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Simplified dynamic analysis of slender tapered thin-walled towers with additional mass and rigidity

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Abstract. A linearly tapered, doubly symmetric thin-walled closed member, such as power-transmission towers and tourist towers, are often characterized by local variation in mass and/or rigidity, due to additional mass and rigidity. On the preliminary stage of design the closed-form solution is more effective than the finite element method. In order to propose approximate solutions, the discontinuous and local variation in mass and/or rigidity is treated continuously by means of a usable function proposed by Takabatake(1988, 1991, 1993). Thus, a simplified analytical method and approximate solutions for the free and forced transverse vibrations in linear elasticity are demonstrated in general by means of the Galerkin method. The solutions proposed here are examined from the results obtained using the Galerkin method and Wilson-θ method and from the results obtained using NASTRAN.

Key words: elastic analysis; dynamic structures; local mass; local rigidity; natural frequency; variable thin-walled member.

1. Introduction

In Japan, power-transmission towers constructed in urban districts and tourist towers are often made of a cantilevered and linearly tapered thin-walled member with a circular or polygonal transverse cross-section. Such towers have discontinuous and local variation in the mass and/or rigidity, due to insulators, insulators crossarms, and floors for observatories and/or lookout restaurants. Those additional mass and rigidity are independent of the structural mass and rigidity of the linearly tapered member.

Takabatake (1990) presented a static closed-form solution for a linearly tapered and doubly symmetric thin-walled member, subjected to arbitrary loads. Gaines and Volterra (1968) discussed frequencies of tapered beams. Wang and Lee (1973) extended the power series solution of Rohde (1953) to the large deflection problem for tapered cantilevers. Prathap and Varadan (1975) presented finite deflection of a tapered cantilever with any arbitrary inertia variation. Bouchet and Biswas (1977) presented nonlinear analysis of towers and stacks by means of FEM. To (1979) presented an explicit expression for mass and stiffness matrices

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of two higher order tapered beam elements for vibration analysis. Gupta (1984, 1985) derived dynamic stiffness and mass matrices in explicit form for a linearly tapered beam element for FEM. However, the general analytical method and approximate solutions for a linearly tapered thin-walled closed member with discontinuous variation in mass and/or rigidity are scarce. Especially, on the preliminary stage of design the closed-form simplified solution is more effective than the finite element method.

The purpose of this paper is to present a simplified dynamic analysis, in linear elasticity, for a cantilevered and linearly tapered thin-walled member with discontinuous additional mass and/or structural rigidity, subjected to the transverse vibrations. The local variation in mass and/or rigidity due to the additional mass and/or additional rigidity, distributed discontinuously, is replaced with a continuous function by means of a usable function proposed by Takabatake (1988, 1991, 1993). The use of the function simplifies a general analytical method for such a member.

First, the governing equation of motion, including discontinuous and local variation in mass and/or rigidity, is presented, in which the transverse shear deformation in such a member is neglected in practice, as shown by Takabatake (1990). Second, the free transverse vibration is presented by means of the Galerkin method; and an approximate solution for the natural frequencies is proposed in closed-form. The validity of the proposed solutions is shown through comparison with the results obtained from the FEM code NASTRAN and the previous results. Third, forced transverse vibration is presented by means of the Galerkin method; and an approximate solution is proposed in closed-form. The analytical method and approximate solution are examined from numerical results obtained using NASTRAN and the previous results.

2. Governing equations of motion

Consider a linearly tapered and doubly symmetric thin-walled closed section, as shown in Fig. 1. The two diameters of the transverse cross section are assumed to be far smaller than the length of the member; and the thickness is also assumed to be far smaller than the size of the transverse cross section. The coordinate axes are prepared as shown in Fig. 1, in which the x-axis takes the axial line, being the centroidal axis; and its original point is taken as the intersection of the generating lines on the middle surface bisecting the thickness at each point. The y-and z-axes are the principal axes of the transverse cross section, prescribed by the value of x.

