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# Geometry-dependent MITC method for a 2-node iso-beam element

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**Abstract.** In this paper, we present an idea of the geometry-dependent MITC method. The simple concept is exemplified to improve a 2-node iso-beam (isoparametric beam) finite element of varying section. We first study the behavior of a standard 2-node iso-beam finite element of prismatic section, which has been widely used with reduced integration (or the equivalent MITC method) in order to avoid shear locking. Based on analytical studies on cantilever beams of varying section, we propose the axial strain correction (ASC) scheme and the geometry-dependent tying (GDT) scheme for the 2-node iso-beam element. We numerically analyze varying section beam problems and present the improved performance by using both ASC and GDT schemes.

**Keywords:** finite elements; iso-beam; MITC method; tying point.

## 1. Introduction

For several decades, finite element method has been dominantly used to analyze various structural engineering problems. Despite of the long history and the success in engineering fields, there have been unsolved issues in finite element method and continuous challenges are still desirable.

Structural finite elements (shells, plates, and beams) have been derived from basic continuum mechanics and standard isoparametric procedures (Bathe 1996). However, the displacement-based finite elements are too stiff in bending-dominated situations when the thickness is small, regardless of the displacement interpolation order. The phenomenon is called as locking and numerous

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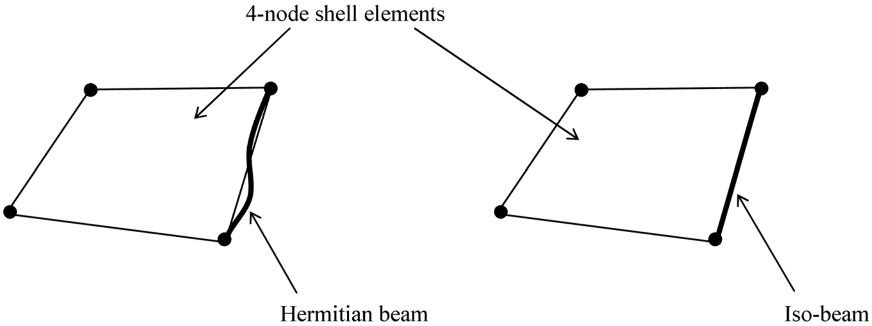


Fig. 1 Coupled use of 2-node beam elements with a 4-node shell element

researches have been performed to overcome the locking in the development of the structural finite elements (Lee and Bathe 2002).

Among various techniques for locking alleviation, the MITC (Mixed Interpolation of Tensorial Components) method or similar assumed strain schemes have been successfully used (Dvorkin and Bathe 1984, Bucalem and Bathe 1993, Choi and Paik 1994, Bathe 1996, Choi *et al.* 1999, Hong *et al.* 2004). Its performance has been presented for various benchmark problems using well-established benchmark procedures. The basic idea of the MITC method is to interpolate displacements and strains separately and "connect" these interpolations at "tying points." The displacement and strain interpolations are chosen so as to satisfy the ellipticity and consistency conditions, and as closely as possible the inf-sup condition (Lee and Bathe 2004).

The key of the MITC method is how to design the tying scheme determined by "strain interpolation functions" and "tying positions", which depend on element types (shell, plate, beam…) and the number of nodes (4-nodes, 9-nodes…). Considering a structural finite element, the tying points of the MITC method are fixed at certain positions in the natural coordinate system irrespective of the element geometry. However, the behavior of the finite elements depends on the positions of the tying points and the dependency has not been well studied.

For finite element analyses of beams, beam elements directly derived from various beam theories have been frequently used and, specially, a 2-node Hermitian beam gives exact solutions with the use of the single element (Bathe and Bolourchi 1979, Bathe 1996). However, the 2-node beam element has a quadratic interpolation along the longitudinal beam direction and this induces incompatibility when the element is coupled with shell elements or 2D- or 3D-solid elements, see Fig. 1.

