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Free vibration analysis of cantilever cylindrical tanks

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Abstract. General free vibration characteristics of cantilevered circular cylindrical tanks are analyzed using the integral equations technique with the cubic spline functions. For computations, the partial differential equations for thin shallow shells as given by Flugge's have been employed after the addition of the inertia forces.

The application of the method is illustrated with a numerical examples of tanks which are free at the top edge and fixed at the bottom. The results obtained by this method have been compared with the available results and a good agreement was found.

Key words: free vibration; cylindrical tanks; cylindrical shells; cantilever cylindrical shells.

1. Introduction

Free vibration characteristics of cylindrical shell is important to designers confronted with aeroelastic and acoustic problem.

Hadid and Bashir (1990) are used integral equation technique with cubic spline function for the static analysis of skew plate. The natural frequencies of cylindrically curved panel were calculated by Hadid and Hasson (1992).

The free vibration analysis of cylindrical shells are studied by Ross (1982) using stiffness method and by Luah and Fan (1989) and Sen and Gould (1974) using finite element method. Solution of vibration analysis problem of thick cylindrical shells are available in Singal and Williams (1988).

In the present method, the partial differential equations of the cylindrical shell are reduced to a set of ordinary differential equations by assuming sinusoidal functions in the circumferential direction. These ordinary equations are then solved using spline integral method.

2. Governing differential equations

The non-dimensional governing differential equations of the free vibration of cylindrical shells in terms of the three displacements u, v and w are:

$$\frac{a^2}{l^2}u''+\frac{1-\nu}{2}u^{\circ\circ}+\frac{1+\nu}{2}\frac{a}{l}v'^{\circ}-\frac{\nu a}{l}w'=-m\omega^2\frac{a^2(1-\nu^2)}{Eh}u$$

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$$\frac{1+\nu}{2}\frac{a}{l}u'^{\circ}+v^{\circ\circ}+\frac{1-\nu}{2}\frac{a^2}{l^2}v''-w^{\circ}=-m\omega^2\frac{a^2(1-\nu^2)}{Eh}v$$

$$\frac{\nu a}{l}u'+v^{\circ}-w-\frac{h^2}{12a^2}\nabla w=-m\omega^2\frac{a^2(1-\nu^2)}{Eh}w \tag{1}$$

where $\nabla w=\frac{a^4}{l^4}w''''+2\frac{a^2}{l^2}w''^{\circ\circ}+w^{\circ\circ\circ\circ}$

u, v and w are displacements in x, θ and z directions respectively. l, a and h are length, radius and thickness of the shell respectively. ω is the circular natural frequency. E, ν and m are modulus of elasticity, poisson's ratio and mass density per unit area of the shell material respectively. Superscripts $'$ and $^{\circ}$ represent the derivative with respect to x and θ direction.

3. Boundary conditions

The free end conditions at the top end at $x=l$ are:

$$u'+\nu l v^{\circ}-\frac{\nu l}{a}w=0$$

$$v'+lu^{\circ}=0$$

$$w''+\nu l^2 w^{\circ\circ}=0$$

$$w'''+(2-\nu)l^2 w'^{\circ\circ}=0$$

The fixed end conditions at the base at $x=0$ are:

$$u=v=w=w'=0$$

4. Free vibration analysis

The partial differential equations are reduced to ordinary differential equations by assuming sinusoidal functions in the circumferential direction, therefore, the displacements u, v and w can be written in the following form:

$$u=u_x\cos n\theta$$

$$v=v_x\sin n\theta$$

$$w=w_x\cos n\theta \tag{2}$$

where u_x, v_x and w_x are unknown functions to be determined.

Substituting the displacements functions given in Eq. (2) and their derivatives in Eq. (1), the following ordinary differential equations are obtained:

$$\frac{a^2}{l^2}u_x''+\frac{1-\nu}{2}n^2u_x+\frac{1+\nu}{2}\frac{a}{l}n\ v_x'-\frac{\nu a}{l}w_x'=-m\omega^2\frac{a^2(1-\nu^2)}{Eh}u$$

$$\frac{1+\nu}{2}\frac{a}{l}n\ u_x'+n^2\ v_x+\frac{1-\nu}{2}\frac{a^2}{l^2}v_x''-n\ w_x=-m\omega^2\frac{a^2(1-\nu^2)}{Eh}v$$

