Structural Engineering and Mechanics, Vol. 29, No. 5 (2008) 469-488
DOI: http://dx.doi.org/10.12989/sem.2008.29.5.469

Non-stochastic interval arithmetic-based finite element analysis for structural uncertainty response estimate

Dongkyu Lee†

Architectural Engineering Research Department, Steel Structure Research Laboratory, Research Institute of Industrial Science & Technology, Korea

Sungsoo Park‡ and Soomi Shin‡†

Department of Architectural Engineering, Pusan National University, Busan, Korea

(Received May 10, 2006, Accepted May 19, 2008)

Abstract. Finite element methods have often been used for structural analyses of various mechanical problems. When finite element analyses are utilized to resolve mechanical systems, numerical uncertainties in the initial data such as structural parameters and loading conditions may result in uncertainties in the structural responses. Therefore the initial data have to be as accurate as possible in order to obtain reliable structural analysis results. The typical finite element method may not properly represent discrete systems when using uncertain data, since all input data of material properties and applied loads are defined by nominal values. An interval finite element analysis, which uses the interval arithmetic as introduced by Moore (1966) is proposed as a non-stochastic method in this study and serves a new numerical tool for evaluating the uncertainties of the initial data in structural analyses. According to this method, the element stiffness matrix includes interval terms of the lower and upper bounds of the structural parameters, and interval change functions are devised. Numerical uncertainties in the initial data are described as a tolerance error and tree graphs of uncertain data are constructed by numerical uncertainty combinations of each parameter. The structural responses calculated by all uncertainty cases can be easily estimated so that structural safety can be included in the design. Numerical applications of truss and frame structures demonstrate the efficiency of the present method with respect to numerical analyses of structural uncertainties.

Keywords: non-stochastic; interval arithmetic; finite element method; structural uncertainty; initial data; interval change function; tolerance error.

1. Introduction

Numerical analyses of structural responses are generally executed with structural parameters such as Young's modulus, cross-sectional areas and lengths of members, and loading conditions. In engineering structure designs, the structural analyses are used to assess behaviors using nominal

† Corresponding author, E-mail: dklee@rist.re.kr
‡ E-mail: parks@pusan.ac.kr
‡† E-mail: shinsumi82@pusan.ac.kr

values of structural parameters and loading conditions. However, in practice, some degree of uncertainty exists in structural parameters and applied loads (Chen and Yang 2000). For example, inaccuracy or error in the material properties of products may occur during manufacturing. After these processes the nominal values of the structural parameters may change due to environmental conditions such as temperature and humidity. In addition, errors in the work of engineers may result in inaccurate magnitudes of structural parameters during the manufacturing stage, or applied loads during experimental tasks (Noh 1998). As a consequence, structural analysis always has some degree of uncertainty. Therefore the concept and application of uncertainty plays an important role in investigations of various engineering and mechanical problems.

Until recently, the majority of scientists and engineers which have researched the problem of uncertainty have utilized stochastic, i.e., probabilistic methods such as heuristic approaches. According to these approaches structural parameters and loading conditions are modeled as random variables in defined fields of structures. However, given that only small amounts of statistical information about initial data are available only in a few specialized cases, the probabilistic approaches can not deliver reliable solutions without sufficient experimental data. This problem is a limitation of the heuristic approach. The application of uncertainty models which are independent of such detailed knowledge has been investigated by some researchers (Karni and Belikoff 1996, McWilliam 2001, Alefeld 1983).

As an alternative of the heuristic method, a branch and bound method such as an interval approach has been considered by many researchers. A famous and traditional example of interval arithmetic was proposed by Archimedes. He considered inscribed polygons and circumscribed them with circles with radius of 1. He then obtained an increasing sequence of lower bounds and at the same time a decreasing sequence of upper bounds for the area of the corresponding disc (Alefeld and Mayer 2000). The intervals as a result of measurement were first introduced by Wiener (1914, 1921). In 1914, he applied intervals to the measurement of distances and in 1921 of time. As a turning point of interval theory, Sunaga (1958) introduced algebraic rules of multidimensional interval operations.

Since the mid-1960s, practical applications of the interval analysis have been introduced by Moore (1962, 1966) for bounding solutions of initial value problems, and since then computational analyses using the interval method have been developed by many researchers. In 1995, Köylouglu *et al*. (1995) and Köyluoglu and Elishakoff (1998) have developed an interval approach utilizing a finite element method to deal with pattern loading and structural uncertainties. Although these works were mainly restricted to narrow intervals and approximate numerical solutions, it was very important that interval analyses were actually applied to measure practical structures, for example, the bounds of complex Eigenvalues of structures with interval parameters were discussed by Yang, Chen and Lian (2001). Rao and Beike (1997) discussed the structural analysis of uncertain structural systems; Skvzypczyk (1997) discussed the fuzzy finite element methods; Chen and Qiu (1994) discussed the interval Eigenvalue problems; and interval analysis was applied to linear mechanical structures by Kulpa, Powmik and Skalna (1998).

Unlike other researches of interval analyses of structural responses, the axial rigidity of truss members and the bending rigidity of frames are regarded as important terms for considering the uncertain response fields of mechanical systems. The uncertain rigidity fields of each member are formulated and use discretized and modified stiffness equations which were first introduced by this study, where multi-degree-of-freedom systems appear in the uncertain linear equations and uncertainties of initial data are expressed as a defined tolerance error. This is used as the selection whether the numerical uncertainties for each uncertainty parameter exist or not. The uncertainty combinations of each parameter can be constructed by a tree graph. The combination classifies the uncertainty of structural responses which are dependent on initial data. An *axial and bending rigidity of uncertainty* and a *displacement coefficient of uncertainty* are formulated by an *interval change function (ICF)* for each, which is composed of upper and lower bounds. When the problem of member uncertainty based on the uncertainty of initial data is addressed, the interval change function is easily applied to the uncertainty problems. Finally, this study represents the dependence between structural parameters and applied loads according to the uncertainty of structural responses. In order to verify the reliability of the proposed method for uncertainty response analyses, numerical examples of truss and frame structures with the interval structural parameters defined are given with respect to the quantities of uncertainty measurement.

The outline of this study is as follows. In Section 2, the theory of interval arithmetic is described. Finite element method formulations using interval change functions derived from the interval arithmetic are shown in Section 3. By using the interval finite element method numerical applications for structural uncertainty analyses of truss and frame structures are studied in Section 4 followed by the conclusions in Section 5.

2. Theory of interval arithmetic

In structural designs and analyses, what causes uncertainty of structural responses is generally divided into internal and external components of structures. The former is the uncertainty by boundary conditions, loading conditions, assumptions of modeling, and analytical assumption, the latter is the uncertainty by the workmanships and natural environmental conditions. Fig. 1 shows the uncertainty of structural response and its reason with respect to initial data of structures.

An interval Analysis is a field of mathematics that accounts for numerical imprecision and physical uncertainty with intervals using set-based operations. In the interval arithmetic, the errors

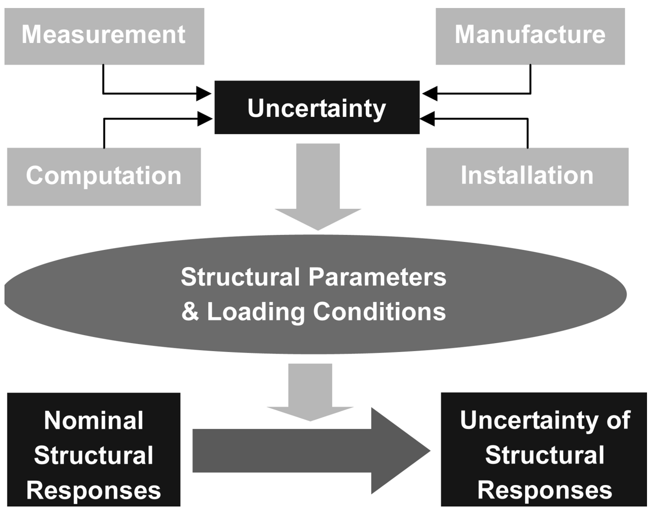


Fig. 1 Structural response uncertainty by uncertainties of initial data

or uncertainties are always denoted by intervals. From this principle, we define intervals firstly. In general, an interval arithmetic operation $\circ$ between intervals a and b is given as

$$a \circ b = hull\{\tilde{a} \circ \tilde{b} | \tilde{a} \in a, \tilde{b} \in b;\ \tilde{a} \circ \tilde{b} \textit{ is defined}\} \tag{1}$$

where the *hull* of a set produces the minimum and maximum bounds. *Interval vectors* and *interval matrices* are nothing more than standard vectors and matrices with intervals instead of scalar values for components and elements (Alefeld and Mayer 2000, Sunaga 1958, Dwyer 1951, Moore 1966).