Iso-beam elements are degenerated form 3D-solid elements. The single element cannot give exact solutions but, as the number of elements used increases, the solutions quickly converge into the exact solutions. The primary applications of the iso-beam elements are for coupled uses with shell elements or 2D- or 3D-solid elements (Bathe 1996). In the cases, the element does not result in the incompatibility because the interpolations of the elements well match with shell elements or 2D- or 3D-solid elements. For example, the 2-, 3- or 4-node iso-beam elements together with the 4-, 9-, 16-node shell elements, respectively, can provide an effective finite element discretization of the stiffened shell structures.

In the following sections, exemplifying a 2-node iso-beam (isoparametric beam) finite element of prismatic section, we first study the role of tying position. The axial strain correction scheme is introduced to improve the axial behavior of varying section beam elements. To find the optimal tying position of the 2-node iso-beam element of varying section, a cantilever beam problem under pure bending is investigated. We then improve the 2-node iso-beam element using a geometry-dependent tying scheme used with the axial strain correction scheme. Finally, various numerical tests are performed and the results are discussed in detail.

In this paper, we restrict our study within the framework of linear elasticity and isotropic material. However, the methods proposed can be generally used without the limitation.

## 2. A 2-node iso-beam finite element of prismatic section

Iso-beam finite elements are very attractive because the formulation is directly derived from three-dimensional continuum mechanic and easily extended for nonlinear analysis as well as simple and general. Locking can be simply removed using reduced integration or the equivalent MITC method[1] (Bathe 1996, Lee and McClure 2006).

The basic kinematic assumption of the beam formulation is that plane cross sections originally normal to the central axis of the beam remain plane and undistorted under deformation but not necessarily perpendicular to the central axis of the deformed beam (Bathe 1996, Lee and McClure 2006).

In this section, we briefly review the formulation of a 2-node iso-beam finite element of prismatic section and the MITC method for beam elements. We study the pure bending behavior of the beam element formulated with the MITC method and investigate the role of tying position.

### 2.1 Interpolation of geometry and displacement fields

The geometry of the $q$-node beam finite element is interpolated by

$$\vec{x}(r,s,t) = \sum_{k=1}^{q} h_k(r)\vec{x}_k + \sum_{k=1}^{q} \frac{t}{2} a_k h_k(r) \vec{V}_t^k + \sum_{k=1}^{q} \frac{s}{2} b_k h_k(r) \vec{V}_s^k \quad (1)$$

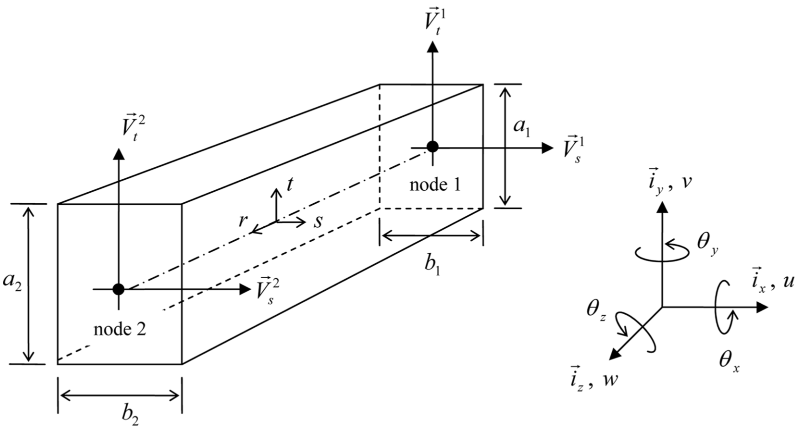


Fig. 2 A 2-node iso-beam finite element

[1]It is important to understand the close relation between reduced integration and the MITC method. The MITC method is a superset of reduced integration, that is, the finite element formulations using reduced integration can be expressed by the MITC method in general. However, the inverse relation does not hold.

where $h_k(r)$ are the interpolation polynomials (shape functions) in usual isoparametric procedures, $\vec{x}_k$ are the Cartesian coordinates of node $k$, $a_k$ and $b_k$ are the cross-sectional dimensions at node $k$, and the unit vectors $\vec{V}_t^k$ and $\vec{V}_s^k$ are the director vectors in directions $t$ and $s$ at node $k$, see Fig. 2. Note that $\vec{V}_t^k$ and $\vec{V}_s^k$ are normal to each other (Bathe 1996).