Takabatake (1983, 1987, 1990) showed that for such a member the transverse shear deformations and the inplane distortions of the transverse cross section are negligible. For simplicity, the boundary conditions are considered to be free at the top ($x=x_T$) and fixed at the base ($x=x_B$). Assuming linear strain-displacement relations, the equations of motion can be written as

$$\delta w : \rho A\ddot{w} + c_w\dot{w} + (EI_y w'')'' = p_z - m_y' \tag{1}$$

$$\delta\theta : \rho I_p\ddot{\theta} + c_0\dot{\theta} - (\rho\Gamma\ddot{\theta}')' + (\rho\Gamma\theta'')'' - (GJ\theta')' = m_x \tag{2}$$

in which $w(x, t)$= the displacement component on the axial line; $\theta(x, t)$= the torsional angle along the x-axis; c_w and c_θ = damping coefficients for the transverse and torsional vibrations, respectively; ρ= the mass density of the member; $P_z(x, t)$= the z-component of an lateral

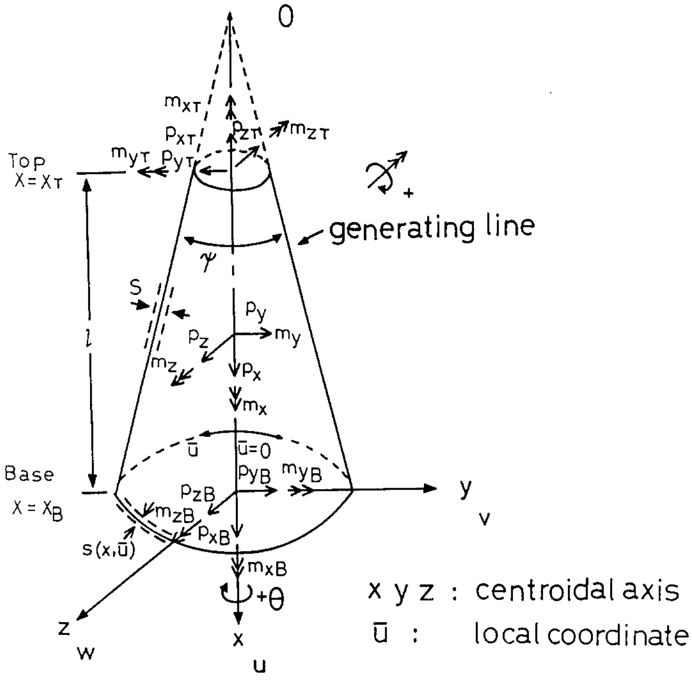

Fig. 1 Linearly tapered doubly symmetric thin-walled member.

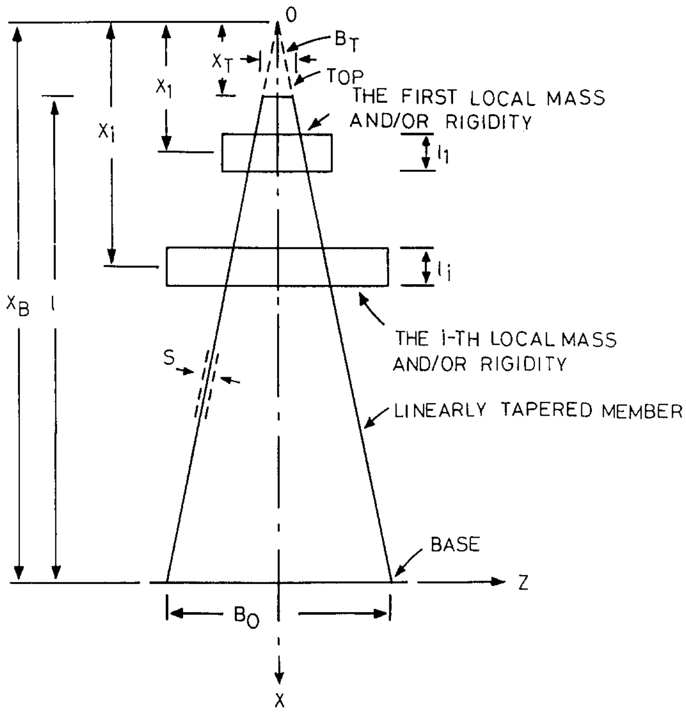

Fig. 2 Variable member with discontinuous and local variation in mass and rigidity.

external load acting along the central axis; $m_x(x, t)$ and $m_y(x,t)$= the external torsional and bending moment acting along the central axis, respectively. These displacements, torsional angle, external loads, and external moments are considered as positive when they point toward each positive direction of the coordinate axes for loads while right-hand rule for moment vectors, as shown in Fig. 1; but the positive external forces and external moments acting on a negative surface are defined inversely. Primes and dots indicate differentiation with respect to x and t, respectively.