$$\frac{\nu a}{l}u'_x+n\ v_x-w_x-\frac{h^2}{12a^2}\nabla w_x=-m\omega^2\frac{a^2(1-\nu^2)}{Eh}w \tag{3}$$

where $\nabla w_x=\frac{a^4}{l^4}w''''_x+2\frac{a^2}{l^2}n^2w''_x+n^4w_x$

Using the same procedure used for the governing partial differential equations the free edge boundary conditions at $x=l$ will be reduced to the following equations:

$$u'_{x1}=-\nu lngv_x+\frac{\nu l}{a}gw_x$$

$$v'_{x1}=lngu_x+\frac{h^2}{a}ngw'_x$$

$$w''_{x1}=\nu l^2n^2gw_x$$

$$w'''_{x1}=(2-\nu)l^2n^2w'_x \tag{4}$$

where u'_{x1}, v'_{x1} w''_{x1} and w'''_{x1} are derivatives of the unknown functions u_x, v_x and w_x at the free edge at $x=l$ and g is a unitary isolation matrix.

To solve Eq. (3) using spline integral method in the x direction, the highest derivations of the unknown functions u_x, v_x and w_x with respect to x are assumed as:

$$u''_x=-s,\ v''_x=-k,\ w''''_x=-q. \tag{5}$$

For non-homogeneous boundary conditions at $x=l$, Eq. (5) can be transformed to non-homogeneous integral equations as:

$$u_x=\int_0^1 b(x,\xi)\ s(\xi)\ d\xi+xu'_{x1}$$

$$v_x=\int_0^1 c(x,\xi)\ k(\xi)\ d\xi+xv'_{x1}$$

$$w_x=\int_0^1 d(x,\xi)\ q(\xi)\ d\xi+\frac{x^2}{2}w''_{x1}+\left(\frac{x^3-3x^2}{6}\right)w'''_{x1} \tag{6}$$

where s, k and q represent the unknown functions given in Eq. (5). b, c and d are Green's functions associated with the following homogeneous boundary conditions:

$$u_x=v_x=w_x=w'_x=0 \text{ at } x=0$$

$$u'_x=v'_x=w''_x=w'''_x=0 \text{ at } x=l$$

Eq. (6) can be solved by approximating the unknown functions as a cubic spline function:

$$u_x=\int_0^1 b(x,\xi)\ S\Delta(\xi)\ d\xi+xu'_{x1}$$

$$v_x=\int_0^1 c(x,\xi)\ S\Delta(\xi)\ d\xi+xv'_{x1}$$

$$w_x=\int_0^1 d(x,\xi)\ S\Delta(\xi)\ d\xi+\frac{x^2}{2}w''_{x1}+\left(\frac{x^3-3x^2}{6}\right)w'''_{x1} \tag{7}$$

where $S\Delta$ is the cubic spline function.

Eq. (7) can be solved using the expressions given in Hajdin and Krajcinovic (1972) for definite integrals. Therefore Eq. (7) can be written in matrix form as follows:

$$\begin{aligned} u_x&=Bs+f_1u'_{x1}\\ v_x&=Ck+f_1v'_{x1}\\ w_x&=Dq+f_2w''_{x1}+f_3w'''_{x1} \end{aligned} \tag{8}$$

where $f_1=[x]$, $f_2=[x^2/2]$, $f_3=[(x^3-3x^2)/6]$, f_1, f_2, f_3 are column matrices of order $M\times 1$, f matrices given the displacements at the various nodes in the x direction for unit influence at the free edge boundary at $x=l$. B, C and D are square matrices of order $M\times M$, M is number of intervals in the x direction.

The derivatives of Eq. (8) required in Eq. (3) can be written as:

$$\begin{aligned} u'_x&=B's+f'_1u'_{x1}\\ v'_x&=C's+f'_1v'_{x1}\\ w'_x&=D'q+f'_2w''_{x1}+f'_3w'''_{x1}\\ w''_x&=D''q+f''_2w''_{x1}+f''_3w'''_{x1} \end{aligned} \tag{9}$$

where $f'_1=[1]$, $f'_2=[x]$, $f''_2=[1]$, $f'_3=[(3x^2-6x)/6]$, $f''_3=[x-1]$.

