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Study of two dimensional visco-elastic problems in generalized thermoelastic medium with heat source

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Abstract. In this paper, a thermo-viscoelastic problem in an infinite isotropic medium in two dimensions in the presence of a point heat source is considered. The fundamental equations of the problems of generalized thermoelasticity including heat sources in a thermo-viscoelastic media have been derived in the form of a vector matrix differential equation in the Laplace-Fourier transform domain for a two dimensional problem. These equations have been solved by the eigenvalue approach. The results have been compared to those available in the existing literature. The graphs have been drawn for different cases.

Keywords: generalized thermoelasticity; viscoelastic media; vector-matrix differential equation; eigenvalue approach; Laplace-Fourier transform.

1. Introduction

In previous years, considerable interest has been shown in the study of plane thermoelastic, magneto-thermoelastic, and magneto-thermo-viscoelastic wave propagation in an infinite random or nonrandom and rotating or non-rotating medium by many authors Chow (1973), Hetnarski (1961), Bahar and Hetnarshki (1978), Bhattacharyya (1986), Schoenberg and Censor (1973) following the classical theory of thermoelasticity which is based on Fourier's law of heat conduction. This law

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predicts an infinite speed of propagation of heat, which is physically absurd, and as a result, many new theories have been proposed to eliminate this absurdity. Lord and Shulman (LS theory) (1967) proposed a modified version of Fourier's law and deduced a theory of thermoelasticity known as the generalized theory of thermoelasticity. This theory with a thermal relaxation time has been used with purpose and profit by many authors. Nayfeh and Nemat-Nasser (1971), Nayfeh and Nemat-Nasser (1972), Roy Choudhuri (1985), Agarwal (1978) to study the effect of thermoelastic, magneto-thermoelastic, and magneto-thermo-viscoelastic plane wave in an infinite rotating or non-rotating medium.

Another theory of thermoelasticity has been proposed by Green and Lindsay (G-L theory) (1972) which has certain special features in contrast with the previous theory proposed by LS. In this theory of Green and Lindsay, Fourier's law of heat conduction remains unchanged, whereas the classical energy equation and the stress strain temperature relations are modified. Two constants α and α^* having the same dimensions of time appear in the governing equations in place of one relaxation time τ in Lord-Shulman's theory.

The governing equations for displacement and temperature fields in the linear dynamical theory of classical thermoelasticity consist of the coupled partial differential equation of motion and the Fourier's law of heat conduction equation. The equation for displacement field is governed by a wave type hyperbolic equation, whereas that for the temperature field is a diffusion type parabolic equation. This amounts to the remark that the classical thermoelasticity predicts a finite speed for predominantly elastic disturbances but an infinite speed for predominantly thermal disturbances, which are coupled together. This means that a part of every solution of the equations extends to infinity. Experimental investigations by Ackerman, Bentman, Fairbank and Gayer (1966), Ackerman and Guyer (1968), Ackerman and Overton, Jr. (1966), von Gutfeld and Nethercot (1966), Taylor, Marris and Elbaum, (1969), Jackson and Walker (1971), and many others, conducted on different solids, have shown that heat pulses do not propagate with infinite speeds. In order to overcome this paradox, efforts were made to modify classical thermoelasticity, on different grounds, for obtaining a wave type heat conduction equation by Kaliski (1965), Norhood and Warren (1969), Suhubi (1975) and Lebon (1982). A comprehensive list on this generalization for the last two decades is available in the works of Chandrasekharaiah (1986, 1998). On going through the literature it is found that hardly any attention has been given to the propagation of plane waves in thermo-visco-elastic medium in the presence of a point heat source. Mukhopadhyay and Bera (1992) have made some works on magneto-visco-elastic media. Sinha and Bera (2003), Baksi, Bera and Debnath (2004) have solved a few problems in generalized thermoelasticity in rotating medium by the method of eigenvalue approach in one dimension and two dimensions respectively. Also, Baksi and Bera (2005) have solved problems of magneto-thermoelasticity in two dimensions and three dimensions by using eigenfunction expansion method. Recent works on Magneto-thermoelasticity are also available from the papers of Ezzat and Karamany (2002), and Librescu, Hasanyan, Qin, and Ambur (2003), Librescu, Hasanyan, and Ambur (2004).

The linear visco-elasticity remains an important area of research as most of the solids and the polymer like materials when subjected to dynamic loading exhibit viscous effect. The stress-strain law for many materials such as polycrystalline metals and high polymers can be approximated by the linear visco-elasticity theory.

In the present paper we have applied technique of eigenvalue approach developed in Das, Lahiri and Giri (1997) to solve a problem of thermo-viscoelasticity in two dimensions. The resulting formulation is applied to three different cases in the presence of heat source. The solutions for the several important cases are given in closed form in the Laplace transform domain. The inversion of the Laplace transform is carried out by using a numerical inversion technique given by Bellman, Kalaba and Lockett (1966). Some of these results have been presented graphically and the effect of relaxation in each case has been shown separately.

2. Formulation of the problem

The present paper deals with the study of the disturbances in an infinite elastic solid containing instantaneous point heat source in a viscoelastic media. It is assumed that the elastic field under consideration is homogeneous, isotropic, and electrically as well as thermally conducting one.

The necessary governing equations in elastic fields are given below:

i) The principle of balance of linear momentum leads to the equations of motion

$$\tau_{ij,j} = \rho \ddot{u}_i \tag{1}$$

ii) The balance of the angular momentum principle implies that

$$\tau_{ij} = \tau_{ij} \tag{2}$$

where ρ =constant mass density

τ_{ij} = component of stress tensor

u_i = components of the displacement vector.