Let $c = (c_1, c_2, \ldots, c_m)^T$ be a structural parameter vector with bound error or uncertainties and $c \in a \circ b$
where

$$c_i \in c_i^I = \{c_i^c - \Delta c_i, c_i^c + \Delta c_i\} \tag{2}$$

then

$$c \in c^I = \{c^c - \Delta c, c^c + \Delta c\} \tag{3}$$

where $\underline{c} = c^c - \Delta c$ is the lower bound of an interval and $\overline{c} = c^c + \Delta c$ is the upper bound of an interval. Also we define the *mid-point* of an interval c^c by

$$c^c = (c_1^c, c_2^c, \ldots, c_m^c)^T \tag{4}$$

We define the *uncertainty* of an interval Δc by

$$\Delta c = (\Delta c_1, \Delta c_2, \ldots, \Delta c_m)^T \tag{5}$$

In a similar way, expressed by $X^I = [\underline{X}, \overline{X}]$, the mid-point and uncertainty of a n-dimensional interval vector $X^I = (X_1^I, X_2^I, \ldots, X_n^I)^T$ can be described by

$$X^c = (X_1^c, X_2^c, \ldots, X_n^c)^T \tag{6}$$

and

$$\Delta X = (\Delta X_1, \Delta X_2, \ldots, \Delta X_n)^T \tag{7}$$

Commonly used notions are the *mid-point* of an interval vector X^c

$$X^c = \frac{\overline{X} + \underline{X}}{2} \tag{8}$$

and the *uncertainty* of an interval vector ΔX

$$\Delta X = \frac{\overline{X} - \underline{X}}{2} \tag{9}$$

A matrix whose elements are the interval parameters is called an interval matrix and expressed by

$A^I = [\underline{A}, \overline{A}]$, in which $\underline{A}$ and $\overline{A}$ consist of each lower and upper bound. Similarly, the mid-point and uncertainty of n-dimensional interval matrix $A^I = (A_1^I, A_2^I, \ldots, A_n^I)^T$ can be expressed by

$$A^c = (A_1^c, A_2^c, \ldots, A_n^c)^T \tag{10}$$

and

$$\Delta A = (\Delta A_1, \Delta A_2, \ldots, \Delta A_n)^T \tag{11}$$

Commonly used notions are the *mid-point* of an interval matrix A^c as follows

$$A^c = \frac{\overline{A} + \underline{A}}{2} \tag{12}$$

and the *uncertainty* of an interval matrix ΔA as follows

$$\Delta A = \frac{\overline{A} - \underline{A}}{2} \tag{13}$$

For many operations, including standard arithmetic operations of *addition*, *subtraction*, *multiplication* and *division*, the resulting set is also an interval that can be conveniently defined in term of end-points of the argument intervals.

Let $X^I = [\underline{X}, \overline{X}]$ and $Y^I = [\underline{Y}, \overline{Y}]$ be the intervals, then the operations are defined by the following formulas.

$$X^I + Y^I = [\underline{X}, \overline{X}] + [\underline{Y}, \overline{Y}] = [\underline{X} + \underline{Y}, \overline{X} + \overline{Y}] \tag{14}$$

$$X^I - Y^I = [\underline{X}, \overline{X}] - [\underline{Y}, \overline{Y}] = [\underline{X} - \overline{Y}, \overline{X} - \underline{Y}] \tag{15}$$

$$X^I \times Y^I = [\underline{X}, \overline{X}] \times [\underline{Y}, \overline{Y}] = [\min(\underline{X} \cdot \underline{Y}, \underline{X} \cdot \overline{Y}, \overline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y}), \max(\underline{X} \cdot \underline{Y}, \underline{X} \cdot \overline{Y}, \overline{X} \cdot \underline{Y}, \overline{X} \cdot \overline{Y})] \tag{16}$$

$$X^I / Y^I = [\underline{X}, \overline{X}] / [\underline{Y}, \overline{Y}] = \begin{cases} [\underline{X}, \overline{X}] \times \left[\frac{1}{\underline{Y}}, \frac{1}{\overline{Y}}\right], & \text{if } 0 \in [\underline{Y}, \overline{Y}] \\ [-\infty, +\infty], & \text{otherwise} \end{cases} \tag{17}$$

3. Interval finite element method

3.1 Formulations of the axial and bending rigidity with displacement coefficient of uncertainty

In plane truss structures, in order to measure the effect of uncertain parameters in compared with uncertainty results of structural responses, it is reasonable to introduce a vector which describes numerical uncertainties of structural parameters and loading conditions. The uncertainties can exist in the values of Young's modulus, cross-sectional areas, length of members and applied loads. When the uncertain initial data values become changed between upper and lower bounds, the discrete structural responses of each degree of freedom immediately vary between certain upper and lower bounds in structural systems.

In order to analyze the influence of the uncertain initial data, it is convenient to divide the uncertain vector into two factors, namely, the *axial rigidity of uncertainty* α, and *the displacement coefficient of uncertainty* β in truss structures.

The axial rigidity α includes relations of numerical uncertainty among Young's modulus E, cross-sectional areas A and length of members L and it is written as

$$\alpha = \frac{EA}{L} = \frac{[E^c - \Delta E, E^c + \Delta E] \times [A^c - \Delta A, A^c + \Delta A]}{[L^c - \Delta L, L^c + \Delta L]}$$

$$= \left[\frac{(E^c - \Delta E)(A^c - \Delta A)}{L^c + \Delta L}, \frac{(E^c + \Delta E)(A^c + \Delta A)}{L^c - \Delta L}\right] \tag{18}$$

If the second order terms are neglected in the formulation (18), i.e., the analysis is a perturbation method, the *mid-point* of α, i.e., nominal value α^c is written as

$$\alpha^c = \frac{E^c A^c}{L^c} \tag{19}$$

The *uncertainty* of α, $\Delta\alpha$ is also expressed as

$$\Delta\alpha = \frac{\Delta E A^c L^c + E^c \Delta A L^c + E^c A^c \Delta L}{(L^c)^2} \tag{20}$$

The displacement coefficient of uncertainty β presents the values of uncertain displacements of each degree-of-freedom and is written with the additional parameter of applied loadings P as

$$\beta = \frac{PL}{EA} = \frac{[P^c - \Delta P, P^c + \Delta P] \times [L^c - \Delta L, L^c + \Delta L]}{[E^c - \Delta E, E^c + \Delta E] \times [A^c - \Delta A, A^c + \Delta A]}$$

$$= \left[\frac{(P^c - \Delta P)(L^c - \Delta L)}{(E^c + \Delta E)(A^c + \Delta A)}, \frac{(P^c + \Delta P)(L^c + \Delta L)}{(E^c - \Delta E)(A^c - \Delta A)}\right] \tag{21}$$

The *mid-point* of an interval β^c may be written by

$$\beta^c = \left(\frac{P^c L^c E^c A^c + \Delta P \Delta L E^c A^c + \Delta P L^c \Delta E A^c + P^c \Delta L \Delta E A^c}{(E^c A^c)^2 + (\Delta E \Delta A)^2 - (\Delta E A^c)^2 - (E^c \Delta A)^2}\right)$$

$$+ \left(\frac{\Delta P L^c E^c \Delta A + P^c \Delta L E^c \Delta A + P^c L^c \Delta E \Delta A + \Delta P \Delta L \Delta E \Delta A}{(E^c A^c)^2 + (\Delta E \Delta A)^2 - (\Delta E A^c)^2 - (E^c \Delta A)^2}\right) \tag{22}$$