On the other hand, the sectional quantities, A, I_y, J, Γ and I_p, are functions of x. Such a thin-walled variable member has often discontinuous and local variation in mass and/or structural rigidity due to additional mass and/or additional rigidity, as shown in Fig. 2. Takabatake (1988, 1991, 1993) demonstrated the effectiveness of using an operator function for the lateral buckling of I beams with web stiffeners and batten plates and for analysis of voided plates and of tube structures. Applying the function to the local variation in mass and/or structural rigidity in current problem, the sectional quantities of the current member may be expressed as follows:

$$\rho(x)A(x) = \rho_0 A_0(x) + \rho_i A_i D(x - x_i) \tag{3}$$
$$\rho(x)I_p(x) = \rho_0 I_{p0}(x) + \rho_i I_{pi} D(x - x_i) \tag{4}$$
$$E(x)I_y(x) = E_0 I_{y0}(x) + E_i I_{yi} D(x - x_i) \tag{5}$$
$$G(x)J(x) = G_0 J_0(x) + G_i J_i D(x - x_i) \tag{6}$$
$$G(x)\Gamma(x) = G_0 \Gamma_0(x) + G_i \Gamma_i D(x - x_i) \tag{7}$$

The first term in the right side of the above equations is the sectional quantity due to only the linearly tapered thin-walled member, and it is indicated with the subscript 0. On the other hand, the second term is the sectional quantity due to the additional mass and/or additional rigidity, in which the subscript i indicates to be based on the i-th local variation. The operator function $D(x\text{-}x_i)$ is defined as

$$D(x - x_i) = \begin{cases} 1 & \text{for } x_i - \dfrac{l_i}{2} \leq x \leq x_i + \dfrac{l_i}{2} \\ 0 & \text{for all others} \end{cases} \tag{8}$$

in which l_i= the length of the i-th local variation; and x_i= the value of x at the midpoint of the i-th local variation. The function is defined as a function of which the Dirac function exists continuously in a prescribed region. For the current problem the function has a value in only the location where the additional mass and/or additional rigidity exist, and replaces local variation in mass and/or rigidity, distributed discontinuously, with a continuous function.

In common with the work of Takabatake (1986, 1990), the variation of the thickness in the axial direction is assumed to be negligible in practice. Then, these sectional quantities for a linearly tapered and doubly symmetric member can be separated into variable and constant as follows:

$$A_o(x) = x\bar{A} \tag{9}$$
$$I_{y0}(x) = x^3\bar{I}_y \tag{10}$$

$$J_0(x) = x^3\bar{J} \tag{11}$$
$$\Gamma_0(x) = x^5\bar{\Gamma} \tag{12}$$
$$I_{p0}(x) = x^3\bar{I}_p \tag{13}$$

in which $\bar{A}, \bar{I}_y, \bar{J}, \bar{\Gamma}$ and $\bar{I}_p$ = sectional constants depending on both the shape of the transverse cross section and the tapering angle, as given in the work of Takabatake (1990). Applying Eqs. (9) - (13) to Eqs. (3) - (7), the current sectional quantities may be written as

$$\rho(x)A(x) = \rho_0\bar{A}\,\alpha_{\rho A}(x) \tag{14}$$
$$\rho(x)I_p(x) = \rho_0\bar{I}_p\alpha_{\rho I}(x) \tag{15}$$
$$E(x)I_y(x) = E_0\bar{I}_y\alpha_{EI}(x) \tag{16}$$
$$G(x)\,J(x) = G_0\bar{J}\,\alpha_{GJ}(x) \tag{17}$$
$$G(x)\Gamma(x) = G_0\bar{\Gamma}\alpha_{G\Gamma}(x) \tag{18}$$

in which $\alpha_{\rho A}, \alpha_{\rho I}$, α_{EI}, α_{GJ} and $\alpha_{G\Gamma}$ are defined as

$$\alpha_{\rho A} = x + \frac{\rho_i A_i}{\rho_0\bar{A}} D(x-x_i) \tag{19}$$

$$\alpha_{\rho I} = x^3 + \frac{\rho_i I_{pi}}{\rho_0\bar{I}_p} D(x-x_i) \tag{20}$$

$$\alpha_{EI} = x^3 + \frac{E_i I_{yi}}{E_0\bar{I}_y} D(x-x_i) \tag{21}$$

$$\alpha_{GJ} = x^3 + \frac{G_i J_i}{G_0\bar{J}} D(x-x_i) \tag{22}$$

$$\alpha_{G\Gamma} = x^5 + \frac{G_i\Gamma_i}{G_0\bar{\Gamma}} D(x-x_i) \tag{23}$$

On the other hand, the sectional quantities for a uniform member including local variation in mass and/or rigidity are given in Appendix.