Along with this will be added the modified form of the equation of heat conduction

$$\kappa_{ij} T_{,ij} = \rho C_e\left(\frac{\partial T}{\partial t} + \alpha_0 \frac{\partial T^2}{\partial t^2}\right) + T_0\left[\frac{\partial}{\partial t} + \tau\frac{\partial^2}{\partial t^2}\right]\beta_{ij}\dot{u}_{i,j} - \left(1 + \tau\frac{\partial}{\partial t}\right)Q \tag{3}$$

The stress-displacement–temperature relation for the viscoelastic medium of Kelvin-Voigt type is

$$\tau_{ij} = \left(\lambda_e + \lambda_v\frac{\partial}{\partial t}\right)\Delta\delta_{ij} + 2\left(\mu_e^{/} + \mu_v^{/}\frac{\partial}{\partial t}\right)e_{ij} - \beta_i(T + \alpha\dot{T})\delta_{ij}$$

where, u_i's are the components of the displacement vector $\vec{u}$, τ_{ij} and e_{ij} are the components of stress tensor and strain tensor respectively, $\Delta = e_{ii}$, the dilatation, λ_e, $\mu_e^{/}$ Lame's elastic constants, λ_v, $\mu_v^{/}$ Lame's viscoelastic constants for the viscoelastic solid, $\beta = (3\lambda_e + 2\mu_e^{/})\alpha_t$, α_t being the coefficient of linear thermal expansion, T is the temperature change above reference temperature T_0, α is the thermal relaxation time parameter and δ_{ij} is the Kronecker delta.

The generalized heat conduction equation as proposed by Lord and Shulman (1967) in two dimensions can be written from (3) as

$$K_y\frac{\partial^2 T}{\partial y^2} + K_z\frac{\partial^2 T}{\partial z^2} = \rho c_v\left[\frac{\partial T}{\partial t} + \alpha_0\frac{\partial^2 T}{\partial t^2}\right] + T_0\beta\left[\frac{\partial}{\partial t} + \tau\frac{\partial^2}{\partial t^2}\right]\left[\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right] - \left[1 + \tau\frac{\partial}{\partial t}\right]Q(y,z,t) \tag{4}$$

We assume that the heat source $Q(y,z,t)$ is instantaneous and acts on the line $y = 0$, $z = 0$, so that

$$Q(y,z,t) = Q_0\delta(y)\delta(z)\delta(t)$$

where Q_0 is the strength of heat source and $\delta(r)$ is the Dirac delta function of r.

The thermal stresses in isotropic infinite elastic solid subject to plane strain in two dimensions are

$$\tau_{11} = \left(\lambda_e + \lambda_v\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) - \beta\left(1 + \alpha\frac{\partial}{\partial t}\right)T$$

$$\tau_{22} = \left(\lambda_e + \lambda_v\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\left(\mu_e^{/} + \mu_v^{/}\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial y} - \beta\left(1 + \alpha\frac{\partial}{\partial t}\right)T$$

$$\tau_{33} = \left(\lambda_e + \lambda_v\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + 2\left(\mu_e^{/} + \mu_v^{/}\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial y} - \beta\left(1 + \alpha\frac{\partial}{\partial t}\right)T$$

$$\tau_{23} = \left(\mu_e^{/} + \mu_v^{/}\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \tag{5}$$

From (1),

$$\left\{(\lambda_e + 2\mu_e^{/}) + (\lambda_v + 2\mu_v^{/})\frac{\partial}{\partial t}\right\}\frac{\partial^2 v}{\partial y^2} + \left(\mu_e^{/} + 2\mu_v^{/}\frac{\partial}{\partial t}\right)\frac{\partial^2 v}{\partial z^2} +$$

$$\left\{(\lambda_e + \mu_e^{/}) + (\lambda_v + \mu_v^{/})\frac{\partial}{\partial t}\right\}\frac{\partial^2 w}{\partial y \partial z} - \beta\left(1 + \alpha\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial y} = \rho\frac{\partial^2 v}{\partial t^2} \tag{6}$$

$$\left(\mu_e^{/} + \mu_v^{/}\frac{\partial}{\partial t}\right)\frac{\partial^2 w}{\partial y^2} + \left\{(\lambda_e + 2\mu_e^{/}) + (\lambda_v + 2\mu_v^{/})\frac{\partial}{\partial t}\right\}\frac{\partial^2 w}{\partial z^2}$$

$$\left\{(\lambda_e + \mu_e^{/}) + (\lambda_v + \mu_v^{/})\frac{\partial}{\partial t}\right\}\frac{\partial^2 v}{\partial y \partial z} - \beta\left(1 + \alpha\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial z} = \rho\frac{\partial^2 w}{\partial t^2} \tag{7}$$

From Eqs. (6) and (7) can be written as

$$\left\{C_1^2 R_H + C_2^2\frac{\partial}{\partial t}\right\}\frac{\partial^2 v}{\partial y^2} + \left(a + b\frac{\partial}{\partial t}\right)\frac{\partial^2 v}{\partial z^2} + \left(d + g\frac{\partial}{\partial t}\right)\frac{\partial^2 w}{\partial y \partial z} - \frac{\beta}{\rho}\left(1 + \alpha\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial y} = \frac{\partial^2 v}{\partial t^2} \tag{8}$$

and

$$\left(a + b\frac{\partial}{\partial t}\right)\frac{\partial^2 w}{\partial y^2} + \left\{C_1^2 + C_2^2\frac{\partial}{\partial t}\right\}\frac{\partial^2 w}{\partial z^2} + \left(d + g\frac{\partial}{\partial t}\right)\frac{\partial^2 v}{\partial y \partial z} - \frac{\beta}{\rho}\left(1 + \alpha\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial z} = \frac{\partial^2 w}{\partial t^2} \tag{9}$$

where

$$C_1^2 = \frac{\lambda_e + 2\mu_e^{/}}{\rho},\ C_2^2 = \frac{\lambda_v + 2\mu_v^{/}}{\rho},\ a = \frac{\mu_e^{/}}{\rho},\ b = \frac{\mu_v^{/}}{\rho},\ d = \frac{\lambda_e + \mu_e^{/}}{\rho},\ g = \frac{\lambda_v + \mu_v^{/}}{\rho}$$