We define the *uncertainty* of an interval $\Delta\beta$ by

$$\Delta\beta = \left(\frac{\Delta P L^c E^c A^c + P^c \Delta L E^c A^c + P^c L^c \Delta E A^c + \Delta P \Delta L \Delta E A^c}{(E^c A^c)^2 + (\Delta E \Delta A)^2 - (\Delta E A^c)^2 - (E^c \Delta A)^2}\right)$$

$$+ \left(\frac{P^c L^c E^c \Delta A + \Delta P \Delta L E^c \Delta A + \Delta P L^c \Delta E \Delta A + P^c \Delta L \Delta E \Delta A}{(E^c A^c)^2 + (\Delta E \Delta A)^2 - (\Delta E A^c)^2 - (E^c \Delta A)^2}\right) \tag{23}$$

In addition to the degrees of freedom of truss structures, frame structures take an additional degree of freedom of an angle of a rotation or a slope. The members of frame structures take properties of constant moments of inertia I, Young's modulus E, lengths of member L, and cross sectional areas A and applied loads P. Note that the cross sectional areas are subject to the moments of inertia. In order to estimate the influence of the above-mentioned uncertain initial data in frame structures, *the bending rigidity with uncertainty* γ, and *the displacement coefficient of uncertainty* χ in frame structures are considered. As related to the formulations (18)~(23) in case of truss structures, those of frame structures are described as the following formulations (24)~(29).

The bending rigidity γ is written as

$$\gamma = \frac{EI}{L} = \frac{[E^c - \Delta E, E^c + \Delta E] \times [I^c - \Delta I, I^c + \Delta I]}{[L^c - \Delta L, L^c + \Delta L]}$$

$$= \left[\frac{(E^c - \Delta E)(I^c - \Delta I)}{L^c + \Delta L}, \frac{(E^c + \Delta E)(I^c + \Delta I)}{L^c - \Delta L} \right] \tag{24}$$

If the second order terms are neglected in formulation (24), i.e., the analysis is a perturbation method, the *mid-point* of γ, i.e., nominal value γ^c is written as

$$\gamma^c = \frac{E^c I^c}{L^c} \tag{25}$$

The *uncertainty* of γ, $\Delta\gamma$ is expressed as

$$\Delta\gamma = \frac{\Delta E I^c L^c + E^c \Delta I L^c + E^c I^c \Delta L}{(L^c)^2} \tag{26}$$

The displacement coefficient of uncertainty χ presents the values of uncertain displacements of each degree-of-freedom and is written as

$$\chi = \frac{PL}{EI} = \frac{[P^c - \Delta P, P^c + \Delta P] \times [L^c - \Delta L, L^c + \Delta L]}{[E^c - \Delta E, E^c + \Delta E] \times [I^c - \Delta I, I^c + \Delta I]}$$

$$= \left[\frac{(P^c - \Delta P)(L^c - \Delta L)}{(E^c + \Delta E)(I^c + \Delta I)}, \frac{(P^c + \Delta P)(L^c + \Delta L)}{(E^c - \Delta E)(I^c - \Delta I)} \right] \tag{27}$$

The *mid-point* of an interval χ^c may be written by

$$\chi^c = \left(\frac{P^c L^c E^c I^c + \Delta P \Delta L E^c I^c + \Delta P L^c \Delta E I^c + P^c \Delta L \Delta E I^c}{(E^c I^c)^2 + (\Delta E \Delta I)^2 - (\Delta E I^c)^2 - (E^c \Delta I)^2} \right)$$

$$+ \left(\frac{\Delta P L^c E^c \Delta I + P^c \Delta L E^c \Delta I + P^c L^c \Delta E \Delta I + \Delta P \Delta L \Delta E \Delta I}{(E^c I^c)^2 + (\Delta E \Delta I)^2 - (\Delta E I^c)^2 - (E^c \Delta I)^2} \right) \tag{28}$$

We define the *uncertainty* of an interval $\Delta\chi$ by

$$\Delta\chi = \left(\frac{\Delta P L^c E^c I^c + P^c \Delta L E^c I^c + P^c L^c \Delta E I^c + \Delta P \Delta L \Delta E I^c}{(E^c I^c)^2 + (\Delta E \Delta I)^2 - (\Delta E I^c)^2 - (E^c \Delta I)^2} \right)$$

$$+ \left(\frac{P^c L^c E^c \Delta I + \Delta P \Delta L E^c \Delta I + \Delta P L^c \Delta E \Delta I + P^c \Delta L \Delta E \Delta I}{(E^c I^c)^2 + (\Delta E \Delta I)^2 - (\Delta E I^c)^2 - (E^c \Delta I)^2} \right) \tag{29}$$

We can let the uncertainty Δ of the interval α, γ, β, and χ be the tolerance errors x, which is $0 \le x < 1$, $x \in R$ about real value R.

3.2 Interval change function

An interval change function is a mathematical formulation which is composed of the upper and lower bounds with respect to the tolerance error x. A basic idea behind the interval change function is quantitatively to calculate the changes of the required results that take place in uncertainty problems, when a small change (i.e., uncertainty) is made by the uncertain parameters against some nominal values in the structural system.

Considering whether numerical uncertainties of the initial data exist or not, a generalized scenario function of the uncertainty $\Im_i$ may be written as follows

$$\Im_i = \frac{b_1^c b_2^c b_3^c \ldots b_q^c}{a_1^c a_2^c a_3^c \ldots a_q^c}[f(x)_i, g(x)_i] \tag{30}$$

where

a^c, b^c : Mid-point of each uncertain parameter

i : 2^{p+q}, The number of uncertainty scenarios

p, q : The number of uncertain parameters

$f(x)_i = \dfrac{(1-x)^q}{(1+x)^p}$: Lower bounds of an interval change function

$g(x)_i = \dfrac{(1+x)^q}{(1-x)^p}$: Upper bounds of an interval change function

x : The tolerance error, $0 \le x < 1$, $x \in R$

When different parts of the structure are uncertain, it is a systematic uncertainty problem of structures. Except for members with certainty, the interval change functions determine geometric and material properties of uncertain members with respect to the structural system. The different combinations of members with and without uncertainties results in different systematic behaviors.

3.2 Interval finite element equation for uncertain structures

Most problems of structural analyses are solved by using conventional finite element methods, where the equilibrium equations governing the displacement fields in linear systems are expressed as follows

$$KU = F \tag{31}$$

where K is a stiffness matrix, U is a vector of displacements for each degree of freedom, and F is a vector of loading conditions. This system appears numerical certainties of structural response values. In order to calculate variations of the structural response when uncertain parameters are combined, it is convenient to substitute displacement fields of the uncertain linear system for the displacements of certain linear system through the axial rigidity of uncertainty, the displacement coefficient of uncertainty and the interval change function. For this purpose, the uncertain displacement is expressed as an interval displacement vector $\hat{U} = [U^c - \Delta U, U^c + \Delta U]$. Similarly, we express the uncertain stiffness matrix as an interval stiffness matrix $\hat{K} = [K^c - \Delta K, K^c + \Delta K]$, and an interval force vector $\hat{F} = [F^c - \Delta F, F^c + \Delta F]$. Here, U^c, K^c, and F^c represent the nominal displacement vector, the stiffness matrix and the force vector, respectively. ΔU, ΔK, and ΔF denote small changes of the displacement vector, the stiffness matrix and the force vector, respectively which result from small variations of the uncertain initial data. Following Eq. (31), the equilibrium equation of the uncertain linear system is as follows

$$[K^c - \Delta K, K^c + \Delta K][U^c - \Delta U, U^c + \Delta U] = [F^c - \Delta F, F^c + \Delta F] \tag{32}$$

The perturbation presents the behavior of a system subjected to small perturbations in design variables. For the linear system represented by Eq. (31), the problem is to generate U. K and F denote a perturbation of the form $K + e\Delta K$ and $F + e\Delta F$, respectively. The constant e is a small parameter. Then it is necessary to determine U when K becomes $K + e\Delta K$ and F becomes $F + e\Delta F$. Let e be small and K and F be smooth, then U is written as a convergent series

$$U = U_0 + eU_1 + e^2 U_2 + \dots \tag{33}$$

$$U_1 = -K^{-1}(\Delta K U_0 - \Delta F) \tag{34}$$

4. Numerical applications

4.1 2D Truss structure with structural uncertainties

The first example in order to demonstrate the proposed method is discussed here. The symmetrical plane truss structure subject to a pair of parting forces P is shown in Fig. 2. All members have the same axial rigidity EA. The truss is composed of six member elements, four