Eq. (8) can be written as:

$$\begin{aligned} s&=B^{-1}(u_x-f_1u'_{x1})\\ k&=C^{-1}(v_x-f_1v'_{x1})\\ q&=D^{-1}(w_x-f_2w''_{x1}+f_3w'''_{x1}) \end{aligned} \tag{10}$$

Substituting Eq. (10) into Eq. (9) gives:

$$\begin{aligned} u'_x&=B'B^{-1}u_x-f_1B'B^{-1}u'_{x1}+u'_{x1}\\ v'_x&=C'C^{-1}v_x-f_1C'C^{-1}v'_{x1}+v'_{x1}\\ w'_x&=D'D^{-1}w_x+\left[f'_2-f_2D'D^{-1}\right]w''_{x1}+\left[f'_3-f_3D'D^{-1}\right]w'''_{x1}\\ w''_x&=D''D^{-1}w_x+\left[f''_2-f_2D''D^{-1}\right]w''_{x1}+\left[f''_3-f_3D''D^{-1}\right]w'''_{x1} \end{aligned} \tag{11}$$

To solve Eq. (3) substituting expressions (11) in Eq. (3), then these equations can be written in matrix form as follows:

$$\begin{bmatrix} \alpha_{11} & \alpha_{12}\\ & \\ \alpha_{21} & \alpha_{22} \end{bmatrix}\begin{bmatrix} u_x\\ v_x\\ w_x\\ u'_{x1}\\ w''_{x1}\\ w'''_{x1} \end{bmatrix}=\lambda[I]\begin{bmatrix} u_x\\ v_x\\ w_x\\ u'_{x1}\\ w''_{x1}\\ w'''_{x1} \end{bmatrix} \tag{12}$$

where:

$$\lambda = -m\omega^2 a^2\left(\frac{1-\nu^2}{Eh}\right).$$

$[I]$ is a unit diagonal matrix of order $(3M+4)\times(3M+4)$ and the order of α_{11}, α_{12}, α_{21} and α_{22} are $(3M\times 3M)$, $(3M\times 4)$, $(4\times 3M)$ an (4×4) respectively.

$$\alpha_{11} = \begin{bmatrix} -\frac{a^2}{l^2}B^{-1} - \frac{1-\nu}{2} & \frac{1-\nu}{2}\frac{a}{l}nC'C^{-1} & -\frac{\nu a}{l}D^{-1} \\ -\frac{1+\nu}{2}\frac{a}{l}nB'B^{-1} & -\frac{1-\nu}{2}\frac{a^2}{l^2}C^{-1} - n^2 & n \\ \frac{\nu a}{l}B'B^{-1} & n & -1 + \frac{h^2a^2}{12l^4}D^{-1} + \frac{n^2}{l^2}D''D^{-1} - \frac{h^2n^4}{12a^2} \end{bmatrix}$$

$$\alpha_{21} = \begin{bmatrix} 0 & \nu lng & -\frac{\nu l}{a}g \\ -lng & 0 & -\frac{h^2}{6a}ngD'D^{-1} \\ 0 & 0 & -\nu l^2n^2g \\ 0 & 0 & -l^2(2-\nu)n^2gD''D^{-1} \end{bmatrix}$$

$$\alpha_{12} = \begin{bmatrix} \frac{a^2}{l^2}f_1B^{-1} & \frac{1+\nu}{2}\frac{a}{l}n(f_1'-f_1C'C^{-1}) & -\frac{\nu a}{l}(f_2'-f_2D'D^{-1}) & -\frac{\nu a}{l}(f_3'-f_3D'D^{-1}) \\ -\frac{1+\nu}{2}\frac{a}{l}n(f_1'-f_1B'B^{-1}) & \frac{1+\nu}{2}\frac{a^2}{l^2}f_1C^{-1} & 0 & 0 \\ \frac{\nu a}{l}(f_1'-f_1B'B^{-1}) & 0 & -\frac{h^2a^2}{12l^4}f_2D^{-1} + \frac{h^2n^2}{6l^2}(f_2''-f_2D''D^{-1}) & -\frac{h^2a^2}{12l^4}f_3D^{-1} + \frac{h^2n^2}{6l^2}(f_3''-f_3D''D^{-1}) \end{bmatrix}$$

$$\alpha_{22} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -\frac{h^2}{6a}ng(f_2'-f_2D'D^{-1}) & -\frac{h^2}{6a}ng(f_3'-f_3D'D^{-1}) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -l^2(2-\nu)n^2g(f_2'-f_2D'D^{-1}) & 1-l^2(2-\nu)n^2g(f_3'-f_3D'D^{-1}) \end{bmatrix}$$

