-off errors in the standard transfer matrix method, we propose a Riccati transformation of state vectors. In this method, Eq. (3) being rearranged and repartitioned, we obtain

$$\begin{Bmatrix} f \\ e \end{Bmatrix}_{i+1}^L = \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}_i \begin{Bmatrix} f \\ e \end{Bmatrix}_i^L \tag{4}$$

where $\{f\}$ involves half of the state variables known at the left hand boundary and $\{e\}$ contains the another half of the state variables.

As pointed out in Honer (1975), a generalized Riccati transformation at section i may be given by

$$\{f\}_i^L = [S]_i \{e\}_i^L + \{p\}_i \tag{5}$$

where matrix $[S]$ is the Riccati transfer matrix and vector $\{p\}$ contains the forcing terms.

From Eqs. (4) and (5), we obtain

$$\{f\}_{i+1}^L = [S]_{i+1} \{e\}_{i+1}^L + \{p\}_{i+1} \tag{6}$$

where

$$[S]_{i+1} = ([T_{11}][S] + [T_{12}])_i ([T_{21}][S] + [T_{22}])_i^{-1} \tag{7}$$

$$\{p\}_{i+1} = ([T_{11}]_i - [S]_{i+1}[T_{21}]_i)\ \{p\}_i \tag{8}$$

Eqs. (7) and (8) are the general recurrence relations for $[S]$ and $\{p\}$. Since the left hand boundary conditions are homogeneous, the initial conditions are

$$[S]_1 = [0] \tag{9}$$

$$\{p\}_1 = \{0\} \tag{10}$$

Using Eqs. (9) and (10), $[S]$ and $\{p\}$ are transferred from left to right through all the structure, hence we have

$$\{f\}_{m+1}^L = [S]_{m+1} \{e\}_{m+1}^L \tag{11}$$

$$\{p\}_1 = \{p\}_2 = \cdots\cdots = \{p\}_{m+1} = \{0\} \tag{12}$$

Eq. (11) at the right boundary demands that for a non-trivial solution the determinant which depends on the boundary conditions must be set equal to zero. This determinant is simply the characteristic equation that gives the natural frequencies.

Finally, the state variables at each section are determined by transferring from right to left through all the structure. Successive application of Eq. (13) gives $\{e\}$ at any section i. Therefore, we could use Eq. (5) to calculate $\{f\}$ at any section i. And now the solution is completed.

$$\{e\}_i^L=([T_{21}][S]+[T_{22}]_i^{-1}\{e\}_{i+1}^L \tag{13}$$

It is worth notice that the transfer matrix $[A]$ in Eq. (3) for the FE-TM method is replaced by the transfer matrix $[S]$ in Eq. (6) for the FE-RTM method. The dimension of matrix $[S]$ is only half that of the matrix $[A]$. The Riccati transfer matrix method would only require about half the storage requirements of the transfer matrix method.

2.3. Determination of natural frequencies

The boundary conditions of the right edge of the structure usually require some components of state variables to be zeros. When these conditions being added, it becomes essential that the determinant of a portion $[Q]$ of the matrix $[S]$ be zero at the correct natural frequency, for a nontrivial solution. For example, if the left edge of the structure is clamped and the right edge of the structure is simple supported, then $\{f\}$ represents a displacement vector at any section, and $\{e\}$ represents a force vector at any section. According to the right boundary condition, we have

$$\begin{Bmatrix} f_1 \\ 0 \end{Bmatrix}_{m+1}^L = \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix}_{m+1} \begin{Bmatrix} 0 \\ e_2 \end{Bmatrix}_{m+1} \tag{14}$$

hence

$$\{0\}=[S_{22}]_{m+1}\{e_2\}_{m+1}^L \tag{15}$$

where $\{f_1\}_{m+1}$ is a portion of the displacement vector $\{f\}$ corresponding to nonzero elements at the right boundary. $\{e_2\}_{m+1}$ is a portion of the force vector $\{e\}$ corresponding to nonzero elements at the right boundary.

For the nontrivial solution of Eq. (15), it is essential that the determinant of matrix $[S_{22}]_{m+1}$ be zero at the correct natural frequency. The matrix $[Q]$ is, therefore in this particular case, the matrix $[S_{22}]_{m+1}$. That is, the natural frequencies are determined from the roots of the polynomial

$$\Delta(\omega)=det[Q(\omega)]=0 \tag{16}$$

In general, the matrix $[Q]$ is obtained from matrix $[S]_{m+1}$ by deleting the columns corresponding to zero elements of $\{e\}_{m+1}^L$ and deleting the rows corresponding to the nonzero elements of $\{f\}_{m+1}^L$.