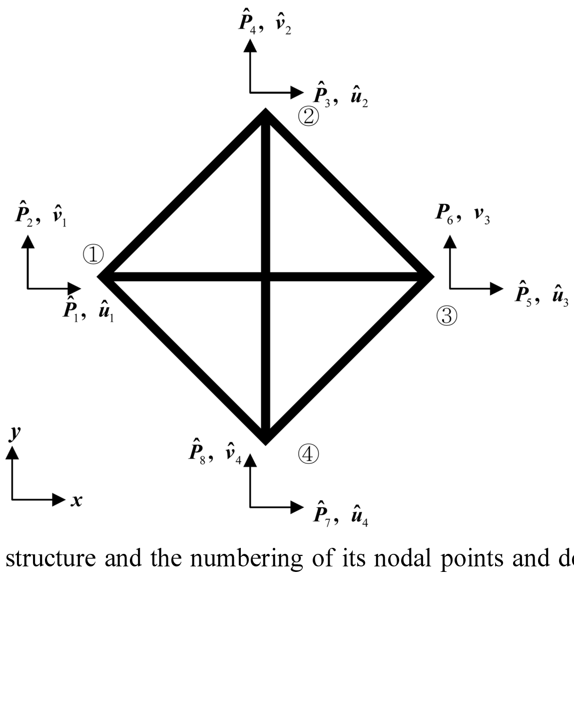


Fig. 2 Plane truss structure and the numbering of its nodal points and degrees of freedom

nodal points, and eight degrees of freedom. There is no connection between two diagonal members.

The global stiffness matrix and the applied force vector are expressed as

$$\hat{K} = \sum_{i=1}^{6} \frac{\hat{E}_i \hat{A}_i}{\hat{L}_i} \hat{K}_i \tag{35}$$

$$\hat{F} = (\hat{P})^T \tag{36}$$

where $\hat{P}$ denotes an applied external force and $\hat{E}_i$ is Young's modulus of the *i*-th member. $\hat{A}_i$ and $\hat{L}_i$ denote a cross-sectional area and a length of the *i*-th member, respectively. The element stiffness matrices $\hat{K}_i$ of the *i*-th member are described in Appendix A. Young's modulus for each member is given by the value of 2.0×10^7 KN/m^2, the cross-sectional area of all members has the same values of 3.6×10^{-1} m^2, and the length L of the 4 members ①-②, ①-③, ②-④, and ③-④ take values of 5 m. It is assumed that the length of a member does not change independently of the others since the geometry of the complete structure is fixed.

The nominal values of the uncertain initial data of structural parameters and applied load are regarded as the mid-point values of the interval formulation. In order to calculate the structural analysis with uncertainties, it is assumed that numerical uncertainties exist in each structural

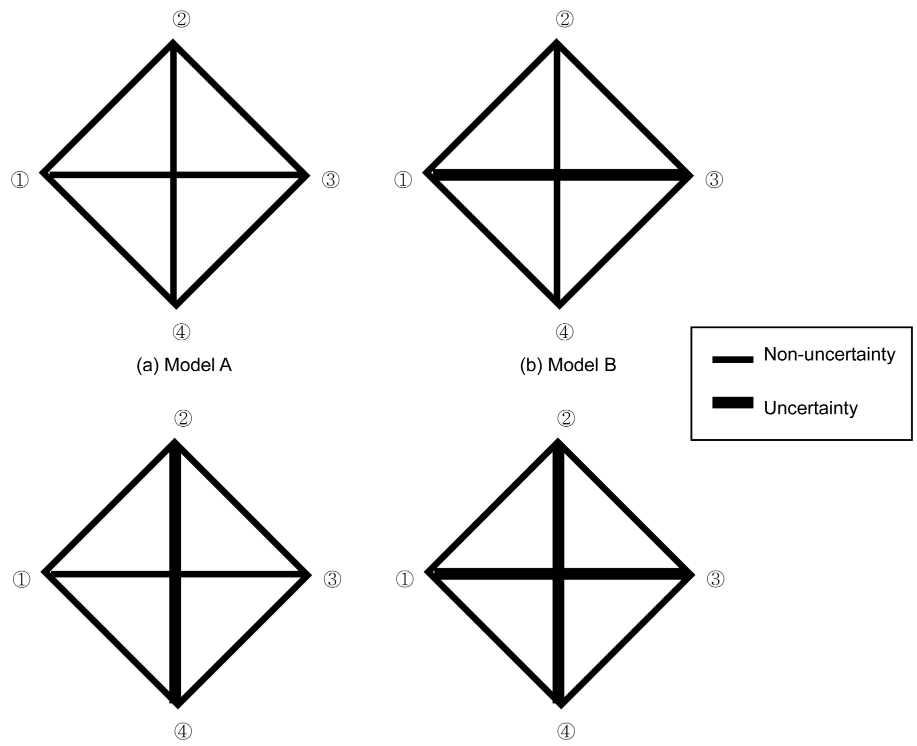


Fig. 3 Analysis models with member uncertainties of initial data: (a) Model *A* has members with nominal values of initial data. (b) In Model *B*, member ① - ③ has only uncertainty of initial data. (c) In Model *C*, member ② - ④ has only uncertainty of initial data. (d) Model *D* is the combination of Model *B* and *C*

parameter and applied load. The uncertainty is given by a tolerance error x and here are 1/100 and 1/1000, i.e., 1% and 0.1% of exact values.

According to members with the numerical uncertainty of initial data, analysis models of A, B, C and D are respectively shown in Fig. 3. The bold and thin lines denote members with uncertainties and non-uncertainties of initial data, respectively.

Applying boundary conditions and symmetry of the structure, i.e., $v_1 = u_2 = v_3 = u_4 = 0$, the 8×8 global stiffness matrix of structural systems is reduced. Therefore, setting nominal values of all structural parameters and loading conditions in Eqs. (35) and (36), the nominal global stiffness matrix and the force vector without uncertainties are obtained as follows:

$$K^c = \frac{E^c A^c}{L^c}\begin{pmatrix} \frac{2+\sqrt{2}}{2} & -\frac{1}{2} & -\frac{\sqrt{2}}{2} & \frac{1}{2} \\ -\frac{1}{2} & \frac{2+\sqrt{2}}{2} & \frac{1}{2} & -\frac{\sqrt{2}}{2} \\ -\frac{\sqrt{2}}{2} & \frac{1}{2} & \frac{2+\sqrt{2}}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{\sqrt{2}}{2} & -\frac{1}{2} & \frac{2+\sqrt{2}}{2} \end{pmatrix} \tag{37}$$

$$F^c = (-P^c, 0, P^c, 0) \tag{38}$$

Substituting Eqs. (37) and (38) into Eq. (31) and solving the resulting equation, nominal nodal displacements by namely Model A using the conventional finite element method are written as follows

$$U^c = \frac{P^c L^c}{E^c A^c}\left(-\frac{1}{2}, \frac{1-\sqrt{2}}{2}, \frac{1}{2}, \frac{\sqrt{2}-1}{2}\right) \tag{39}$$

For the purpose of the structural analysis of Model B, the local element stiffness matrix of member ①-③ with the uncertainties of initial data is expressed by considering an interval change function $\Im_i$ as

$$\hat{K}^B_{1-3} = \frac{E^c A^c}{\sqrt{2}L^c}\left[\frac{(1-x)^2}{(1+x)}, \frac{(1+x)^2}{(1-x)}\right]\begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \tag{40}$$

Similarly, in the analysis of Model C, the local stiffness matrix of member ②-④ with numerical uncertainties of initial data, is also expressed by considering an interval change function $\Im_i$ as follows

$$\hat{K}^C_{2-4} = \frac{E^c A^c}{\sqrt{2}L^c}\left[\frac{(1-x)^2}{(1+x)}, \frac{(1+x)^2}{(1-x)}\right]\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \tag{41}$$