Eq. (12) can be reduced to eigenvalue problem determinate as follows:

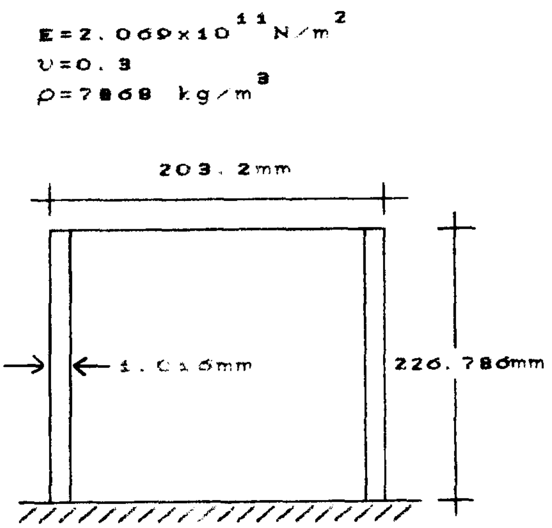

Fig. 1 Cylindrical tank.

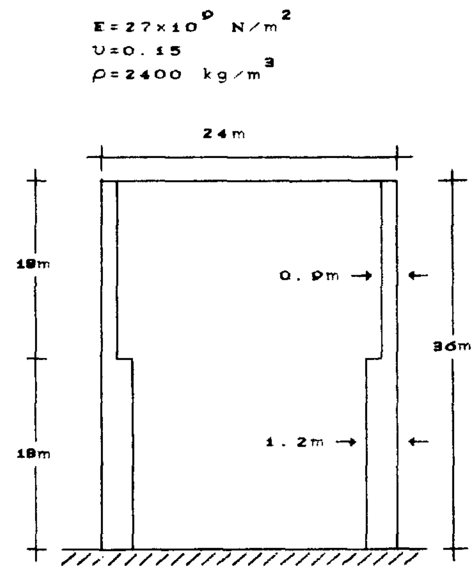

Fig. 2 Tank with variable thickness.

$$\Big[[\alpha]-\lambda[I]\Big]\delta=o \tag{13}$$

where α is the global matrix of order $(3M+4)\times(3M+4)$ given in Eq. (12) and δ is the displacements vectors u_x, v_x, w_x, u'_x, w''_{x1} and w'''_{x1} of order $(3M+4)\times 1$.

5. Results and discussions

The results of the general free vibration analysis of cylindrical tanks shown in Fig. 1 using four and six elements are given in Table 1 together with those obtained by Sen and Gould (1974) using finite element method and Luah and Fan (1989) using spline finite element method. Remarkable agreement in the results is observed. However, the torsional mode obtained by present method was not reported in Sen and Gould (1974).

The natural frequencies of tank with variable thickness shown in Fig. 2 are analyzed. The boundary conditions at the step junction are considered to be continuous. The results are presented in Table 2. The values of natural frequencies were found to be in good agreement in comparison with those obtained by finite elements method (SAP 1988).

The natural frequencies for the first three modes of vibration of the fixed-free cylindrical tank as shown in Fig. 1 are plotted in Fig. 3 together with those of Sen and Gould (1974) obtained using finite element method and Weingarten's experimental results (Weingarten 1964). The agreement between the results are encouraging. For $n=2$ and 3, however, the experimental results are lower than those predicted by the theory. As pointed out by Weingarten, this is due to inadequate clamping of the shell in the experiment.

The problem of convergence was studied for the first three modes of vibration of the cylindrical tank shown in Fig. 1. The results of the study are plotted in Fig. 4.