Instead of resorting to a trial and error procedure in solving Eq. (16), we adopt the Newton-Raphson iteration technique.

Differentiating Eqs. (4)-(6) each with respect to ω, we obtain Eqs. (17)-(19).

$$\begin{Bmatrix} \dot{f} \\ \dot{e} \end{Bmatrix}_{i+1}^L = \begin{bmatrix} \dot{T}_{11} & \dot{T}_{12} \\ \dot{T}_{21} & \dot{T}_{22} \end{bmatrix}_i \begin{Bmatrix} f \\ e \end{Bmatrix}_i^L + \begin{bmatrix} T_{11} & T_{12} \\ T_{21} & T_{22} \end{bmatrix}_i \begin{Bmatrix} \dot{f} \\ \dot{e} \end{Bmatrix}_i^L \tag{17}$$

$$\{\dot{f}\}_i^L=[\dot{S}]_i\{e\}_i^L+[S]_i\{\dot{e}\}_i^L \tag{18}$$

$$\{\dot{f}\}_{i+1}^L=[\dot{S}]_{i+1}\{e\}_{i+1}^L+[S]_{i+1}\{\dot{e}\}_{i+1}^L \tag{19}$$

where the dot represents the differentiation with respect to ω. From Eqs. (17), (18) and Eq. (5), we obtain Eq. (20) and Eq. (21)

$$\begin{aligned}\{\dot{f}\}_{i+1}^L=&([\dot{T}_{11}]_i[S]_i+[T_{11}]_i[\dot{S}]_i+[\dot{T}_{12}]_i)\{e\}_i^L \\ &+([T_{11}]_i[S]_i+[T_{12}]_i)\{\dot{e}\}_i^L\end{aligned} \tag{20}$$

$$\{\dot{e}\}^L_{i+1}=([\dot{T}_{21}]_i[S]_i+[T_{21}]_i[\dot{S}]_i+[\dot{T}_{22}]_i)\{e\}^L_i +([T_{21}]_i[S]_i+[T_{22}]_i)\{\dot{e}\}^L_i \tag{21}$$

or written in

$$\{\dot{f}\}^L_{i+1}=[\dot{V}]_i\{e\}^L_i+[V]_i\{\dot{e}\}^L_i \tag{22}$$

$$\{\dot{e}\}^L_{i+1}=[\dot{W}]_i\{e\}^L_i+[W]_i\{\dot{e}\}^L_i \tag{23}$$

where

$$[V]_i=[T_{11}]_i[S]_i+[T_{12}]_i \tag{24}$$

$$[\dot{V}]_i=[\dot{T}_{11}]_i[S]_i+[T_{11}]_i[\dot{S}]_i+[\dot{T}_{12}]_i \tag{25}$$

$$[W]_i=[T_{21}]_i[S]_i+[T_{22}]_i \tag{26}$$

$$[\dot{W}]_i=[\dot{T}_{21}]_i[S]_i+[T_{21}]_i[\dot{S}]_i+[\dot{T}_{22}]_i \tag{27}$$

From Eq. (13) and Eq. (26), we obtain

$$\{e\}^L_i=[W]^{-1}_i\{e\}^L_{i+1} \tag{28}$$

and from Eq. (23)

$$\{\dot{e}\}^L_i=[W]^{-1}_i\{\dot{e}\}^L_{i+1}-[W]^{-1}_i[\dot{W}]_i\{e\}^L_i \tag{29}$$

is obtained.
Substitution of Eqs. (28), (29) in Eq. (22) leads to

$$\{\dot{f}\}^L_{i+1}=(-[V]_i[W]^{-1}_i[\dot{W}]_i[W]^{-1}_i+[\dot{V}]_i[W]^{-1}_i)\{e\}^L_{i+1}+[V]_i[W]^{-1}_i\{\dot{e}\}^L_{i+1} \tag{30}$$

From Eqs. (7), (24), (26) and Eq. (30), we obtain Eq. (31)

$$\{\dot{f}\}^L_{i+1}=(-[S]_{i+1}[\dot{W}]_i[W]^{-1}_i+[\dot{V}]_i[W]^{-1}_i)\{e\}^L_{i+1}+[S]_{i+1}\{\dot{e}\}^L_{i+1} \tag{31}$$