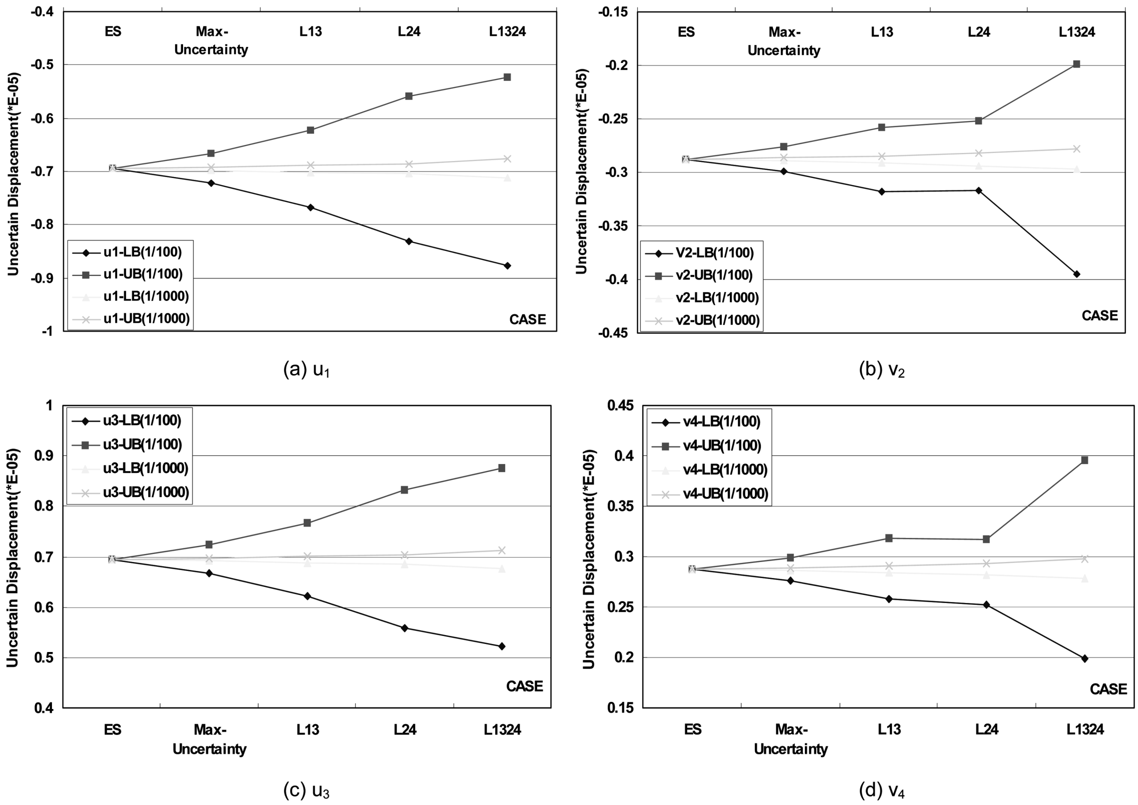


Fig. 4 Comparisons of nodal displacements in analysis models (*A*, *B*, *C*, *D*)

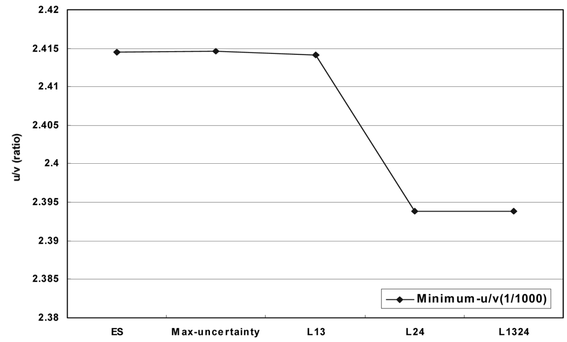
In 2-dimensional-truss systems, the graphs of generalized interval change functions are given in Appendix B.

In case of Model *D*, the local element stiffness matrices of Model *B* and Model *C* are combined and it produces a local element stiffness matrix of Model *D* with the uncertainties of member ①-③ and ②-④.

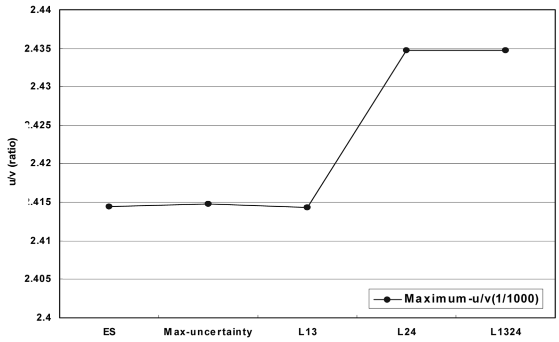
For each analysis model, four solutions are represented: *ES*, i.e., the exact solution, *L13*, i.e., the solution with uncertainty of member ①-③, *L24*, i.e., the solution by uncertainty of member ②-④, and *L1324*, i.e., solution with uncertainty of members ①-③ and ②-④. The uncertain nodal displacements with the upper and lower bounds of the Model *A*, *B*, *C*, and *D* are shown in Fig. 4. Here, *Max-uncertainty* presents behaviors of Model *A*. When all *E*, *A*, *P*, and *L* take uncertainties in Eq. (39), maximums of lower and upper bounds of nodal displacements are directly produced by Eq. (39) and they denote *Max-uncertainty.* The tolerance error x of uncertainties are set to 1/100 and 1/1000. *LB* and *UB* are respectively lower and upper bounds.

In Fig. 5, uncertain nodal displacement values of the defined analysis models are expressed as some ratio between horizontal and vertical nodal displacements in analysis model of *A*, *B*, *C*, *D* and different tolerance errors.

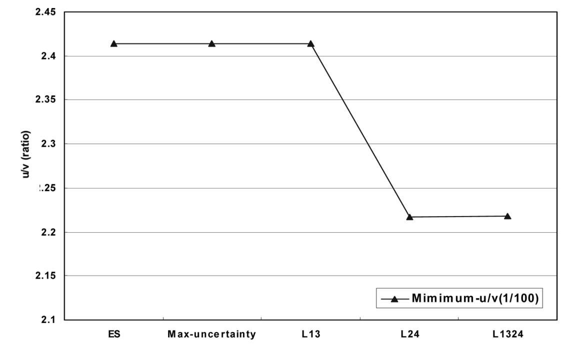
In tolerance error x = 1/1000, the errors of uncertain nodal displacements of each analysis model in comparisons with those of *ES*, are illustrated in Fig. 6. Here, *G-uncert 1~4* describe uncertainties using final formulation of *ES* solution explicitly and the number of 1~4 denotes the number of



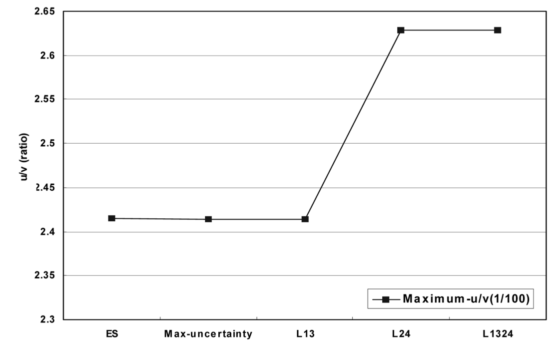
(a) Minimum - u/v (x=1/1000)



(b) Maximum - u/v (x=1/1000)



(c) Minimum - u/v (x=1/100)



(d) Maximum - u/v (x=1/100)

Fig. 5 Some ratio of horizontal displacements in compared with vertical displacements in different analysis models (*A*, *B*, *C*, *D*) and tolerance errors (0.1% and 1.0%)

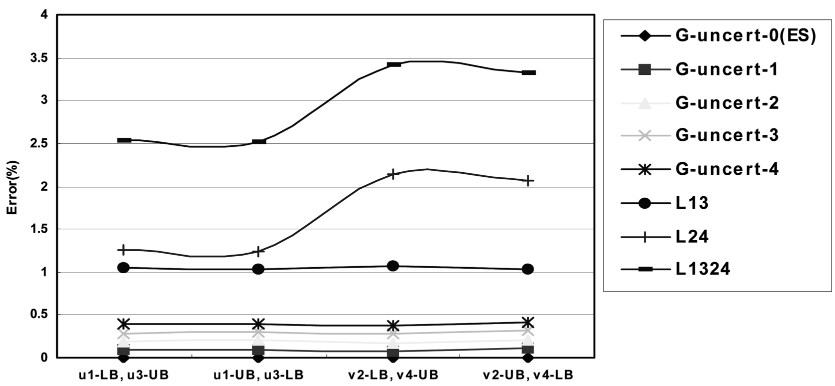


Fig. 6 Uncertainty solution errors of each model in comparisons with exact solutions (ES)

uncertainties of initial data of Young's modulus *E*, cross sectional areas *A*, lengths of members *L* and load *P*.