3. Free transverse vibrations

Consider the free transverse vibrations of a linearly tapered thin-walled closed member. The method of separation of the variables is employed assuming that

$$w(x,\,t) = X(x)f(t) \tag{24}$$

Then, the equation of free transverse vibrations for $X(x)$ is

$$[E(x)\,I_y(x)X'']'' - \omega^2\rho(x)\,A(x)\,X = 0 \tag{25}$$

in which ω^2 = a constant. The substitution of Eqs. (14) and (16) into the above equation yields

$$\alpha_{EI} X'''' + 2\alpha_{EI}' X''' + \alpha_{EI}'' X'' - k^4 \alpha_{\rho A} X(x) = 0 \tag{26}$$

in which the constant k^2 is defined as

$$k^2 = \sqrt{\frac{\omega^2 \rho_0 \bar{A}}{E_0 \bar{I}_y}} \tag{27}$$

Natural frequencies for a variable member with discontinuous and local variation in mass and/or rigidity are proposed by means of the Galerkin method. $X(x)$ is expressed by a power series expansion as follows:

$$X(x) = \sum_{n=1} c_n \phi_n(x) \tag{28}$$

in which c_n = unknown coefficients; and ϕ_n = shape functions. The Galerkin equations for Eq. (26) are written as

$$\delta c_n : \sum_{m=1} c_m [A_{nm} - k^4 B_{nm}] = 0 \tag{29}$$

in which the coefficients, A_{nm} and B_{nm}, are defined as

$$A_{nm} = \int_{X_I}^{X_B} (\alpha_{EI} \phi_m'''' + 2\alpha_{EI}' \phi_m''' + \alpha_{EI}'' \phi_m'') \phi_n dx \tag{30}$$

$$B_{nm} = \int_{X_I}^{X_B} \alpha_{\rho A} \phi_m \phi_n dx \tag{31}$$

The integrals including the operator function for a function $f(x)$ are calculated as follows:

$$\int_{X_I}^{X_B} D(x - x_i) f(x) dx = \int_{X_i - \frac{l_i}{2}}^{X_i + \frac{l_i}{2}} f(\xi)\, d\xi \tag{32}$$

$$\int_{X_I}^{X_B} D^{(n)}(x - x_i) f(x) dx = \int_{X_i - \frac{l_i}{2}}^{X_i + \frac{l_i}{2}} (-1)^n f^{(n)}(\xi) d\xi \tag{33}$$

in which the superscripts enclosed within parentheses indicate the differential order. Since in practice $l_i \ll l$, the above equations can be approximated as

$$\int_{X_I}^{X_B} D(x - x_i) f(x) dx \cong l_i f(x_i) \tag{34}$$

$$\int_{X_I}^{X_B} D^{(n)}(x - x_i) f(x) dx \cong l_i (-1)^n f^{(n)}(x_i) \tag{35}$$

Hence, the coefficients, A_{nm} and B_{nm}, may be rewritten as

$$A_{nm} = \int_{X_T}^{X_B} (x^3\phi_m'''' + 6x^2\phi_m''' + 6x\phi_m'')\phi_n dx + \sum_{i=1} \frac{E_i I_{yi}}{E_0 \bar{I}_{y0}} l_i \phi_m''(x_i)\phi_n''(x_i) \tag{36}$$

$$B_{nm} = \int_{X_T}^{X_B} x\phi_m\phi_n dx + \sum_{i=1} \frac{\rho_i A_i}{\rho_0 \bar{A}} l_i \phi_m(x_i)\phi_n(x_i) \tag{37}$$

The natural frequencies ω_n for current variable members are obtained by solving Eq. (29).

The natural frequencies for the variable members are now obtained numerically from the above procedure. Then, consider a closed-form approximate expression for the natural frequencies. Assuming that the natural frequencies are dominated by the diagonal terms in the square matrices A_{nm} and B_{nm}, the approximate natural frequencies ω_n are obtained as

$$\omega_n = \sqrt{\frac{A_{nn}}{B_{nn}}} \sqrt{\frac{E_0 \bar{I}_y}{\rho_0 \bar{A}}} \tag{38}$$

The above equation will be considered as a closed-form solution, because the constant, A_{nn} and B_{nn}, are calculated easily and previously by the use of computer.