From Eq. (1), the displacement of the element is given by

$$\vec{u}(r,s,t) = \sum_{k=1}^{q} h_k(r)\vec{u}_k + \sum_{k=1}^{q}\frac{t}{2}a_k h_k(r)\{\vec{\theta}_k \times \vec{V}_t^k\} + \sum_{k=1}^{q}\frac{s}{2}b_k h_k(r)\{\vec{\theta}_k \times \vec{V}_s^k\} \tag{2}$$

in which $\vec{u}_k$ is the nodal displacement vector of node $k$ in the global Cartesian coordinate system and the rotation vector at node $k$ is

$$\vec{\theta}_k = \begin{bmatrix} \theta_x^k \\ \theta_y^k \\ \theta_z^k \end{bmatrix} \tag{3}$$

For a 2-node beam finite element, $q$ is 2 and the shape functions are

$$h_1 = \frac{1}{2}(1-r), \quad h_2 = \frac{1}{2}(1+r) \tag{4}$$

The linear part of the covariant strain components are directly calculated by

$$e_{ij} = \frac{1}{2}(\vec{g}_i \cdot \vec{u}_{,j} + \vec{g}_j \cdot \vec{u}_{,i}) \tag{5}$$

where

$$\vec{g}_i = \frac{\partial \vec{x}}{\partial r_i}, \quad \vec{u}_{,i} = \frac{\partial \vec{u}}{\partial r_i} \quad \text{with} \quad r_1 = r, r_2 = s, r_3 = t \tag{6}$$

The three covariant strain components are considered in beam finite elements

$$e_{rr} = \vec{g}_r \cdot \vec{u}_{,r}, \quad e_{rs} = \frac{1}{2}(\vec{g}_r \cdot \vec{u}_{,s} + \vec{g}_s \cdot \vec{u}_{,r}), \quad e_{tr} = \frac{1}{2}(\vec{g}_t \cdot \vec{u}_{,r} + \vec{g}_r \cdot \vec{u}_{,t}) \tag{7}$$

### *2.2 The MITC method*

Using the MITC method for beam finite elements, the transverse shear and normal covariant strain components are interpolated

$$\bar{e}_{ij}(r,s,t) = \sum_{k=1}^{n_{ij}} \bar{h}_k(r) e_{ij}(r^k, s, t) \tag{8}$$

where $\bar{e}_{ij}(r,s,t)$ are assumed covariant strain components in the beam finite element, $e_{ij}(r^k,s,t)$ are the covariant strain components of Eq. (5) calculated from the displacement-based beam finite element at tying point $(r^k,s,t)$ and $\bar{h}_k(r)$ are the assumed interpolation functions satisfying

$$\bar{h}_k(r^l) = \delta_k^l \ (\delta_i^j = 1 \text{ if } i = j \text{ and 0 otherwise}), \ k, l = 1, \ldots, n_{ij} \tag{9}$$

In Eq. (9), $n_{ij}$ is the number of tying points for the strain component $\bar{e}_{ij}$.

The assumed strain components and the tying points depend on the displacement interpolation functions used (or the number of element nodes). As an example, for the 2-node iso-beam finite element, we use one tying point at $r = 0$ for the covariant strains $e_{rs}$ and $e_{tr}$ but two tying points at $r = \pm 1/\sqrt{3}$ for the covariant strains $e_{rs}$, $e_{tr}$ and $e_{rr}$ need to be used for the 3-node element in general.

### *2.3 Pure bending behavior*

In this section, we study the behavior of the 2-node iso-beam finite element of prismatic section under pure bending. Considering a cantilever beam structure of length $L$ in Fig. 3, a moment $M_z$ is applied at free tip. The boundary condition of this beam problem at the clamped tip is

$$u = v = w = \theta_x = \theta_y = \theta_z = 0 \tag{10}$$

The cantilever is modeled by one finite element and, from the geometry, we have

$$x_1 = 0, y_1 = 0, z_1 = 0 \quad \text{and} \quad x_2 = L, y_2 = 0, z_2 = 0 \tag{11}$$

in which the subscripts are the node numbers.