As can be seen, the errors of uncertainty structural responses of explicit analysis models of *G-uncert-1~4* are less than those of implicit analysis models of *L13*, *L24*, and *L1324*. In other words, the error degree of the explicit uncertainty analysis model takes under 0.5% in comparisons with ES solution, however, implicit analysis models such as Model *B*, *C*, and *D* have the errors more than

Table 1 Horizontal displacements, maximum uncertainty and some ratio (tolerance error = 1/1000)

	Joint		U^C (E-05)	ΔU (E-05)	$\Delta U/\Delta E$ (E-11)	$\Delta U/\Delta A$ (E-05)	$\Delta U/\Delta L$ (E-05)	$\Delta U/\Delta P$ (E-05)
	I	*J*						
A	·	·	0.6944	0.0027	0.135	7.500	0.540	0.135
B	1	3	0.6944	0.0073	0.366	20.361	1.466	0.367
C	2	4	0.6944	0.0087	0.438	24.361	1.754	0.439
D	1(2)	3(4)	0.6944	0.0176	0.882	49.028	3.530	0.883

Table 2 Vertical displacements, maximum uncertainty and some ratio (tolerance error = 1/1000)

	Joint		U^C (E-05)	ΔU (E-05)	$\Delta U/\Delta E$ (E-11)	$\Delta U/\Delta A$ (E-05)	$\Delta U/\Delta L$ (E-05)	$\Delta U/\Delta P$ (E-05)
	I	*J*						
A	·	·	0.2876	0.0011	0.055	3.0555	0.220	0.055
B	1	3	0.2876	0.0031	0.153	8.5277	0.614	0.154
C	2	4	0.2876	0.0061	0.307	17.056	1.228	0.307
D	1(2)	3(4)	0.2876	0.0099	0.492	27.361	1.970	0.493

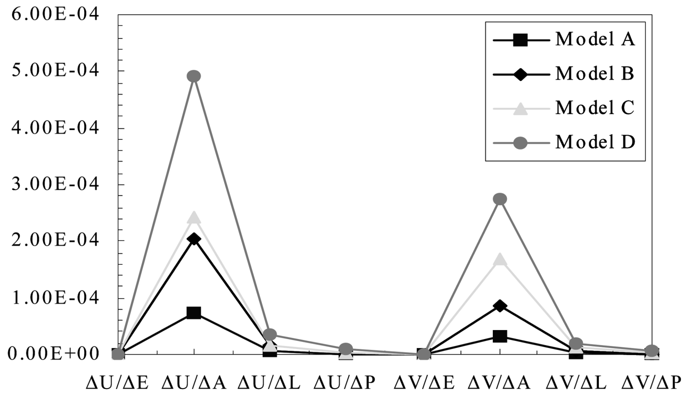


Fig. 7 Sensitivities and uncertainty relationships between initial data and structural behaviors for analysis model *A*, *B*, *C*, and *D*: tolerance errors = 1% for the left hand graph, tolerance error = 0.1 % for the right one

1% and less than about 2.5% in case of vertical displacements. When the defined tolerance error is more than 0.1%, the uncertainty effects of structural responses would be also increased. The tolerance errors have to be selected under some quantitative criteria such as statistic or experimental data of experience. From Figs. 4, 5, and 6, it can be found that the uncertainty of structural responses depends on the numerical uncertainty of initial data and combination characteristics of each member with initial data of uncertainty or no uncertainty in the structural system.

Tables 1, 2, and Fig. 7 illustrate the relationship between uncertainties of initial data and structural responses.

It can be seen that in truss systems the 2nd-order parameters, i.e., cross-sectional areas *A*, have more influence on the uncertainty of structural responses than 1st-order parameters, i.e., Young's modulus E, length of members L and applied forces P. The sensitivities result in.

4.2 2D Frame structure with structural uncertainties

Frame structures are mechanical systems more general than the truss of the first example. It is composed of elastic elongated beams jointed at nodes using both stiff joints which are not allowed in truss structures and also rotary joints like truss structures. In frames a possibility of bending of beams has to be taken into account, since the beams can carry also bending moments. The nodes of the frame can be supported by full supports which give no degrees of freedom to the supported node, sliding supports which allow the node to move along a specified line or within a specified plane. The full support denotes a fixed one. The sliding one includes a hinged or a free support.

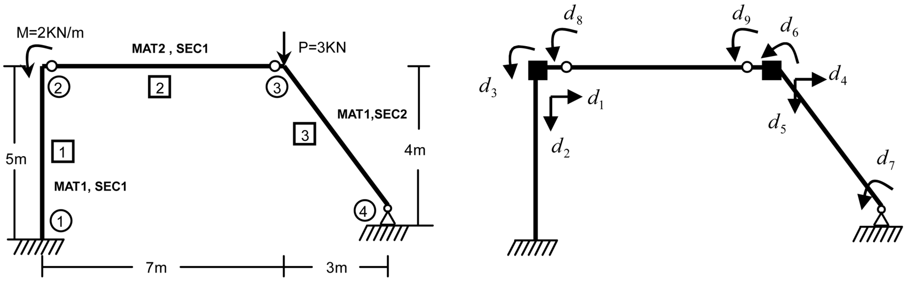
The second example of the frame structure is shown in Fig. 8 and the frame has a fixed support at the joint 1 and a hinged one at the joint 4, and two external loads, i.e., a bending moment M at the joint 2 and a vertical concentrated load P at the joint 3. The frame has DOF (degree of freedom) = 9 as shown in Fig. 8(b). Initial loading conditions are that M = 2 KN·m and P = 3 KN. The Initial data consist of two factors: Material parameter such as Young's modulus E and a geometrical parameter such as cross sectional areas A, length of members L, and moments of inertia I. The Young's modulus in this example is taken as $E_1 = 3.4 \times 10^7$ KN/m^2 (MAT1) and $E_2 = 3.4 \times 10^5$ KN/m^2 (MAT2). The cross sectional area and moment of inertia take A_1 = 0.24 m^2 and I_1 = 0.0072 m^4 (SEC1) and A_2 = 0.09 mand I_2 = 0.005 m^4 (SEC2), respectively. It is assumed that the tolerance error x = 0.01 (1%) is equally applied to all initial data.

The number of cases of numerical uncertainties of initial data is 32 and is shown in Table 3. Here, the number 1 denotes uncertainties and the number 0 no uncertainties. The number of the combination depends on the number of initial data.

Fig. 9 shows lower and upper bounds of structural responses and those arithmetic means with regard to each case, in compared with solutions without uncertainties.

From Fig. 9, it can be seen that the variational structural response results do not depend on the number of initial data with uncertainties but types of the uncertainty combinations of initial data. Here, the combination of Case 29 takes the most uncertainty variation with respect to structural responses. The fact can be also investigated by displacement coefficient of uncertainty χ in terms of the bending rigidity γ.