Let us introduce the following non-dimensional quantities

$$y^* = \frac{C_1 y}{k_y},\ z^* = \frac{C_1 z}{k_y},\ t^* = \frac{C_1^2 t}{\sqrt{k_y k_z}},\ v^* = \frac{\rho C_1^3 v}{k_y \beta_2 T_0},\ \omega^* = \frac{\rho C_1^3 \omega}{k_y \beta_3 T_0},\ T^* = T_0 T$$

With the introduction of the above non-dimensionless variables in the Eq. (5) becomes after dropping star

$$\tau_{11} = T_0\left[\left(\lambda_e + \bar{\lambda}_v\frac{\partial}{\partial t}\right)\left(\bar{\beta}\frac{\partial v}{\partial y} + \bar{\beta}\frac{\partial w}{\partial z}\right) - \beta\left(1 + \bar{\alpha}\frac{\partial}{\partial t}\right)T\right]$$

$$\tau_{22} = T_0\left[\bar{\beta}\left(\lambda_e + \bar{\lambda}_v\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \bar{\beta}\left(\mu_e^{/} + \mu_v^{/}\frac{\partial}{\partial t}\right)\frac{\partial v}{\partial y} - \beta\left(1 + \bar{\alpha}\frac{\partial}{\partial t}\right)T\right]$$

$$\tau_{33} = T_0\left[\bar{\beta}\left(\lambda_e + \bar{\lambda}_v\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right) + \bar{\beta}\left(\mu_e^{/} + \mu_v^{/}\frac{\partial}{\partial t}\right)\frac{\partial w}{\partial y} - \beta\left(1 + \bar{\alpha}\frac{\partial}{\partial t}\right)T\right]$$

$$\tau_{23} = T_0\left[\left(\mu_e^{/} + \mu_v^{/}\frac{\partial}{\partial t}\right)\left(k\bar{\beta}\frac{\partial v}{\partial z} + \frac{\bar{\beta}}{k}\frac{\partial w}{\partial y}\right)\right] \tag{10}$$

where $\bar{\lambda}_v = \lambda_v\dfrac{C_1^2}{\sqrt{k_y k_z}}$, $\bar{\beta} = \dfrac{\beta}{\rho C_1^2}$, $\bar{\alpha} = \alpha\dfrac{C_1^2}{\sqrt{k_y k_z}}$, $\bar{\mu}_v' = \bar{\mu}_v\dfrac{C_1^2}{\sqrt{k_y k_z}}$, $k = \dfrac{k_y}{k_z}$.

Then Eqs. (8), (9) and (4) will reduce to

$$\beta\left\{C_1^2 + \bar{C}_2^2\frac{\partial}{\partial t}\right\}\frac{\partial^2 v}{\partial y^2} + \left(a + \bar{b}\frac{\partial}{\partial t}\right)k^2\beta\frac{\partial^2 v}{\partial z^2} + \left\{d + \bar{g}\frac{\partial}{\partial t}\right\}\frac{\partial^2 w}{\partial y\partial z} - C_1^2\beta\left(1 + \bar{\alpha}\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial y} = C_1^2\beta k\frac{\partial^2 v}{\partial t^2} \tag{11}$$

$$\left(a + \bar{b}\frac{\partial}{\partial t}\right)\frac{\partial^2 w}{\partial y^2} + \left\{C_1^2 + \bar{C}_2^2\frac{\partial}{\partial t}\right\}k^2\frac{\partial^2 w}{\partial z^2} + \left\{d + \bar{g}\frac{\partial}{\partial t}\right\}k^2\beta\frac{\partial^2 v}{\partial y\partial z} - C_1^2\beta\left(1 + \bar{\alpha}\frac{\partial}{\partial t}\right)\frac{\partial T}{\partial z} = C_1^2\beta k\frac{\partial^2 w}{\partial t^2} \tag{12}$$

$$\frac{\partial^2 T}{\partial y^2} + k\frac{\partial^2 T}{\partial z^2} = (k)^{1/2}\left[\frac{\partial}{\partial t}\left(1 + \alpha_0^{/}\frac{\partial}{\partial t}\right)T + \varepsilon(k)^{1/2}\frac{\partial}{\partial t}\left(1 + \tau^{/}\frac{\partial}{\partial t}\right)\left(\frac{\partial v}{\partial y} + \frac{\partial w}{\partial z}\right)\right.$$

$$\left. - Q_0^{/}\left(1 + \tau^{/}\frac{\partial}{\partial t}\right)\delta\left(\frac{k_y}{C_1}y\right)\delta\left(\frac{k_z}{C_1}z\right)\delta\left(\frac{\sqrt{k_y k_z}}{C_1^2}t\right)\right] \tag{13}$$

where $\bar{b} = b\dfrac{C_1^2}{\sqrt{k_y k_z}}$, $\bar{C}_2 = C_2^2\dfrac{C_1^2}{\sqrt{k_y k_z}}$, $\bar{g} = \dfrac{C_1^2}{\sqrt{k_y k_z}}$, $\bar{\alpha} = \alpha\dfrac{C_1^2}{\sqrt{k_y k_z}}$.

$\tau' = \tau\dfrac{C_1^2}{\sqrt{k_y k_z}}$, $Q_0^{/} = \dfrac{Q_0}{\rho c_v T_0 C_1^2}$, $\varepsilon = \dfrac{\beta T_0}{\rho^2 c_v C_1^2}$

3. Formulation of vector matrix differential equation

We assume that at $t = 0$, the body is at rest, in an undeformed and unstressed state and is maintained at the reference temperature so that

$$v(y,z,0) = 0 = v_t(y,z,0);\; w(y,z,0) = 0 = w_t(y,z,0) \text{ and } T(y,z,0) = 0 = T_t(y,z,0)$$

Let us apply joint Laplace and Fourier double integral transforms with respect to t and y respectively in the form

$$\hat{\bar{f}}(\xi, z, p) = \frac{1}{2\pi}\int_0^\infty \exp(-pt)\int_{-\infty}^\infty f(y,z,t)\exp(i\xi y)dydt$$

where p and ξ are the Laplace and Fourier transform variables respectively.