4. Numerical results for free transverse vibrations

The exactness of approximate solution proposed for the free transverse vibrations of a linearly tapered member with additional mass and/or rigidity will be proven from numerical computation. The variable member is assumed to be circular section; the total height l = 38.6 m; thickness s=0.016m; diameter at the base B_0=1.25 m; Young's modulus E=205.9 GN/m^2 (21Gkgf/m^2); Poisson's ratio ν=0.3; and mass density ρ_0=7.87KN.sec^2/m^4 (802 kgf.sec^2/m^4). A variable member is obtained by changing arbitrarily the tapering ratio B_l/B_0 from 1.0 to 0.2, in which B_l=the diameter at the top. The tapering ratio B_l/B_0=1.0 means the uniform member. For simplicity, the shape functions for the current cantilevered variable members use approximately natural functions for the transverse vibrations of cantilevered uniform member. Namely,

$$\phi_n = \mathrm{ch}[k_n(x_B - x)] - \cos[k_n(x_B - x)] - \alpha_n\{\mathrm{sh}[k_n(x_B - x)] - \sin[k_n(x_B - x)]\} \tag{39}$$

The numerical results are shown in Table 1. In this table the columns with ADMASS=0 and ADST=0 show natural frequencies of tapered thin-walled members without additional mass and rigidity. On the other hand, the columns with ADMASS=1 and ADST=0 show natural frequencies of tapered members with local variation in only mass due to the following two additional local masses: for the first mass (i=1) $x_1 - x_T$=2.6 m, l_1=0.4 m, and $\rho_1 A_1/(\rho_0 A_0^*)$=0.5; while for the second mass (i=2) $x_2 - x_T$=5.6 m, l_2=0.4 m, and $\rho_2 A_2/(\rho_0 A^*)$=1.0, in which A_0^*=the sectional area at the base. On the other hand, the columns with AD MASS=1 and ADST=1 show natural frequencies of tapered members with local variation in mass and rigidity due to the above additional mass and the following additional rigidity placed on the above positions: $E_i I_{yi}/(E_0 I_{y0}^*)$=1.0 for the first and second rigidity (i=1 and

Table 1 Natural frequencies of transverse vibrations.

		Natural Frequencies (rad/sec)								
		First			Second			Third		
Shape	ADMASS	0	1	1	0	1	1	0	1	1
B_1/B_0	ADST	0	0	1	0	0	1	0	0	1
1.0	Approximate	5.34	5.22	5.22	33.44	33.22	33.24	93.64	93.41	93.75
	Galerkin	5.34	5.22	5.23	33.44	33.12	33.14	93.64	93.00	93.34
	NASTRAN	5.34	5.22	5.34	33.43	33.22	33.44	93.57	93.38	93.61
0.8	Approximate	5.50	5.36	5.34	31.51	31.27	31.30	85.97	85.74	86.15
	Galerkin	5.48	5.34	5.34	31.30	31.05	31.09	85.29	85.03	85.62
	NASTRAN	5.48	5.34	5.34	31.28	31.05	31.07	85.22	85.01	85.26
0.6	Approximate	5.78	5.61	5.61	29.98	29.71	29.75	79.51	79.26	79.77
	Galerkin	5.67	5.49	5.49	29.01	28.70	28.81	76.46	76.14	77.33
	NASTRAN	5.67	5.49	5.49	28.99	28.70	28.73	76.35	76.06	76.41
0.4	Approximate	6.27	6.02	6.02	29.00	28.70	28.74	74.68	74.41	75.04
	Galerkin	5.97	5.71	5.72	26.55	26.13	26.42	67.03	66.60	69.23
	NASTRAN	5.99	5.71	5.71	26.52	26.12	26.16	66.74	66.32	66.81
0.2	Approximate	7.17	6.77	6.77	28.83	28.46	28.51	72.05	71.75	72.52
	Galerkin	6.51	6.09	6.11	23.96	23.27	24.18	57.33	56.66	66.62
	NASTRAN	6.51	6.08	6.09	23.87	23.20	23.28	55.89	55.11	55.84

2, respectively), in which $I_{y0}^*=$ the moment of inertia at the base.

The approximate solutions proposed here show good in agreement both with the results obtained from Eq. (29) and with the results obtained using FEM code NASTRAN, in which the total number of bean element used in NASTRAN is 60.

5. Forced transverse vibrations

The equation of motion for the variable member with local mass and rigidity becomes

$$\rho_o \bar{A}\alpha_{\rho A}\ddot{w} + c_w\dot{w} + E_0\bar{I}_y(\alpha_{EI}w'')'' = p_z - m_y' \tag{40}$$

The general solution of the above equation is assumed to be of the form

$$w(x,\ t) = \sum_{n=1}^{\infty} w_n(t)\,\phi_n(x) \tag{41}$$

in which $w_n(t)$=unknown functions with respect to time; and $\phi_n(x)$=natural functions satisfying both Eq. (25) and the specified boundary conditions of the member. Substituting Eq. (41) into Eq. (40) and using Eq. (25), we have

$$\sum_{m=1} \phi_m \rho_0\bar{A}\ \alpha_{\rho A}[\ddot{w}_m(t) + 2h_m^*\omega_m\dot{w}_m(t) + \omega_m^2 w_m(t)] = p_z - m_y' \tag{42}$$

in which h_m^* is defined by

$$\frac{c_m}{\rho_0 \bar{A} \alpha_{\rho 4}} = 2h_m^* \omega_m \tag{43}$$

with the m-th damping coefficients c_m of c_w.