This is a pure bending problem and the exact analytical solution corresponds to

$$e_{rs} = e_{sr} = 0 \quad \text{and} \quad \varepsilon_{xy} = \varepsilon_{zx} = 0 \tag{12}$$

in the whole domain.

When the iso-beam finite element is used for the cantilever problem as shown in Fig. 3, we have the conditions

$$\vec{x}_k = \begin{Bmatrix} x_k \\ 0 \\ 0 \end{Bmatrix}, \quad \vec{u}_k = \begin{Bmatrix} 0 \\ v_k \\ 0 \end{Bmatrix}, \quad \vec{\theta}_k = \begin{bmatrix} 0 \\ 0 \\ \theta_z^k \end{bmatrix}, \ \vec{V}_s^k = \vec{i}_y, \ \vec{V}_t^k = \vec{i}_z \ \text{and} \ a_k = a, b_k = b \ \text{for all} \ k \tag{13}$$

where $\vec{i}_y$ and $\vec{i}_z$ are the unit base vectors in the global Cartesian coordinate system.

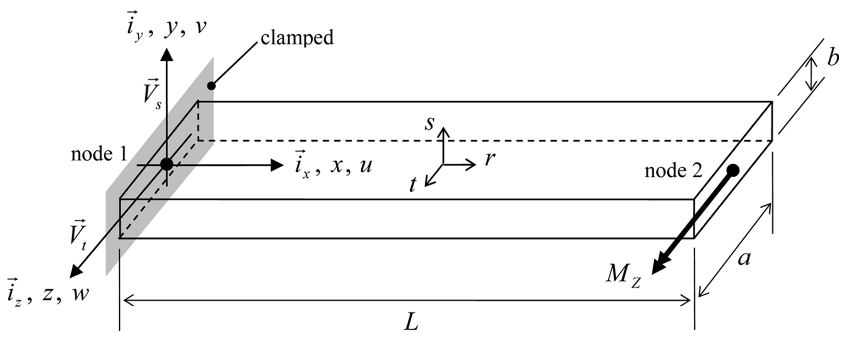


Fig. 3 A 2-node iso-beam finite element of prismatic section ($a_1 = a_2 = a$ and $b_1 = b_2 = b$)

From Eqs. (1) and (2), we then obtain the geometry and displacement interpolations

$$\vec{x} = \begin{Bmatrix} \sum_{k=1}^{2} h_k x_k \\ \frac{s}{2}b \\ \frac{t}{2}a \end{Bmatrix}, \quad \vec{u} = \begin{Bmatrix} -\frac{s}{2}b\sum_{k=1}^{2} h_k \theta_z^k \\ \sum_{k=1}^{2} h_k v_k \\ 0 \end{Bmatrix} \tag{14}$$

Using Eq. (7), the covariant strain field of the displacement-based 2-node iso-beam finite element are determined

$$e_{rr} = \left(\sum_{k=1}^{2}\frac{\partial h_k}{\partial r}x_k\right)\left(-\frac{s}{2}b\cdot\sum_{k=1}^{2}\frac{\partial h_k}{\partial r}\theta_z^k\right) = -\frac{b}{8}Ls\theta_z^2 \tag{15a}$$

$$e_{rs} = \frac{1}{2}\left\{\left(\sum_{k=1}^{2}\frac{\partial h_k}{\partial r}x_k\right)\left(-\frac{b}{2}\sum_{k=1}^{2}h_k\theta_z^k\right)+\frac{b}{2}\sum_{k=1}^{2}\frac{\partial h_k}{\partial r}v_k\right\} = \frac{b}{8}\left\{v_2-\frac{L}{2}(1+r)\theta_z^2\right\} \tag{15b}$$

$$e_{tr} = 0 \tag{15c}$$

Of course, it is well known that this strain field induces shear locking.