The Eqs. (14-16) reduce to the form

$$[-\xi^2(C_1^2+\bar{C}_2^2 p) - C_1^2 kp^2]\beta\hat{\bar{v}} + (a+\bar{b}p)k^2\beta\frac{d^2\hat{\bar{v}}}{dz^2} - i\xi(d+\bar{g}p)\frac{d\bar{w}}{dz} + C_1^2\beta i\xi(1+\bar{\alpha}p)\hat{\bar{T}} = 0 \tag{14}$$

$$-\xi^2(a+\bar{b}p)\hat{\bar{w}} + \{C_1^2+\bar{C}_2^2 p\}k^2\frac{d^2\hat{\bar{w}}}{dz^2} - i\xi(d+\bar{g}p)k^2\beta\frac{d\hat{\bar{v}}}{dz} - C_1^2k^2(1+\bar{\alpha}p)\frac{d\hat{\bar{T}}}{dz} = C_1^2kp^2\hat{\bar{w}} \tag{15}$$

$$-\xi^2\hat{\bar{T}} + k\frac{d^2\hat{\bar{T}}}{dz^2} = \sqrt{k}p\{1+\alpha_0' p\}\bar{T} + \sqrt{k}\varepsilon p(1+\tau' p)\left(-i\xi\hat{\bar{v}} + \frac{d\hat{\bar{w}}}{dz}\right) - Q_0'\frac{C_1^3(1+\tau' p)}{(2\pi k_y^3 k_z)^{1/2}}\delta\left(\frac{k_z}{C_1}z\right) \tag{16}$$

where $\hat{\bar{v}}, w$ and $\hat{T}$ tend to zero as $|\xi|\to\infty$ and $\hat{\bar{v}}, w$, $\hat{T}$, $\frac{\partial\bar{v}}{\partial y}, \frac{\partial\bar{w}}{\partial y}, \frac{\partial\bar{T}}{\partial y}$ tend to zero as $|y|\to\infty$.

If $\lambda_v = 0, \mu_v = 0$ Eqs. (14) and (15) reduce to those of Das *et al.* (1997) and Baksi *et al.* (2004) in non-rotating medium.

The Eqs. (14)-(16) can be written in the vector matrix differential equation as follows

$$\frac{d\tilde{v}}{dz} = M\tilde{v} + \tilde{f}(z) \tag{17}$$

where $\tilde{v} = [\hat{\bar{v}}, \hat{\bar{w}}, \hat{\bar{T}}, \hat{\bar{v}}', \hat{\bar{w}}', \hat{\bar{T}}']^T$ and $\tilde{f}(z) = \left[0,0,0,0,0,\left(-\frac{Q_0'C_1^3(1+\tau' p)}{(2\pi k_y^3 k_z)^{1/2}\cdot k}\right)\delta\left(\frac{k_z}{C_1}z\right)\right]$ where prime denotes the differentiation with respect to z.

The matrix M is given by

$\begin{bmatrix}\bar{O} & \bar{I}\\ M_1 & M_2\end{bmatrix}$, where $M_1 = \begin{bmatrix} m_{41} & 0 & m_{43}\\ 0 & m_{52} & 0\\ m_{61} & 0 & m_{63}\end{bmatrix}$, $M_2 = \begin{bmatrix} 0 & m_{45} & 0\\ m_{54} & 0 & m_{56}\\ 0 & m_{65} & 0\end{bmatrix}$ and $\bar{O}$ and $\bar{I}$ are the null and unit matrices of order three, respectively and

$$m_{41} = \frac{[-\xi^2(C_1^2+\bar{C}_2^2p) - C_1^2kp^2]\beta}{(a+\bar{b}p)k^2}, \quad m_{43} = \frac{-i\xi C_1^2(1+\bar{\alpha}p)}{(a+\bar{b}p)k^2}, \quad m_{45} = \frac{i\xi(d+\bar{g}p)}{(a+\bar{b}p)k^2\beta} \tag{18}$$

$$m_{52} = \frac{\xi^2(a+\bar{b}p) + C_1^2kp^2}{\{C_1^2+\bar{C}_2^2p\}k^2}, \quad m_{54} = \frac{i\xi\{d+C_1^2R_H+\bar{g}p\}\beta}{\{C_1^2+\bar{C}_2^2p\}k^2}, \quad m_{56} = \frac{C_1^2(1+\beta)}{\{C_1^2+\bar{C}_2^2p\}} \tag{19}$$

$$m_{61} = \frac{-p(1+\tau' p)i\xi\varepsilon_2}{\sqrt{k}}, \quad m_{63} = \frac{\sqrt{k}p\{1+\alpha_0' p\} + \xi^2}{k}, \quad m_{65} = \frac{\sqrt{k}p\{1+\tau' p\}\varepsilon_3}{k} \tag{20}$$

If $\lambda_v = 0, \mu_v = 0$ the values of m_{ij} exactly same as the values of c_{ij} in Baksi and Bera (2005) in the absence of magnetic field.

4. Solution of vector matrix differential equation

The characteristic equation of the matrix A takes the form

$$\lambda^6 - \lambda^4(m_{41} + m_{52} + m_{63} + m_{45}m_{54} + m_{56}m_{65}) + \lambda^2(m_{52}m_{63} + m_{41}m_{52} + m_{41}m_{63} - m_{43}m_{61} + m_{45}m_{54}m_{63} - m_{43}m_{45}m_{65} + m_{41}m_{56}m_{65} - m_{45}m_{56}m_{61}) - (m_{41}m_{52}m_{63} - m_{43}m_{52}m_{61}) = 0 \tag{21}$$