Table 1 Natural frequency (Hz) of a fixed-free cylindrical shell

n	Mode number	Present method 4 elements (Hz)	Present method 6 elements (Hz)	FEM* 12 elements (Hz)	SFEM[$] 6 elements (Hz)
0	Torsional mode	3508	3507	–	3506
	1	5463	5491	5486	5479
	2	7950	7956	8055	7953
	3	8168	8028	8123	8017
1	1	2083	2084	2033	2032
	2	5757	5702	5431	5412
	3	7266	7191	6986	6943
2	1	1058	1025	982	981
	2	3698	3571	3409	3396
	3	6023	5855	5783	5718
3	1	659	640	565	563
	2	2425	2387	2243	2228
	3	4545	4472	4378	4310
4	1	614	599	487	485
	2	1841	1776	1598	1587
	3	3555	3473	3318	3278
5	1	757	728	621	620
	2	1563	1506	1295	1287
	3	2906	2838	2632	2605
6	1	989	951	863	862
	2	1507	1462	1258	1251
	3	2572	2493	2250	2229
7	1	1288	1244	1170	1168
	2	1636	1583	1419	1413
	3	2424	2372	2126	2108
8	1	1633	1596	1531	1528
	2	1897	1830	1710	1704
	3	2509	2434	2217	2201

*Finite element method; [$]Spline finite element method.

Table 2 Natural frequency (Hz) of a fixed-free surge tank[†]

Mode No.	Spline integral 7 ele.	Spline integral 9 ele.	Spline integral 11 ele.	Finite element 15 ele.	Finite element 19 ele.	Finite element 23 ele.
1	23.25	23.26	23.26	25.36	25.36	25.36
2	44.54	44.54	44.54	41.29	42.79	43.47
3	44.82	44.83	44.83	44.22	44.28	44.30
4	45.22	45.21	45.21	44.85	44.91	44.95
5	46.14	46.11	46.09	45.71	45.80	45.88
6	46.95	46.94	46.93	49.31	49.39	49.49
7	51.54	51.03	50.92	55.15	55.06	55.10
8	59.81	58.12	57.66	65.43	64.82	59.81

[†]Fig. 2

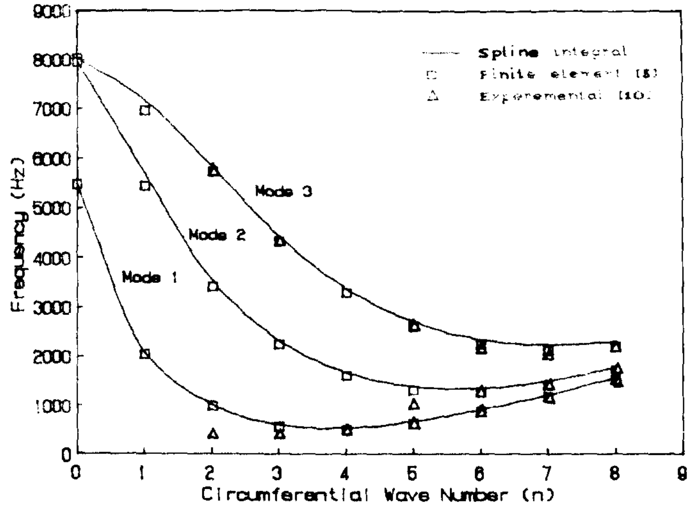

Fig. 3 Natural frequencies of a fixed-free cylindrical tank.

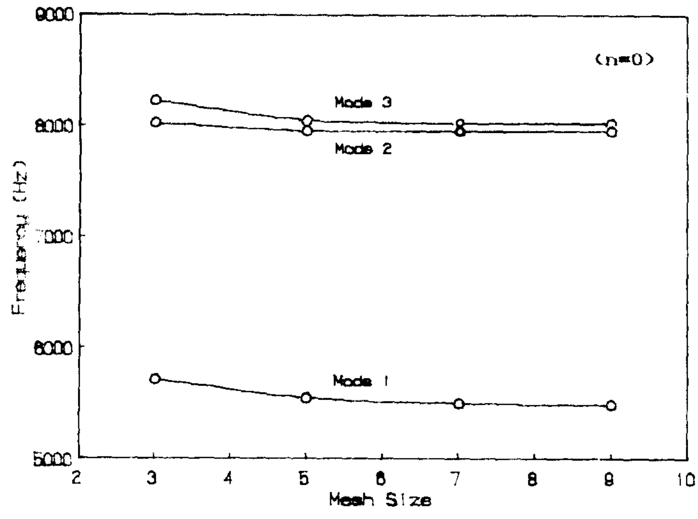

Fig. 4 Natural frequencies of a fixed-free cylindrical tank.

The general form of some typical circumferential and longitudinal mode shapes of the cylindrical tank shown in Fig. 1 is given in Fig. 5. The circumferential mode shape is defined by the number of n. The longitudinal mode shape is defined by M, which represents the number of nodes (for radial displacements) in the axial direction.