The assumed covariant strain field of the beam finite element using the MITC method, which is equivalent to one-point reduced integration, can be simply obtained

$$\bar{e}_{rr} = -\frac{bLs}{8}\theta_z^2, \quad \bar{e}_{rs} = \frac{b}{8}\left\{v_2-\frac{L}{2}(1+\bar{r})\theta_z^2\right\}, \quad \bar{e}_{tr} = 0 \quad \text{with} \quad \bar{r} = 0 \tag{16}$$

where $\bar{r}$ is the position of the tying point in the natural coordinate system.

We then use

$$\bar{\varepsilon}_{ij}(\vec{n}_i\otimes\vec{n}_j) = \bar{e}_{rr}(\vec{g}^r\otimes\vec{g}^r)+\bar{e}_{rs}(\vec{g}^r\otimes\vec{g}^s+\vec{g}^s\otimes\vec{g}^r)+\bar{e}_{tr}(\vec{g}^t\otimes\vec{g}^r+\vec{g}^r\otimes\vec{g}^t) \tag{17}$$

where $\bar{\varepsilon}_{ij}$ is the assumed strain tensor defined in the local Cartesian coordinate system of the beam $((\vec{n}_r, \vec{n}_s, \vec{n}_t))$ and $\vec{g}^i$ are the contravariant base vectors satisfying $\vec{g}_i\cdot\vec{g}^j = \delta_i^j$. Since $\vec{n}_r = \vec{i}_x, \vec{n}_s = \vec{i}_y$ and $\vec{n}_t = \vec{i}_z$ in this beam problem, the strain components are obtained

$$\bar{\varepsilon}_{xx} = -\frac{bs}{2L}\theta_z^2, \quad \bar{\varepsilon}_{xy} = \frac{1}{2L}\left\{v_2-\frac{L}{2}(1+\bar{r})\theta_z^2\right\}, \ \bar{\varepsilon}_{zx} = 0 \tag{18}$$

In a matrix form

$$\begin{Bmatrix} \varepsilon_{xx} \\ 2\varepsilon_{xy} \end{Bmatrix} = \mathbf{B}\begin{Bmatrix} v_2 \\ \theta_z^2 \end{Bmatrix} \quad \text{with} \quad \mathbf{B} = \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} = \begin{bmatrix} 0 & -\frac{bs}{2L} \\ \frac{1}{L} & -\frac{1}{2}(1+\bar{r}) \end{bmatrix} \tag{19}$$

Using the standard procedure of finite element method

$$\mathbf{K} = \int_V \mathbf{B}^T \mathbf{D} \mathbf{B} dV, \quad \text{and} \quad \mathbf{D} = \begin{bmatrix} E & 0 \\ 0 & G \end{bmatrix} \tag{20}$$

the corresponding equilibrium equation is

$$\begin{bmatrix} GV(B_{21})^2 & GVB_{21}B_{22} \\ GVB_{21}B_{22} & E\int_V (B_{12})^2 dV + GV(B_{22})^2 \end{bmatrix} \begin{Bmatrix} v_2 \\ \theta_z^2 \end{Bmatrix} = \begin{Bmatrix} 0 \\ M_z \end{Bmatrix} \tag{21}$$

where $V$ is the volume of the beam.

In the pure bending problem considered, from the first row of Eq. (21), we have

$$GVB_{21}(B_{21}v_2 + B_{22}\theta_z^2) = 0 \tag{22}$$

which exactly corresponds to $\bar{\varepsilon}_{xy} = 0$ in the second equation of Eq. (18).

Therefore, the solution of Eq. (21) needs to satisfy Eq. (22), that is

$$B_{21}v_2 + B_{22}\theta_z^2 = 0 \quad \text{or} \quad \frac{v_2}{\theta_z^2} = \frac{L}{2}(1 + \bar{r}) \tag{23}$$

Here, it is very important to recognize that the ratio of the tip displacements ($v_2/\theta_z^2$) is determined by $\bar{r}$, the tying position of the transverse shear strain in the MITC method.