Since the mass coefficient $\alpha_{\rho 4}$ is a function of x, Eq. (42) cannot be transformed into an uncoupled form by means of the orthogonality relations for natural functions. Multiplying both sides of Eq. (42) by ϕ_n; integrating between x_T to x_B: and noticing that h_m^* at each point of current member may be replaced with the usual damping constants h_m for the member as a whole; we have

$$\delta w_n : \sum_{m=1} K_{mn} [\ddot{w}_m(t) + 2h_m \omega_m \dot{w}_m(t) + \omega_m^2 w_m(t)] = Q_n(t) \tag{44}$$

in which the notations K_{mn} and $Q_n(t)$ are defined as

$$K_{mn} = \int_{X_T}^{X_B} \rho_0 \bar{A} \alpha_{\rho 4} \phi_m \phi_n \, dx \tag{45}$$

$$Q_n(t) = \int_{X_T}^{X_B} (p_z - m_y') \phi_n \, dx \tag{46}$$

Eq. (44) are coupled differential equations with respect to time due to the additional mass and can be solved by means of the linear acceleration method.

Then, consider an approximate solution for the transverse vibrations for practical uses. Assuming that the behavior of the member is now dominated by the uncoupled terms(i. e., $m=n$) in Eq. (44), Eq. (44) can be approximated as the following uncoupled second order differential equation:

$$\delta w_m : K_{mm} [\ddot{w}_m(t) + 2h_m \omega_m \dot{w}_m(t) + \omega_m^2 w_m(t)] = Q_m(t) \tag{47}$$

The general solution of the above equation is

$$w_m(t) = exp(-h_m \omega_m t)[C_1 \sin\omega_{Dm} t + C_2 \cos\omega_{Dm} t] + \frac{1}{K_{mm}\omega_{Dm}} \int_0^t exp[-h_m \omega_m (t-\tau)] \sin\omega_{Dm}(t-\tau) Q_m(\tau) dt \tag{48}$$

in which C_1 and C_2 = constants which are determined from the initial conditions; and ω_{Dm} = the natural frequencies of the damped variable member, as given by

$$\omega_{Dm} = \omega_m \sqrt{1 - h_m^2} \tag{49}$$

Eq. (48) may be treated as a closed-form solution, because the integral calculation included may be easily computed by computer.

Next, consider the following harmonic external load:

$$p_z(x, t)=p_{z0}(x)\sin\omega_p t \tag{50}$$

in which $p_{z0}(x)=$ a distribution function in the x direction of the external load; and $\omega_p=$ a frequency of the external load. Then, $Q_m(t)$ are defined as

$$Q_m(t)=Q_m^* \sin\omega_p t \tag{51}$$

in which Q_m^* are defined as

$$Q_m^* = \int_{x_l}^{x_B} p_{z0}\phi_m dx \tag{52}$$

If at t=0 the initial displacement and velocity are zero, then Eq. (48) is written as

$$\begin{aligned} \mathrm{w_m} = \frac{1}{K_{mm}\,\omega_{Dm}} \frac{Q^*_m}{2} \Bigg\{ & \frac{h_m\omega_m\cos\omega_p t+(\omega_p+\omega_{Dm})\sin\omega_p t}{(h_m\omega_m)^2+(\omega_p+\omega_{Dm})^2} \\ & - \frac{h_m\omega_m\cos\omega_p t+(\omega_p-\omega_{Dm})\sin\omega_p t}{(\mathrm{h}_m\omega_m)^2+(\omega_p-\omega_{Dm})^2} \\ & + \exp(-\mathrm{h}_m\omega_m \mathrm{t})\left[\frac{(\omega_p+\omega_{Dm})\sin\omega_{Dm}\mathrm{t}-\mathrm{h}_m\omega_m\cos\omega_{Dm}\mathrm{t}}{(h_m\omega_m)^2+(\omega_p+\omega_{Dm})^2}\right. \\ & \left. + \frac{(\omega_p-\omega_{Dm})\sin\omega_{Dm}t+h_m\omega_m\cos\omega_{Dm}t}{(h_m\omega_m)^2+(\omega_p-\omega_{Dm})^2}\right]\Bigg\} \end{aligned} \tag{66}$$