Comparing Eq. (19) with Eq. (31), we obtain

$$[\dot{S}]_{i+1}=-[S]_{i+1}[\dot{W}]_i[W]^{-1}_i+[\dot{V}]_i[W]^{-1}_i \tag{32}$$

Eqs. (32), (25) and (27) are the general recurrence relations for $[\dot{S}]$.
On the left hand boundary, from Eq. (18) we have

$$\{\dot{f}\}^L_1=[\dot{S}]_1\{e\}^L_1+[S]_1\{\dot{e}\}^L_1 \tag{33}$$

With the initial condition $\{\dot{f}\}^L_1=\{0\}$ and $[S]_1=[0]$, from Eq. (33), we obtain Eq. (34)

$$[\dot{S}]_1=[0] \tag{34}$$

With the Eqs. (32), (25), (27) and (7), $[\dot{S}]$ are transferred from left to right through all the structure, we then obtain $[\dot{S}]_{m+1}$, and the matrix $[\dot{Q}]$ as well.

The recurrence relation between the trial frequencies based on the Newton-Raphson method is

$$\omega_{new}=\omega_{trial}-\frac{det[Q]}{(det[\dot{Q}_1]+det[\dot{Q}_2]+\cdots+det[\dot{Q}_p])} \tag{35}$$

where p is the order of the matrix $[Q]$. In Eq. (28) the determinants are evaluated at $\omega=\omega_{trial}$

Fig. 2 Clamped beam.

and the coefficients of the matrices $[Q_1]$, $[Q_2]$, $\cdots$, $[Q_p]$ are identical to those of matrix $[Q]$ except for the following coefficients which are related to the coefficients of matrix $[\dot{Q}]$:

$$Q_1(i,\ 1)=\dot{Q}(i,\ 1), \qquad Q_2(i,\ 2)=\dot{Q}(i,\ 2),\ \cdots$$

$$Q_p(i,\ p)=\dot{Q}(i,\ p) \qquad i=1,\ 2,\ \cdots,\ p \tag{36}$$

Hence $[Q]$, $[Q_1]$, $[Q_2]$, $\cdots$, $[Q_p]$ are known, they can be directly used in Eq. (35) to calculate the natural frequencies systematically.

The number of steps for convergence when using this method depends on the closeness of the initial frequency to the true natural frequency. In the vicinity of a root, the convergence is quadratic. The Newton-Raphson iteration technique requires a derivative of the function at each step, so the computation time is double per step. However this increase in computation time per step is offset by the fewer number of steps for the same final accuracy. An additional advantage of the Newton-Raphson method is that it is a single point method requiring only one initial trial value. Besides, it has a known sufficient condition for convergence given by $|\omega_{trial}-\omega_{true}|<=d/(n-1)$ (Lancaster 1964), where d is the separation between the true natural frequency under consideration and its nearest neighbouring natural frequency, and n is the degree of the polynomial $\Delta(\omega)$ under consideration.

3. Numerical examples

In order to investigate the accuracy and the computation efficiency of our method, we develop a program FERTMV-W based on this method on the microcomputer IBMPC-AT and the numerical results are compared with those obtained by the regular finite element method and the ordinary FE-TM method.

3.1. Example 1

With the FE-RTM method, we calculate the natural frequencies of a beam in bending vibration shown in Fig. 2, where the physical parameters of the beam are as follows: length$=$20 m, flexural strength $EJ=6.4\times10^7$ kg$\cdot$m^2 and $\rho F=9.6236\times10^2$ kg$\cdot$sec^2/m^2, here ρ is the mass density and F the area. The beam is divided into 40 elements. The convergence factor $eps=0.00001$, here $eps=(\omega_k-\omega_{k-1})/\omega_k$. The natural frequencies calculated are listed in Table 1. The solutions of the FE method and the ordinary FE-TM method are also listed in Table 1. Table 2 and Table 3 show the trial frequencies, the calculated natural frequencies, the number of iterative steps and the computation time for the FE-RTM method and the FE-TM method. Table 4 shows the comparision of the mode displacements at the right boundary in our example (the maximum mode displacement is normalized 1). From the above results, it can be concluded that FE-RTM method has lower round-off errors and higher computation efficiency. Especially, the mode displacements at the right boundary in our example should be zero, while in the FE-TM method, owing to round-off errors, they are not. This discrepancy becomes more serious for the higher mode. Our FE-RTM method can reduce the propagation of round-off errors produced in the

Table 1 Comparison of natural frequencies for clamped beam (rad/s)