When applying the MITC method in the 2-node iso-beam finite element, the fixed tying position $\bar{r} = 0$ is used in spite of that the beam section is varying or not. This is exactly equivalent to the one-point reduced integration technique in the beam element. The ratio of the tip displacements in the solution is then given

$$\frac{v_2}{\theta_z^2} = \frac{L}{2} \tag{24}$$

and this is the exactly same to the analytical ratio of the prismatic cantilever beam structure under the pure bending condition by the tip moment $M_z$ only.

Therefore, it is clear that, for prismatic beam problems, the point $\bar{r} = 0$ is the optimal tying position which can result in the analytical ratio of the tip displacements. However, we here can have a question, "Is the same point is optimal for varying section beam problems?" Indeed, this question was the motivation of this study.

## 3. A 2-node iso-beam finite element of varying section

In this section, studying a cantilever beam problem of varying section under axial stretching and pure bending, we find the optimal tying positions of the beam finite element with linearly varying width. Of course, the assumptions of the Timoshenko beam theory are considered.

As shown in Fig. 4, we consider a cantilever beam structure with linearly varying width along the beam

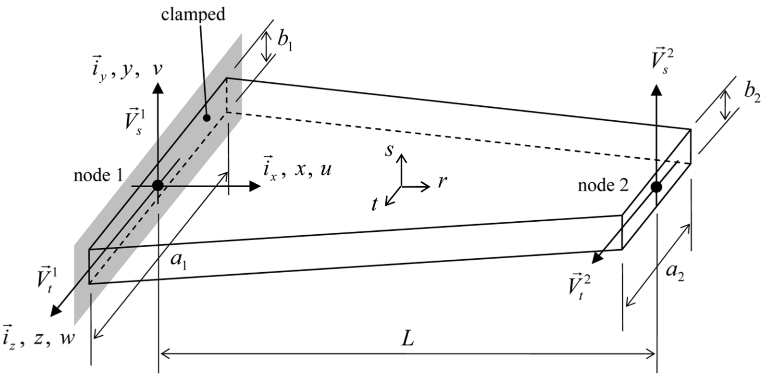

Fig. 4 A 2-node iso-beam finite element with linearly varying width ($b_1 = b_2 = b$)

$$a(x) = a_1 - mx \quad \text{with} \quad m = \frac{a_1 - a_2}{L}, \quad a_1 = a(0), \quad a_2 = a(L) \tag{25}$$

but the height of the beam section ($b$) is constant ($b_1 = b_2 = b$).

The cantilever problem is modeled using one element and the beam is clamped at $x = 0$

$$u_1 = v_1 = w_1 = \theta_x^1 = \theta_y^1 = \theta_z^1 = 0 \tag{26}$$

From the geometry, we have

$$x_1 = 0, y_1 = 0, z_1 = 0, x_2 = L, y_2 = 0, z_2 = 0 \tag{27}$$

and the director vectors are

$$\vec{V}_s^1 = \vec{V}_s^2 = \vec{i}_y, \quad \vec{V}_t^1 = \vec{V}_t^2 = \vec{i}_z \tag{28}$$

For this cantilever beam, the interpolation of geometry is

$$\vec{x} = \begin{Bmatrix} \sum_{k=1}^{2} h_k x_k \\ \frac{s}{2} b \\ \frac{t}{2} \sum_{k=1}^{2} a_k h_k \end{Bmatrix} \tag{29}$$

The corresponding 3D covariant base vectors are given by Eq. (6)

$$\vec{g}_r = \begin{Bmatrix} \frac{L}{2} \\ 0 \\ \frac{t}{2}(a_2 - a_1) \end{Bmatrix}, \quad \vec{g}_s = \begin{Bmatrix} 0 \\ \frac{b}{2} \\ 0 \end{Bmatrix}, \quad \vec{g}_t = \begin{Bmatrix} 0 \\ 0 \\ \frac{(1+r)a_2}{4} \end{Bmatrix} \tag{30}$$

and, from $\vec{g}_i \cdot \vec{g}^j = \delta_i^j$, we obtain the 3D contravariant base vectors

$$\vec{g}^r = \begin{Bmatrix} \frac{2}{L} \\ 0 \\ 0 \end{Bmatrix}, \quad \vec{g}^s = \begin{Bmatrix} 0 \\ \frac{2}{b} \\ 0 \end{Bmatrix}, \quad \vec{g}^t = \begin{Bmatrix} -\frac{2t(a_2 - a_1)}{L(a+r)a_2} \\ 0 \\ \frac{4}{(1+r)a_2} \end{Bmatrix} \tag{31}$$