6. Numerical results for forced transverse vibrations

The approximate solution proposed here is validated by comparing it with numerical results obtained using the Galerkin method and NASTRAN. Consider a thin-walled member with circular section; the total height l=38.6 m; thickness s=0.016 m; diameter at the base B_0=1.25 m; Young's modulus E=205.9 GN/m^2 (21Gkgf/m^2); Poisson's ratio ν=0.3; and mass density ρ_0=7.87KN · sec^2/m^4 (802 kgf.sec^2/m^4). A variable member is obtained by changing arbitrarily the tapering ratio B_l/B_0 from 1.0 to 0.2. For simplicity, consider the following triangularly distributed harmonic forces;

$$\mathrm{p}_z(\mathrm{x},\ \mathrm{t})=\mathrm{p}_{z0}^* \frac{(x_B-x)}{l}\sin\omega_p \mathrm{t} \tag{55}$$

in which p_{z0}^*=3315 N/m(0.338 tf/m) and ω_p=2.668 rad/sec. For simplicity, the natural functions for current variable members use approximately the natural functions of cantilevered uniform members.

Table 2 shows the maximum dynamic deflection and the ratio of the value obtained from the proposed solutions to NASTRAN for the above tapering ratios, in which the value of the notations "ADMASS" and "ADST" takes 0 for exclusion and 1 for inclusion of the fo-

Table 2 Maximum dynamic deflection of variable member.

Shape B_l/B_0	ADMASS / ADST	0 / 0	1 / 0	1 / 1
		Maximum deflection		
		Maximum(ratio)	Maximum(ratio)	Maximum(ratio)
1.0	Approximate	0.463 m (0.98)	0.545 m (1.00)	0.545 m (1.00)
	Wilson-θ method	0.463 m (0.98)	0.548 m (1.01)	0.548 m (1.01)
	NASTRAN	0.475 m	0.544 m	0.544 m
0.8	Approximate	0.589 m (0.99)	0.525 m (0.99)	0.526 m (0.99)
	Wilson-θ method	0.587 m (0.98)	0.525 m (0.98)	0.525 m (0.98)
	NASTRAN	0.597 m	0.533 m	0.533 m
0.6	Approximate	0.658 m (0.96)	0.680 m (0.98)	0.680 m (0.97)
	Wilson-θ method	0.657 m (0.96)	0.678 m (0.96)	0.678 m (0.96)
	NASTRAN	0.683 m	0.704 m	0.704 m
0.4	Approximate	0.747 m (0.92)	0.773 m (0.93)	0.771 m (0.93)
	Wilson-θ method	0.755 m (0.94)	0.776 m (0.93)	0.774 m (0.93)
	NASTRAN	0.808 m	0.831 m	0.830 m
0.2	Approximate	0.853 m (0.84)	0.915 m (0.86)	0.906 m (0.85)
	Wilson-θ method	0.923 m (0.92)	0.954 m (0.89)	0.945 m (0.89)
	NASTRAN	1.012 m	1.067 m	1.064 m

llowing two additional mass and rigidity: for the first local variation $x_l - x_T = 2.6$ m, $l_1 = 0.4$ m, $\rho_l A_l/(\rho_0 A_0^*) = 0.5$, and $E_l I_{yl}/(E_0 I_{y0}^*) = 2.0$; while for second local variation $x_2 - x_T = 5.6$ m, $l_2 = 0.4$ m, $\rho_2 A_2/(\rho_0 A_0^*) = 1.0$, and $E_2 I_{y2}/(E_0 I_{y0}^*) = 2.0$. The approximate solution shows good in agreement with the results obtained from the other methods, but its maximum tra nsverse displacement is slightly smaller than results obtained from the other methods. The accuracy of the approximate solution is improved when the shape of the member approaches to the uniform member. The reason is that the natural functions used here for variable members are the ones for the uniform member.

Fig. 3 shows the dynamic transverse displacements at the top for the variable member with $B_l/B_0 = 0.4$ and the above-mentioned two additional masses and rigidities. Here the solid curve indicates the results obtained using the approximate solution given in Eqs. (48) and (54); the broken curve indicates the results obtained from Eq. (44) by means of the Wilson-θ method; and the solid curve with circles shows the results obtained from NASTRAN.