Contrary to response results of truss structures, the solutions of frame become described by both



(a) Analytical model of frame structure and initial data

(b) DOF in the defined frame: total DOF=9

Fig. 8 Frame structure

Table 3 Cases of uncertainty combinations of initial data: 1 = uncertainty, 0 = no uncertainty: MAT1 (E_1) MAT2 (E_2) SEC1 (I_1, A_1) SEC2 (I_2, A_2,) L_1 = 5 m L_2 = 7 m P_1 = 2 kN·m P_2 = 3 kN Error = 0.01 (1%)

Case	E_1, E_2	I_1, I_2	A_1, A_2	L_1, L_2	P_1, P_2	Case	E_1, E_2	I_1, I_2	A_1, A_2	L_1, L_2	P_1, P_2
1	1	1	1	1	1	17	0	1	1	1	1
2	1	1	1	1	0	18	0	1	1	1	0
3	1	1	1	0	1	19	0	1	1	0	1
4	1	1	1	0	0	20	0	1	1	0	0
5	1	1	0	1	1	21	0	1	0	1	1
6	1	1	0	1	0	22	0	1	0	1	0
7	1	1	0	0	1	23	0	1	0	0	1
8	1	1	0	0	0	24	0	1	0	0	0
9	1	0	1	1	1	25	0	0	1	1	1
10	1	0	1	1	0	26	0	0	1	1	0
11	1	0	1	0	1	27	0	0	1	0	1
12	1	0	1	0	0	28	0	0	1	0	0
13	1	0	0	1	1	29	0	0	0	1	1
14	1	0	0	1	0	30	0	0	0	1	0
15	1	0	0	0	1	31	0	0	0	0	1
16	1	0	0	0	0	32	0	0	0	0	0

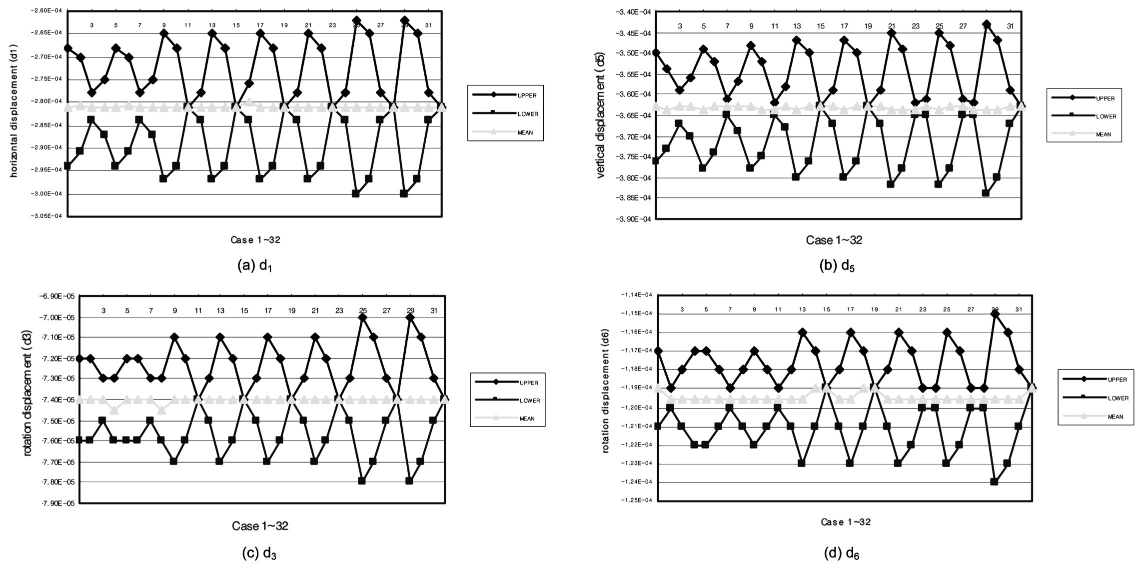


Fig. 9 Nodal displacements by numerical uncertainty of initial data: (a) d_1 = horizontal displacement (b) d_5 = vertical displacement (c) and (d) = rotation displacement

bending rigidity and axial rigidity. Therefore, it seems that mean values of lower and upper bounds of solutions in each case are not likely to be equal to approximated solutions of typical finite element analysis of Case 32 with all non-uncertainty as shown in Fig. 9. In order to estimate relationship between initial data and uncertainty solutions, the numerical sensitivities of structural uncertainty responses in terms of initial data can be investigated and are shown in Table 4.

Table 4 Sensitivity of (a) axial displacements $U(d_1, d_5)$ and (b) rotation displacements $R(d_3, d_6)$ in terms of initial data of material and geometrical data: MAT1 (E_1), MAT2 (E_2), SEC1 (I_1, A_1), SEC2 (I_2, A_2), Case 1 with error = 1/100 (1%)

(a) Sensitivity of axial displacements (m) by initial data

ΔU (E-05)	$\Delta U/\Delta E_1$ (E-11)	$\Delta U/\Delta E_2$ (E-09)	$\Delta U/\Delta I_1$ (E-00)	$\Delta U/\Delta I_2$ (E-00)	$\Delta U/\Delta A_1$ (E-03)	$\Delta U/\Delta A_2$ (E-03)	$\Delta U/\Delta L$ (E-04)	$\Delta U/\Delta P$ (E-04)
1.3	3.824	3.0	0.181	0.26	5.416	14.44	2.6 /1.857	4.333 /6.5

ΔR (E-05)	$\Delta R/\Delta E_1$ (E-11)	$\Delta R/\Delta E_2$ (E-09)	$\Delta R/\Delta I_1$ (E-00)	$\Delta R/\Delta I_2$ (E-00)	$\Delta R/\Delta A_1$ (E-03)	$\Delta R/\Delta A_2$ (E-03)	$\Delta R/\Delta L$ (E-05)	$\Delta R/\Delta P$ (E-04)
0.2	0.588	0.588	0.028	0.04	0.833	2.222	0.4 /0.286	0.666 /1.0

From Table 4, the results of sensitivities by initial data are $dU/dI > dU/dA > dU/dL > dU/dP > dU/dE$. In the frame structure, it can be seen that the variation of moment of inertia I changed by the numerical uncertainty is the most influent factor in comparisons with truss structure of the first example in which the variation of cross sectional areas A is the most important factor in the structural system with uncertainty.

5. Conclusion

In this study an interval-based finite element analysis is proposed in order to estimate the quantitative uncertainty of structural responses in linear systems. Contrary to typical interval finite element methods, the present approach allows the uncertainty with interval forms (with tolerances) of initial data to be substituted into both load and element stiffness terms. Therefore this approach is identified with the numerical process of typical finite element analysis and it provides numerical efficiency when applied to uncertainty. The problems of combining structural members with uncertainties in the initial data can also be investigated for reliable uncertainty analyses.

For systems possessing larger uncertainties, an interval finite element analysis yields increasingly large errors. Practically, in many cases the structure parameter errors or uncertainties are small. However their integration will yield a larger error. Therefore in this study it is proposed that 2-order parameter A errors should be minimized during the truss structure design process, and also higher order parameter I errors in design process of frame structures. This method allows the engineering practice to account for uncertainty in load and stiffness and to calculate very sharp bounds on the system response for all possible scenarios of uncertainty. Numerical applications of a truss and frame demonstrate the numerical efficiency of the proposed method for appropriate uncertainty estimates of structural behaviors.

This study is limited to the truss and beam structures of two-dimensional systems. These structures are related to axial and bending rigidity, respectively, which are derived in this study. The present method can not be used to deal with plate and shell structures since they take additional degrees of freedom in a three-dimensional system. Future works includes studying axial and bending rigidity problems, the problems of uncertainty structural responses by shear or torsion rigidity using the proposed interval finite element method.

References

Alefeld, G. (1983), *Introductions to Interval Computations*, New York, Academic Press.

Alefeld, G. and Claudio, D. (1998), "The basic properties of interval arithmetic, its software realizations and some applications", *Comput. Struct.*, **67**, 3-8.

Alefeld, G. and Herzberger, J. (1983), *Introduction to Interval Computations*, Academic Press, New York.

Alefeld, G. and Mayer, G. (2000), "Interval analysis: Theory and applications", *J. Comput. Appl. Math.*, **121**, 421-464.

Chen, S.H. and Qiu, Z.P. (1994), "A new method for computing the upper and lower bounds on frequencies of structures with interval parameters", *Mech. Res. Commun.*, **2**, 583-592.

Chen, S.H. and Qiu, Z.P. (1994), "Perturbation method for computing eigenvalue bounds in vibration system with interval parameters", *Commun. Numer. Math. Eng.*, **10**, 121-134.

Chen, S.H. and Yang, X.W. (2000), "Interval finite element method for beam structures", *Finite Elem. Anal. Des.*, **34**, 75-88.

Dwyer, P.S. (1951), *Linear Computation*, Wiley.

Geschwindner, L.F. and Disque, R.O. (1994), *Load and Resistance Factor Design of Steel Structures*, Prentice-hall.