### *3.1 Axial strain correction*

One 2-node iso-beam finite element gives the exact solution for the behavior of axial stretching when the beam section is prismatic. However, for the beam problems of varying section, the single element cannot give the exact solution. It is not hard to improve the prediction accuracy of the beam element under stretching and this is discussed in this section.

We first find the analytical solution of the cantilever beam problem of length $L$, where an axial loading ($P_x$) at free tip is applied as shown in Fig. 5(a). Since the area of the beam section is varying, we solve

$$EA(x)\frac{du}{dx} = P_x, \quad u(0) = 0 \quad \text{with} \quad A(x) = ba(x) \tag{32}$$

and the solution is

$$u(x) = \frac{P_x}{E}\frac{1}{mb}\{\ln a_1 - \ln(a_1 - mx)\} \tag{33}$$

The free tip displacement in axial direction is

$$u(L) = \frac{P_x}{E}\frac{1}{mb}\ln(a_1/a_2) \tag{34}$$

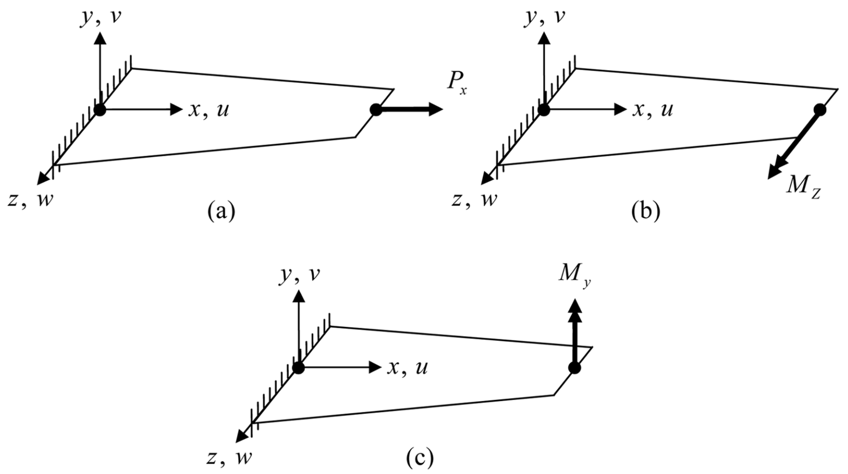


Fig. 5 Cantilever beam problems with varying section (Beam length = $L$)

To investigate the behavior of the iso-beam finite element, we model the cantilever using one beam element and then, under axial force only, the tip displacements are

$$\vec{u}_2 = \begin{Bmatrix} u_2 \\ 0 \\ 0 \end{Bmatrix}, \quad \vec{\theta}_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \tag{35}$$

Using Eqs. (26) and (2), the displacement interpolation of the 2-node iso-beam finite element is

$$\vec{u} = \begin{Bmatrix} \sum_{k=1}^{2} h_k u_k \\ 0 \\ 0 \end{Bmatrix} \tag{36}$$

The resulting covariant strain components are

$$e_{rr} = \frac{L}{4}u_2, \quad e_{rs} = e_{tr} = 0 \tag{37}$$

We then use Eq. (17) and the strain components in the local Cartesian coordinate system are obtained

$$\varepsilon_{xx} = \frac{u_2}{L}, \quad \varepsilon_{xy} = \varepsilon_{zx} = 0 \tag{38}$$

Here, note that the axial strain is constant even if the beam section is varying along the longitudinal axis of the beam.