7. Conclusions

The analytical method and approximate solutions for a dynamic linearly tapered thin-walled member with discontinuous and local variation in mass and/or rigidity, due to the additional mass and/or rigidity, have been presented in general by means of a usable function. The solutions proposed here will be applicable to the preliminary stage of design for such a variable member. The accuracy of the approximate solutions proposed here for variable members will

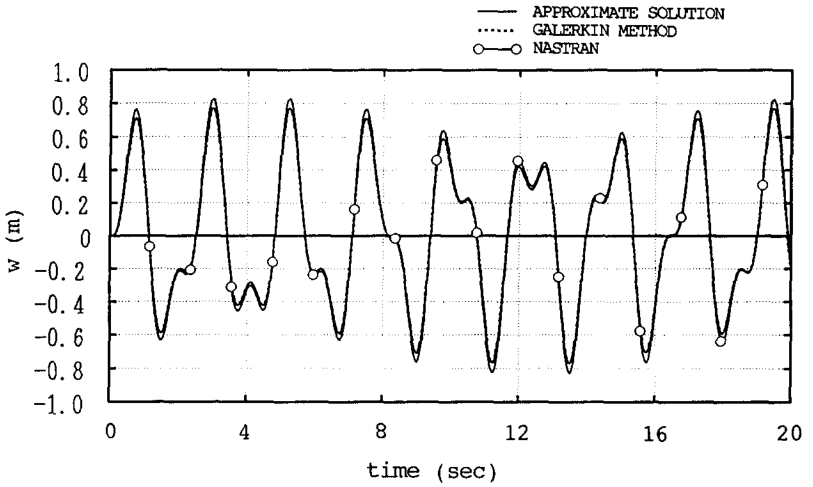

Fig. 3 Dynamic transverse displacements for member with $B_1/B_0 = 0.4$ and with additional mass and rigidity.

be improved by using suitable natural functions for the variable members instead of the natural functions of uniform members, used approximately here. The torsional behavior of current member is also presented by the use of method proposed here.

Notations

A, A_0^*	sectional area, and sectional area at the base;
$\bar{A}$, $\bar{I}_y$, $\bar{J}$, $\bar{\Gamma}$, $\bar{I}_p$	sectional constants depending on both the shape of the transverse cross section and the tapering angle;
ADMASS	parameter indicating the additional local mass;
ADST	parameter indicating the additional local rigidity;
c_w, c_θ	damping coefficients;
$D(x-x_i)$	operator function;
E, G	Young's modulus and shear modulus;
I_y, I_p	moment of inertia and polar moment, respectively;
J, Γ	torsional constant and warping constant, respectively;
l_i	length of the i-th local variation of mass and/or rigidity;
m_x, m_y, p_z	external moments and external load;
w	displacement component on axial line;
x_i	location of the i-th local variation;
$\alpha_{\rho A}$, $\alpha_{\rho I}$, α_{EI}, α_{GJ}, $\alpha_{G\Gamma}$	coefficients of sectional constants;
θ	torsional angle;
ρ	mass density;
ρ_i	mass density of the i-th local variation of mass;
ϕ_n	natural functions;
ω_n	natural frequencies

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Appendix

For a uniform member with the local variation in the mass and rigidity Eqs. (14)-(18) must be rewritten as

$$\rho(x)A(x)=\rho_0 A_0\,\alpha_{\rho A}(x) \tag{56}$$

$$\rho(x)I_p(x)=\rho_0 I_{p0}\,\alpha_{\rho I}(x) \tag{57}$$

$$E(x)I_y(x)=E_0 I_{y0}\,\alpha_{EI}(x) \tag{58}$$

$$G(x)J(x)=G_0 J_0 \alpha_{GJ}(x) \tag{59}$$

$$G(x)\Gamma(x)=G_0\,\Gamma_0\,\alpha_{G\Gamma}(x) \tag{60}$$

in which

$$\alpha_{\rho A}=1+\frac{\rho_i A_i}{\rho_0 A_0}\,D(x-x_i) \tag{61}$$

$$\alpha_{\rho J}=1+\frac{\rho_i I_{pi}}{\rho_0 I_{p0}}D(x-x_i) \tag{62}$$

$$\alpha_{EI}=1+\frac{\rho_i I_{yi}}{\rho_0 I_{y0}}D(x-x_i) \tag{63}$$

$$\alpha_{GJ}=1+\frac{G_i J_i}{G_0 J_0}D(x-x_i) \tag{64}$$

$$\alpha_{G\Gamma}=1+\frac{G_i \Gamma_i}{G_0 \Gamma_0}D(x-x_i) \tag{65}$$

in which the subscript 0 indicates to be the value of the uniform member.