Mode number	Exact solution	FE	FE-RTM	FE-TM
1	14.4239	14.429	14.424	14.424
2	39.7587	39.769	39.759	39.761
3	77.9526	77.998	77.950	77.847
4	128.847	129.04	128.85	128.99
5	192.486	193.81	192.62	193.24

Table 2 The trial frequencies and convergence results for FE-RTM

Mode number	Trial frequency (rad/s)	Calculated frequency (rad/s)	Number of iterations	Computation time (sec)
1	10.0	14.424	6	9
2	25.0	39.759	7	10
3	49.0	77.950	6	9
4	90.0	128.85	6	9
5	170.0	192.63	6	9

Table 3 The trial frequencies and convergence results for FE-TM

Mode number	Trial frequency (rad/s)	Calculated frequency (rad/s)	Number of iterations	Computation time (sec)
1	10.0	14.424	20	12
2	25.0	39.761	23	14
3	49.0	77.847	19	12
4	90.0	128.99	18	11
5	170.0	193.24	19	12

Table 4 The mode displacement W at the right boundary

Mode number	FE-RTM	FE-TM
1	0.0	0.0001
2	0.0	−0.0010
3	0.0	−0.0228
4	0.0	−0.0632
5	0.0	−0.0946

ordinary FE-TM method.

3.2. Example 2

A cantilevered square plate shown in Fig. 3 is analysed in the example. The plate chosen is $300\times300\times10$ cm with a specific weight of 76158 KN/m^3, $\mu=0.3$, $E=2.058\times10^8$ KN/m^2. In the numerical calculation, a half of the plate is divided 20×10 elements. True natural frequencies were obtained after only a few iterations. Table 5, Table 6, Table 7, Table 8 and Table 9 show

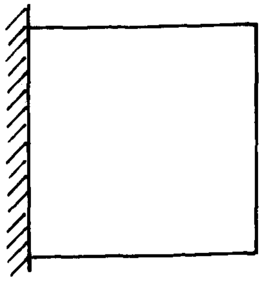

Fig. 3 Cantilever plate.

Table 5 Comparison of natural frequencies for clamped plate (rad/s)

Mode number	FE solution	FE-RTM solution	FE-TM solution
1	60.264	60.272	60.272
2	147.03	147.08	147.10
3	364.58	364.65	364.71
4	460.97	460.88	460.95
5	529.67	529.78	529.51
6	913.04	913.15	912.47

Table 6 The trial frequencies and convergence results for FE-RTM

Mode number	Trial frequency (rad/s)	Calculated frequency (rad/s)	Number of iterations	Computation time (sec)
1	50.0	60.272	5	37
2	120.0	147.08	6	45
3	300.0	364.65	5	37
4	400.0	460.88	5	37
5	500.0	529.78	4	30
6	700.0	913.15	7	52

Table 7 The trial frequencies and convergence results for FE-TM

Mode number	Trial frequency (rad/s)	Calculated frequency (rad/s)	Number of iterations	Computation time (sec)
1	50.0	60.272	17	50
2	120.0	147.10	20	58
3	300.0	364.71	18	52
4	400.0	460.95	17	50
5	500.0	529.51	14	40
6	700.0	912.47	23	67

a comparison among the FE-RTM solutions, FE-TM solutions and the FE solutions, where FE-RTM, FE-TM and FE methods are applied to 20×10 same mesh pattern. Similar results as in Example 1 are obtained.

Table 8 The maximum mode flexure moment M at the right boundary (the maximum mode moment M at all nodes is normalized 1)

Mode number	FE-RTM	FE-TM
1	0.0	0.0015
2	0.0	0.0024
3	0.0	0.0116
4	0.0	0.0346
5	0.0	0.0763
6	0.0	0.0956

Table 9 Comparison of computation time for clamped plate

Method by applying	Computation time (sec)
FE method	600
FE-TM method	317
FE-RTM method	238

4. Conclusion

A combination of Riccati transfer matrix method and finite element method has been proposed for obtaining natural frequencies of structures. Results for vibration frequencies of the plate show that the method enables the user to successfully calculate the frequency from one assumed value of frequency. The method has the advantage of reducing the size of a matrix to much less than that obtained by the FE method or the ordinary FE-TM method; it also has an additional advantage in that one does not need to calculate so many values of the determinants and plot them versus assumed values of the frequencies. Other advantages of the method presented are that it allows a greater degree of error in the selection of trial frequencies and gives quadratic convergence in calculating natural frequencies.

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