The roots of the characteristic Eq. (21) which are also the eigenvalues of the matrix M are of the form $\lambda = \pm\lambda_1$, $\lambda = \pm\lambda_2$, $\lambda = \pm\lambda_3$. The right eigenvector $\overline{X} = [x_1, x_2, x_3, x_4, x_5, x_6]^T$ corresponding to eigenvalue λ can be calculated as

$$\overline{X} = \begin{bmatrix} \lambda(m_{45}m_{54} + m_{43}) - m_{43}m_{52} \\ \lambda(\lambda^2 m_{56} + m_{43}m_{54} - m_{41}m_{56}) \\ \lambda^4 - \lambda^2(m_{41} + m_{52} + m_{45}m_{54}) + m_{41}m_{52} \\ \lambda^3(m_{45}m_{56} + m_{43}) - \lambda m_{43}m_{52} \\ \lambda^4 m_{56} + \lambda^2(m_{43}m_{54} - m_{41}m_{56}) \\ \lambda^5 - \lambda^3(m_{41} + m_{52} + m_{45}m_{54}) + \lambda m_{41}m_{52} \end{bmatrix} \tag{22}$$

From the Eq. (21), we can now easily calculate the eigenvector $\overline{X}$ corresponding to the eigenvalue $\lambda = \lambda_i$.

For our further reference we shall use the following notations

$$X_1 = [\overline{X}]_{\lambda=\lambda_1},\ \ X_2 = [\overline{X}]_{\lambda=-\lambda_1},\ \ X_3 = [\overline{X}]_{\lambda=\lambda_2},\ \ X_4 = [\overline{X}]_{\lambda=-\lambda_2},\ \ X_5 = [\overline{X}]_{\lambda=\lambda_3},\ \ X_6 = [\overline{X}]_{\lambda=-\lambda_3} \tag{23}$$

The left eigenvector $\overline{Y} = [y_1, y_2, y_3, y_4, y_5, y_6]$ corresponding to eigenvalue λ can be calculated as

$$\overline{Y} = \begin{bmatrix} \lambda^2 m_{61} + \lambda(-m_{52}m_{61} - m_{45}m_{54}m_{61} + m_{41}m_{54}m_{65}) \\ \lambda_{52}(\lambda^2 m_{65} + m_{45}m_{61} - m_{65}m_{41}) \\ \lambda^5 - \lambda^3(m_{41} + m_{52} + m_{45}m_{54} + m_{56}m_{65}) + \lambda(m_{41}m_{52} - m_{45}m_{61}m_{65} + m_{41}m_{56}m_{65}) \\ \lambda^2(m_{61} + m_{54}m_{65}) - m_{52}m_{61} \\ \lambda^3 m_{65} + \lambda(m_{45}m_{61} - m_{41}m_{65}) \\ \lambda^4 - \lambda^2(m_{41} + m_{52} + m_{45}m_{54}) + m_{41}m_{52} \end{bmatrix}$$

For simplicity, we shall denote them as

$$Y_1 = [\overline{Y}]_{\lambda=\lambda_1},\ \ Y_2 = [\overline{Y}]_{\lambda=-\lambda_1},\ \ Y_3 = [\overline{Y}]_{\lambda=\lambda_2},\ \ Y_4 = [\overline{Y}]_{\lambda=-\lambda_2},\ \ Y_5 = [\overline{Y}]_{\lambda=\lambda_3},\ \ Y_6 = [\overline{Y}]_{\lambda=-\lambda_3} \tag{24}$$

Assuming the regularity condition at infinity as in Das *et al.*, the solution of the Eq. (20) can be written as (Appendix I)

$$\overline{V}(\xi,z,p) = a_2(z)X_2\exp(-\lambda_1 z) + a_4(z)X_4\exp(-\lambda_2 z) + a_6(z)X_6\exp(-\lambda_3 z) \tag{25}$$

where

$$a_i(z) = -\frac{1}{Y_iX_i}\int_{-\infty}^{\infty}[\lambda_i^4 - \lambda_i^2(m_{41}+m_{52}+m_{45}m_{54}) + m_{41}m_{52}]Q_0^{/}\frac{C_1^3(1+\tau^{/}p)}{(2\pi k_y^3 k_z)^{1/2}}\delta\left(\frac{k_z}{C_1}z\right)e^{-\lambda_i n}\,dn$$

After evaluation of the integral,

$$a_i(z) = \frac{-Q_0^{//}}{Y_iX_i}[\lambda_i^4 - \lambda_i^2(m_{41}+m_{52}+m_{45}m_{54}) + m_{41}m_{52}], \quad s>0, \quad i = 2,4,6 \tag{26}$$

where

$$Q_0^{//} = Q_0^{/}\frac{C_1^4(1+\tau^{/}p)}{(2\pi k_y^3 k_z^3)^{1/2}}$$

Writing (a_2, a_4, a_6) as (A_1, A_2, A_3) the deformations $\hat{\overline{V}}(\xi,z,p)$, $\hat{\overline{W}}(\xi,z,p)$ and temperature $\hat{\overline{T}}(\xi,z,p)$ can be compactly written from Eq. (25) as

$$\hat{\overline{V}}(\xi,z,p) = \sum_{i=1}^{3}A_i[-\lambda_i^2(m_{45}m_{56}+m_{43}) - m_{43}m_{52}]\exp(-\lambda_i z) \tag{27}$$

$$\hat{\overline{W}}(\xi,z,p) = \sum_{i=1}^{3}A_i[-\lambda_i\{\lambda_i^2 m_{56} + (m_{43}m_{54} - m_{41}m_{56})\}]\exp(-\lambda_i z) \tag{28}$$

and

$$\hat{\overline{T}}(\xi,z,p) = \sum_{i=1}^{3}A_i[\{\lambda_i^4 - \lambda_i^2(m_{41}+m_{52}+m_{45}m_{54}) + m_{41}m_{52}\}]\exp(-\lambda_i z) \tag{29}$$