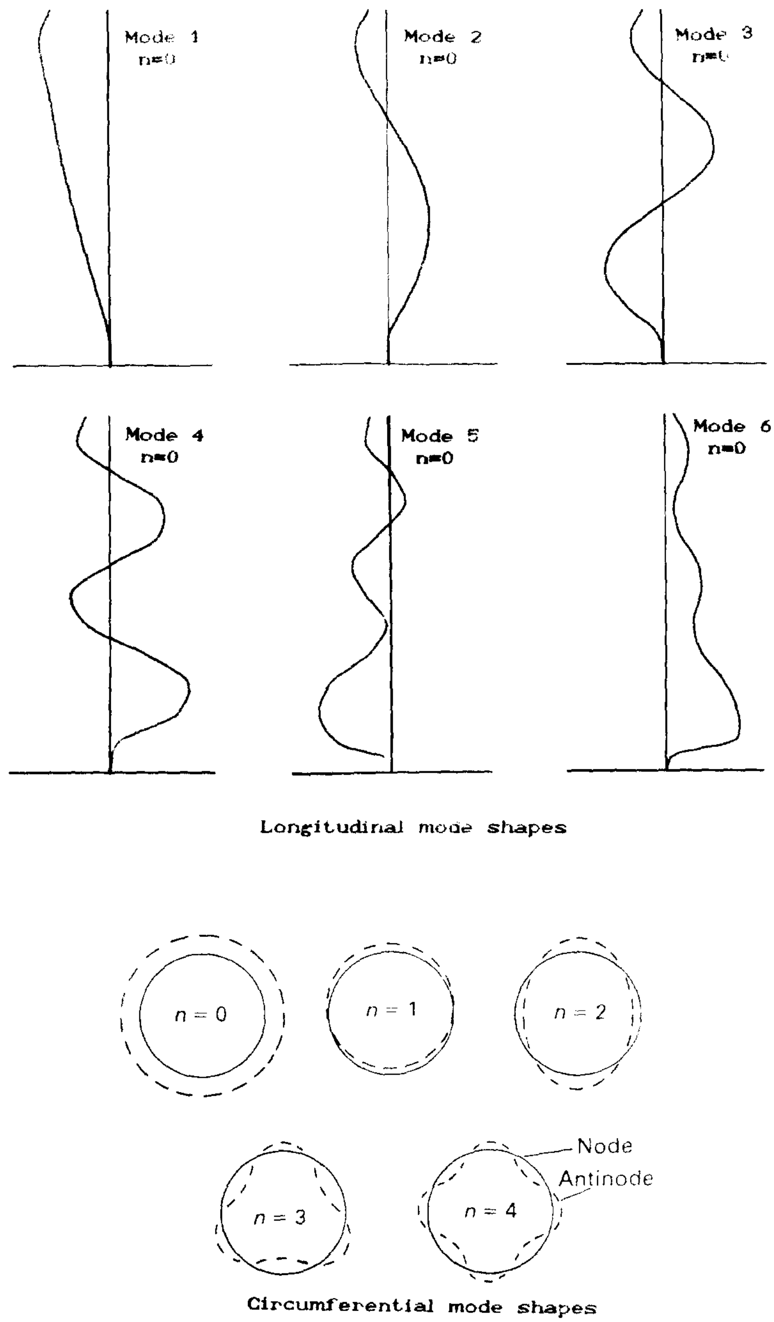

Fig. 5 Typical longitudinal and circumferential mode shapes.

6. Conclusions

The formulation of the spline integral method for free vibration analysis of shell of revolution has been developed. This method inherits both the efficiency of the cubic spline interpolation and the generality and flexibility of the integral equations technique. From the theoretical considerations and the illustrated numerical examples, the following conclusions can be made.

(1) The present formulation is accurate and efficient in analysis a variety of shells of revolution. As can be seen from the examples presented, highly accurate results could be obtained with relatively few elements.

(2) The method is capable of predicting the frequencies in any of the membrane, flexural or torsional modes.

(3) It was found from the results, that the present method has a remarkable convergence property for structural mechanic problems and this property arises from the use of cubic spline.

(4) Shells with variable thickness can be modeled and analyzed by the present method.

(5) This method is very modest in consuming computer time and core storage, and suitable for personal computers.

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