Jansson, C. (2001), "Quasic-convex relation based on interval arithmetic", *Linear Algebra and Its Applications*, **342**, 27-53.

Karni, R. and Belikoff, S. (1996), "Concurrent engineering design using interval methods", *Int. Trans. Opl. Res.*, **3**, 77-87.

Kim, U.H. and Yoo, Y.H. (1999), *Matrix Structural Analysis*, Korea, Kong-Sung Press.

Klir, G.J. and Folger, T.A. (1998), *Fuzzy set. Uncertainty and Information*, Prentice-Hall International.

Kulisch, U. (1969), *Grundzuge der Intervallrechnung, Jahrbuch Ueberblicke Mathematik*, Vol. 2, Bibliographisches Institut, Mannheim.

Kulpa, Z., Pownuk, A. and Skalna, I. (1998), "Analysis linear mechanical structures with uncertainties by means of interval methods", *Comput. Assisted Mech. Eng. Sci.*, **5**.

Köyluoglu, U. and Elishakoff, I. (1998), "A comparison of stochastic and interval finite elements applied to shear frames with uncertain stiffness properties", *Comput. Struct.*, **67**(1-3), 91-98.

Köyluoglu, U., Cakmak, A.S. and Nielsen, S.R.K. (1995), "Interval algebra to deal with pattern loading and structural uncertainty", *J. Eng. Mech.*, 1149-1157.

McWilliam, S. (2001), "Anti-optimization of uncertain structures using interval analysis", *Comput. Struct.*, **79**, 421-430.

Moore, R.E. (1962), *Interval Arithmetic and Automatic Error Analysis in Digital Computing*, PhD Thesis, Stanford University.

Moore, R.E. (1966), *Interval Analysis*, Prentice-Hall, Englewood Cliffs, NJ.

Moore, R.E. (1979), "Methods and applications of interval analysis", *SIAM Studies in Applied Mathematics*, SIAM, Philadelphia, PA.

Noh, H.I. (1998), *Reinforce Concrete Structure*, Sanup-Dose, 17-18.

Qiu, Z. and Elishakoff, I. (1998), "Anti-optimization of structures with large uncertain-but-non-random parameters via interval analysis", *Comput. Meth. Appl. Mech.*, **152**, 361-372.

Qu, Y. (1999), "On using alpha-cuts to evaluate fuzzy equations", *Fuzzy Set Syst.*, **38**, 309-312.

Rao, S.S. and Berke, L. (1997), "Analysis of uncertain structural systems using interval analysis", *AIAA J.*, **35**(4), 727-735.

Sen, M. and Powers, J. (2001), *LECTURE NOTES ON 24. MATHEMATICAL METHODS*, 86-109.

Skrzypczyk, J. (1997), "Fuzzy finite element methods-A new methodology", *Comput. Meth. Mech.*, **4**, 1187-1194.

Sunaga, T. (1958), "Theory of an interval algebra and its application to numerical analysis", *RAAG Memoirs*, **2**, 29-46.

Wiener, N. (1914), "A contribution to the theory of relative position", *Proc. Cambridge Philos. Soc.*, **17**, 441-449.

Wiener, N. (1921), "A new theory of measurement: a study in the logic of mathematics", *Proceedings of the London Mathematical Society.*

Yang, T.Y. (1986), *Finite Element Structural Analysis*, Prentice-Hall, Inc. Englewood Cliffs, N.J.

Yang, X., Chen, S. and Lian, H. (2001), "Bounds of complex eigenvalues of structures with interval parameters", *Eng. Struct.*, **23**, 557-563.

Appendix A

The matrices appearing in Eq. (35) are as follows:

(1) Element ①-②:

$$\hat{K}_1' = \begin{pmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{pmatrix}$$

(2) Element ②-③ and ①-④:

$$\hat{K}_2' = \hat{K}_3' = \begin{pmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ -1/2 & 1/2 & 1/2 & -1/2 \\ 1/2 & -1/2 & -1/2 & 1/2 \end{pmatrix}$$

(3) Element ③-④:

$$\hat{K}_4' = \begin{pmatrix} 1/2 & 1/2 & -1/2 & -1/2 \\ 1/2 & 1/2 & -1/2 & -1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \\ -1/2 & -1/2 & 1/2 & 1/2 \end{pmatrix}$$

(4) Element ①-③:

$$\hat{K}_5' = \begin{pmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

(5) Element ②-④:

$$\hat{K}_6' = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 0 & 0 & 0 \\ 0 & -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{pmatrix}$$

Appendix B

The interval change functions of Model *B*, *C*, and *D* are shown as follows:

(1) Model *B*:

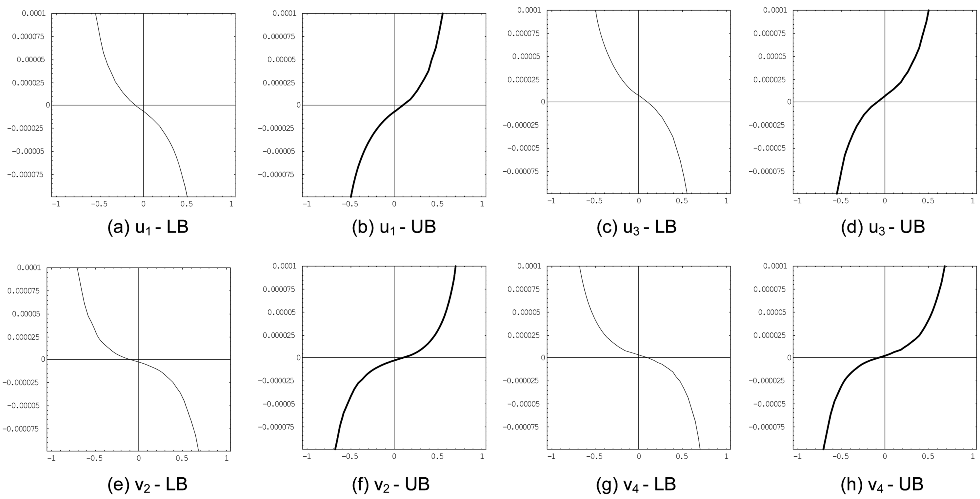
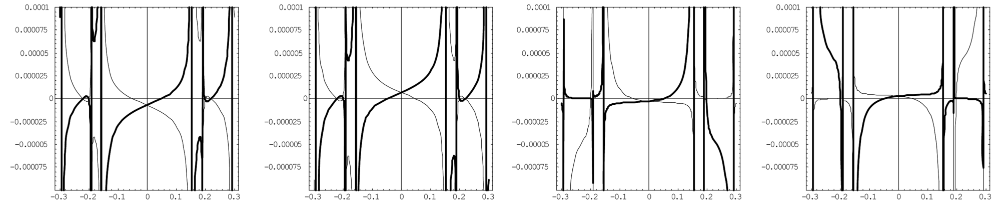


Fig. 10 Model B: interval change function with lower and upper bounds of uncertain nodal displacement in tolerance error = 1/100

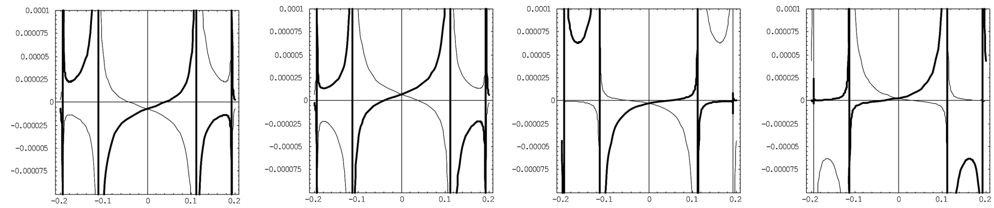
(2) Model *C*:



(a) u_1 - LB, UB (b) u_3 - LB, UB (c) v_2 - LB, UB (d) v_4 - LB,

Fig. 11 Model C: interval change function with lower and upper bounds of uncertain nodal displacement in tolerance error = 1/100

(3) Model *D*:



(a) u_1 - LB, UB (b) u_3 - LB, UB (c) v_2 - LB, UB (d) v_4 - LB, UB

Fig. 12 Model *D*: interval change function with lower and upper bounds of uncertain nodal displacement in tolerance error = 1/100