From Eq. (8), the stresses in the Laplace-Fourier transform domain can be written as

$$\hat{\overline{\tau}}_{11} = T_0\left[(\lambda_e + \overline{\lambda}_v p)\left(\overline{\beta}_2 i\xi\overline{v} + \overline{\beta}_3\frac{\partial\hat{\overline{w}}}{\partial z}\right) - i\beta_1(1+\overline{\alpha}p)\hat{\overline{T}}\right] \tag{30}$$

$$\hat{\overline{\tau}}_{22} = T_0\left[(\lambda_e + \overline{\lambda}_v p)\left(\overline{\beta}_2 i\xi\overline{v} + \overline{\beta}_3\frac{\partial\hat{\overline{w}}}{\partial z}\right) + \overline{\beta}_2(\mu_e' + \mu_v' p)i\xi\hat{\overline{v}} - \beta_2(1+\overline{\alpha}p)\hat{\overline{T}}\right] \tag{31}$$

$$\hat{\overline{\tau}}_{33} = T_0\left[(\lambda_e + \overline{\lambda}_v p)\left(\overline{\beta}_2 i\xi\overline{v} + \overline{\beta}_3\frac{\partial\hat{\overline{w}}}{\partial z}\right) + \overline{\beta}_3(\mu_e' + \mu_v' p)\frac{\partial\hat{\overline{w}}}{\partial z} - \beta_3(1+\overline{\alpha}p)\hat{\overline{T}}\right] \tag{32}$$

$$\hat{\overline{\tau}}_{23} = T_0\left[(\mu_e' + \mu_v' p)\left(\frac{\beta_3}{k}i\xi\overline{w} + k\overline{\beta}_2\frac{\partial\hat{\overline{v}}}{\partial z}\right)\right] \tag{33}$$

where $\hat{\overline{v}}, \hat{\overline{w}}$ and $\hat{\overline{T}}$ are obtained from Eqs. (27), (28) and (29).

We now write down from (29)-(33), the expressions of the temperature and the stresses from the Laplace-Fourier transform domain to the Laplace transform domain as

$$[\overline{T}, \overline{\tau}_{11}, \overline{\tau}_{22}, \overline{\tau}_{33}](y,z,p) = \sqrt{\frac{2}{\pi}}\int_0^{\infty}[\hat{\overline{T}}, \hat{\overline{\tau}}_{11}, \hat{\overline{\tau}}_{22}, \hat{\overline{\tau}}_{33}, \hat{\overline{h}}_x](z,p)\cos(\xi y)\,d\xi \tag{34}$$

since $\hat{\overline{T}}$, $\hat{\overline{\tau}}_{11}$, $\hat{\overline{\tau}}_{22}$, $\hat{\overline{\tau}}_{33}$, $\hat{\overline{h}}_x$ are even functions of ξ.

and

$$\bar{\tau}_{23}(y,z,p) = \sqrt{\frac{2}{\pi}}\int_0^\infty [\hat{\bar{\tau}}_{23}](z,p)\sin(\xi y)d\xi \tag{35}$$

since $\hat{\bar{\tau}}_{23}$ is an odd function of ξ.

5. Numerical solution

The Fourier-Laplace inversion of the expressions for temperature and stresses in the space-time domain are very complex and we prefer to develop efficient computer software for the purpose of inversion of these double integral transforms. As such, as in the previous case, for the inversion of Laplace transform we follow the method of Bellman (1966) and choose seven values of the time $t = t_i$, $i = 1, 2, 3, 4, 5, 6, 7$, at which the stresses are to be determined, where, t_i are the roots of the shifted Legendre polynomial of degree seven, *vide* Bellman (1966). Simultaneous calculations for the inversion of the Fourier-transform were done by evaluating the infinite integrals (34) and (35) numerically by seven-point Gaussian quadrature (Appendix II) formula for several prescribed values of y and z.

The copper material is chosen for numerical computation. The values of the dimensionless constants are taken as

$$c_1^2 = 0.1,\ \bar{c}_2^2 = .2,\ d = 0.1,\ \bar{g} = 0.2,\ a = 0.1,\ \bar{b} = 0.3,\ \varepsilon_2 = 0.1,\ \varepsilon_3 = 0.1$$

We now present our results in the form of graphs (Figs. 1-4) to compare with the cases CTE, ETE, TRDTE for the stresses field when time variable t = 0.025775, 0.138382, 0.352509, 0.693147, 1.21376, 2.04612 and 3.67119 are labeled in the abscissa and for particular values of the space variables $y = 20$ and $z = 1$. The material constants α, α_0 and thermal relaxation parameter τ were taken as for different cases (i) CTE (ii) ETE (iii) TRDTE as follows:

(i)	CTE	$\alpha = 0$,	$\alpha_0 = 0$,	$\tau = 0$
(ii)	ETE	$\alpha = 0$,	$\alpha_0 = 10^{-5}$,	$\tau = 10^{-5}$

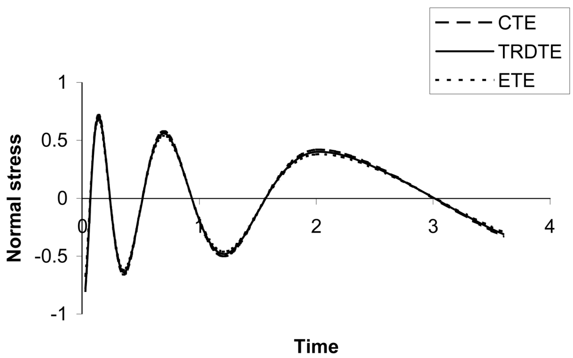

Fig. 1 Distribution of the normal stress τ_{11} versus time

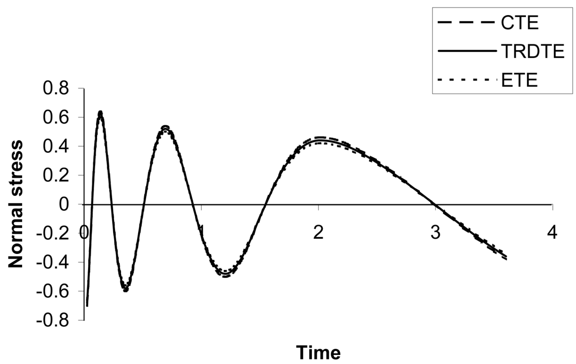

Fig. 2 Distribution of the normal stress τ_{22} versus time

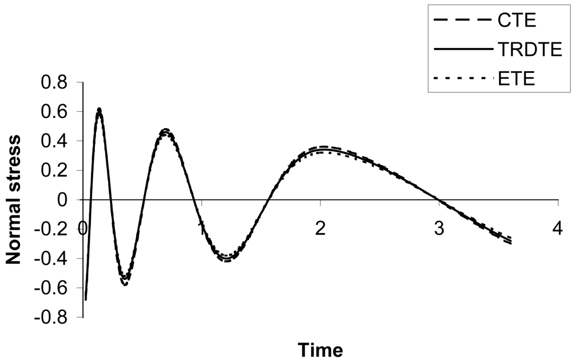

Fig. 3 Distribution of the normal stress τ_{33} versus time

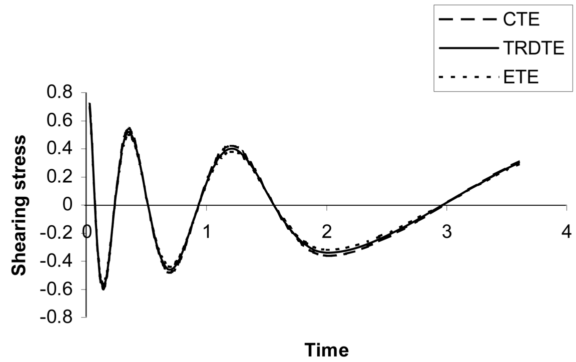

Fig. 4 Distribution of the shearing stress τ_{23} versus time

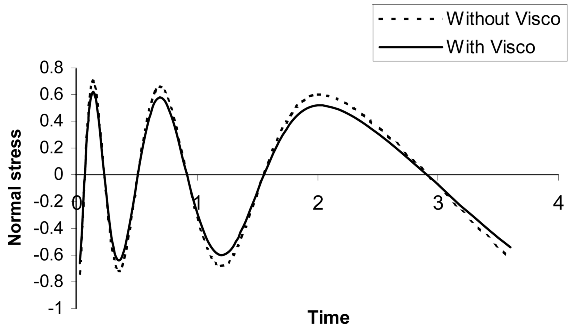

Fig. 5 Comparison of normal stress in the presence and absence of viscoelastic parameters

(iii) TRDTE $\alpha = 10^{-5}$, $\alpha_0 = 10^{-7}$, $\tau = 0$

The behavior of the stresses etc. for $t \to 0$ can be estimated form the initial value theorem

$$\lim_{t \to 0} \phi(t) = \lim_{p \to \infty} p\overline{\phi}(t)$$

6. Conclusion

i) The graphs drawn almost coincide with the graphs drawn in generalized thermoelasticity, if we put $\lambda_v = 0$, $\mu_v = 0$ given in Fig. 5.
ii) The nature of the propagation of stresses for τ_{11}, τ_{22} and τ_{33} is identical but that of the stress τ_{23} is different. τ_{11}, τ_{22} and τ_{33} start from the negative value where as τ_{23} starts from the positive value.
i) The values of the stresses have slight difference in the three cases CTE, TRDTE and ETE.
ii) The amplitudes of stresses are initially high and diminish as time increases.
iii) It is clear from the Fig. 5 that the in the viscoelastic medium the amplitude of the stress reduces as expected.

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Appendix: I

Solution of the vector-matrix differential equation

Let us consider a vector-matrix differential equation

$$\frac{d\vec{V}}{dx} = \mathbf{M}\vec{V} + \vec{f}(\mathbf{x}) \tag{1}$$

with the condition

$$\vec{V}(x_0) = \vec{C} \tag{2}$$

where M is an $n \times n$ constant real matrix, $\vec{C}$ is a given constant real n-vector and $\vec{f}$ is a real n-vector function. Let

$$\vec{V} = \vec{X}\exp(\lambda x) \tag{3}$$

be a solution of the homogeneous equation

$$\frac{d\vec{V}}{dx} = \mathrm{M}\vec{V} \tag{4}$$

where λ is a scalar and $\vec{X}$ is an n-vector independent of x. Substituting (3) in (4) we get

$$(M\vec{X} - \lambda\vec{X})e^{\lambda x} = \theta \Rightarrow \mathrm{M}\vec{X} - \lambda\vec{X} = \theta \Rightarrow \mathrm{M}\vec{X} = \lambda\vec{X}$$

This may be interpreted that λ is an eigenvalue of the matrix M and $\vec{X}$ is the corresponding right eigenvector. Let $\lambda_1, \lambda_2, \lambda_3, \ldots\ldots, \lambda_n$, be n distinct eigenvalues of the matrix M and $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n$ be the corresponding right eigenvectors of the matrix M. Then the vectors $\vec{X}_1, \vec{X}_2, \ldots, \vec{X}_n$ are linearly independent and so they form a basis of the space Γ^n, where Γ denotes the field of complex numbers. We can find scalars b_1, b_2, …, b_n such that

$$\vec{C} = b_1\vec{X}_1 + b_2\vec{X}_2 + \ldots + b_n\vec{X}_n$$

Let us choose

$$c_i = b_i e^{-\lambda_i x_0} \quad (i = 1, 2, \ldots n)$$

Let

$$\vec{u}(x) = \sum_{i=1}^{n} c_i \vec{X}_i e^{\lambda_i x} \tag{5}$$

Thus $\vec{u}(x)$ is a solution of the differential Eq. (4) and

$$\vec{u}(\dot{x}_0) = \sum_{i=1}^{n} c_i \vec{X}_i e^{\lambda_i x_0} = \sum_{i=1}^{n} b_i \vec{X}_i = \vec{C}$$

Now, let

$$\vec{w}(x) = \sum_{i=1}^{n} a_i(x) \vec{X}_i e^{\lambda_i x} \tag{6}$$

be a solution of Eq. (1), where $a_1(x)$, $a_2(x)$, ... , $a_n(x)$ are scalar functions of x such that $a_i(x_0) = 0$, $i = 1(1)n$.
Differentiating (6) with respect to x, we get

$$\vec{w}'(x) = \sum_{i=1}^{n} a_i'(x) \vec{X}_i e^{\lambda_i x} + \sum_{i=1}^{n} a_i(x) \lambda_i \vec{X}_i e^{\lambda_i x} \tag{7}$$

Substituting (6) and (7) in (1) we have

$$\sum_{i=1}^{n} a_i'(x) \vec{X}_i e^{\lambda_i x} + \sum_{i=1}^{n} a_i(x) \lambda_i \vec{X}_i e^{\lambda_i x} = \sum_{i=1}^{n} a_i(x) M \vec{X} e^{\lambda_i x} + \vec{f}(x)$$

Or,

$$\sum_{i=1}^{n} a_i'(x) \vec{X}_i e^{\lambda_i x} = \sum_{i=1}^{n} a_i(x) [M \vec{X}_i - \lambda_i \vec{X}] e^{\lambda_i x} + \vec{f}(x) = \vec{f}(x) \tag{8}$$

Multiplying by $\vec{Y}_j e^{-\lambda_j x}$ where ($\vec{Y}_1, \vec{Y}_2, \ldots, \vec{Y}_n$ are left eigenvectors of M corresponding to the eigenvalues λ_1, λ_2, ... , λ_n) we get

$$\sum_{i=1}^{n} a_i'(x) [\vec{Y}_j \vec{X}_i e^{(\lambda_i - \lambda_j) x}] = \vec{Y}_j \vec{f}(x) e^{-\lambda_j x}$$

Or,

$$a_j'(x) \vec{Y}_j \vec{X}_j = \vec{Y}_j \vec{f}(x) e^{-\lambda_j x}, \quad [\vec{Y}_j \vec{X}_i = 0 \text{ for } i \neq j]$$

$$a_j'(x) = \frac{1}{\vec{Y}_j \vec{X}_j} \vec{Y}_j \vec{f}(x) e^{-\lambda_j x}$$

Or,

$$a_j(x) = \int_{x_0}^{x} (\vec{Y}_j \vec{X}_j)^{-1} \vec{Y}_j \vec{f}(s) e^{-\lambda_j s} ds$$

$$[a_j(x_0) = 0 \text{ for } j = 1(1)n]$$

Now, we take

$$\vec{V}(x) = \vec{u}(x) + \vec{w}(x)$$

Differentiating we get

$$\begin{aligned}\vec{V}'(x) &= \vec{u}'(x) + \vec{w}'(x)\\ &= A\vec{u}(x) + A\vec{w}(x) + \vec{f}(x)\\ &= A[\vec{u}(x) + \vec{w}(x)] + \vec{f}(x)\\ &= A\vec{V}(x) + \vec{f}(x)\end{aligned}$$

and

$$\vec{V}(x_0) = \vec{u}(x_0) + \vec{w}(x_0) = \vec{C}$$

Hence, $\vec{V}(x) = \vec{u}(x) + \vec{w}(x)$ is the unique solution of the differential Eq. (1) satisfying the condition (2).

Appendix: II

Numerical inversion of the laplace transform

Let the Laplace transform $F(p)$ of $u(t)$ is given by

$$F(p) = \int_0^\infty e^{-pt} u(t)\,dt \qquad p \geq 0 \tag{1}$$

We assume that $u(t)$ is sufficiently smooth to permit the approximate method we apply.
Putting

$$x = e^{-t} \tag{2}$$

in (1), we get

$$F(p) = \int_0^1 xp^{-1} g(x)\,dx \tag{3}$$

where $u(-\log x) = g(x)$

Applying the Gaussian quadrature formula in (II.3) yields

$$\sum_{i=1}^{N} W_i x_i^{p-1} g(x_i) = F(p) \tag{4}$$

where x_i are the roots of the shifted Legendre polynomial $P_N(x) = 0$ and W_i are the corresponding coefficients. Thus x_i and W_i are known.

Eq. (II.4) can be written as

$$W_1 x_1^{p-1} g(x_1) + W_2 x_2^{p-1} g(x_2) + \ldots + W_N x_N^{p-1} g(x_N) = F(p) \tag{5}$$

We now put $p = 1, 2, \ldots, N$ in Eq. (5), then the resulting equations become

$$\begin{gathered}W_1 g(x_1) + W_2 g(x_2) + \ldots + W_N g(x_N) = F(1)\\ W_1 x_1 g(x_1) + W_2 x_2 g(x_2) + \ldots + W_N x_N g(x_N) = F(2)\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ W_1 x_1^{N-1} g(x_1) + W_2 x_2^{N-1} g(x_2) + \ldots + W_N x_N^{N-1} g(x_N) = F(N)\end{gathered} \tag{6}$$

Thus

$$\begin{bmatrix} g(x_1) \\ g(x_2) \\ \cdots \\ g(x_N) \end{bmatrix} = \begin{bmatrix} W_1 & W_2 & W_N \\ W_1 x_1 & W_2 x_2 & W_N x_N \\ & \ldots\ldots\ldots\ldots\ldots & \\ W_1 x_1^{N-1} & W_2 x_2^{N-1} & W_N x_N^{N-1} \end{bmatrix}^{-1} \begin{bmatrix} F(1) \\ F(2) \\ \cdots \\ F(N) \end{bmatrix}$$

Hence, $g(x_1)$, $g(x_2)$ … , $g(x_N)$ are known.
Now

$$U(-\log x_1) = g(x_1),\ U(-\log x_2) = g(x_2),\ \ldots,\ U(-\log x_N) = g(x_N)$$

For $N = 7$

Roots x_i of the shifted Legendre Polynomial	$u(-\log x_i) = g(x_i)$
$x_1 = -0.94910791$	3.671194951
$x_2 = -0.74153119$	2.046127431
$x_3 = -0.40584515$	1.213 762484
$x_4 = 0$	0.69314718
$x_5 = 0.40584515$	0.352508528
$x_6 = 0.74153119$	0.138382
$x_7 = 0.94910791$	0.025775394.