Introducing a correction factor $k$ for axial strain, we can assume

$$\bar{e}_{rr} = k e_{rr} \tag{39}$$

and then obtain $\bar{\varepsilon}_{xx} = k\frac{u_2}{L}$.

Using the standard finite element procedure,

$$\left[\int_V E\left(\frac{k}{L}\right)^2 dV\right]u_2 = P_x \tag{40}$$

and, finally, the relation between the tip loading and displacement is

$$u_2 = \frac{P_x}{E}\frac{L^2}{k^2 V} \tag{41}$$

We let $u(L)$ in Eq. (34) equal to $u_2$ in Eq. (41) and, finally, the correction factor is given

$$k = \sqrt{\frac{2(a_1/a_2 - 1)}{(a_1/a_2 + 1)\ln(a_1/a_2)}} \tag{42}$$

Fig. 6 shows the function of Eq. (42) depending on $a_1/a_2$. The graph in Fig. 6 presents that, for

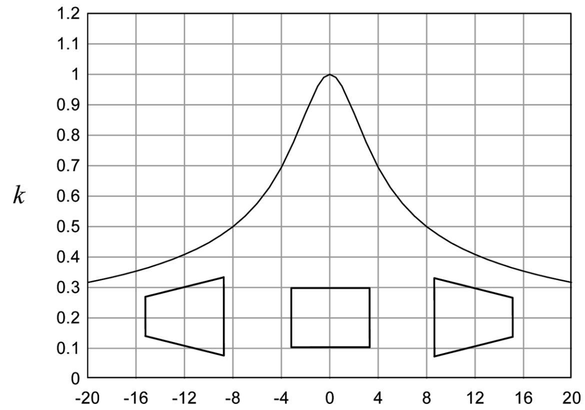


Fig. 6 Correction factor for axial covariant strain depending on $a_1/a_2$. ($0 < k < \leq 1$)

prismatic beams ($a_1 = a_2$), the correction factor is equal to 1, that is, the correction is not necessary.

It is important to note that we derived Eq. (42) for single finite element model of the cantilever problem but, for multi-element models, the equation can be applied. Then, $a_1$ and $a_2$ are the cross-sectional dimensions of each finite element at nodes 1 and 2 in Eq. (1). The convergence results depending on the number of elements used will be discussed in the numerical examples.

### 3.2 Geometry-dependent tying position

Let us consider a cantilever beam of varying section under tip moment in Fig. 5(b). For this problem, the governing equations are (Baker 1996)

$$EI_{zz}(x)\frac{d^2v}{dx^2} = M_z, \quad \theta_z(0) = \left.\frac{dv}{dx}\right|_{x=0} = 0, \quad v(0) = 0 \tag{43}$$

in which $E$ is Young's modulus and $I_{zz}(x)$ is the moment of inertia of the varying beam section about the $z$-axis, $I_{zz}(x) = \frac{b^3}{12}a(x)$.

The solutions of Eq. (43) are given

$$\theta_z(x) = \frac{dv(x)}{dx} = \frac{12M_z}{b^3E}\frac{1}{m}\{\ln a_1 - \ln(a_1 - mx)\} \tag{44a}$$

$$v(x) = \frac{12M_z}{b^3E}\left[\frac{x}{m}\{1 + \ln a_1 - \ln(a_1 - mx)\} + \frac{a_1}{m^2}\{\ln(a_1 - mx) - \ln a_1\}\right] \tag{44b}$$

The ratio between deflection and rotation at the free tip is

$$\frac{v(L)}{\theta_z(L)} = \frac{(a_1 - mL)\{\ln(a_1 - mL) - \ln a_1\} + mL}{m\{\ln a_1 - \ln(a_1 - mL)\}} = \frac{L}{\ln(a_1/a_2)} - \frac{L}{a_1/a_2 - 1} \tag{45}$$

which depends on how rapidly the beam section is varying, $a_1/a_2$.

In this case, when we model the cantilever problem using one beam element, the displacement field of the finite element discretization is

$$\vec{u} = \begin{Bmatrix} -\frac{1}{2}b\sum_{k=1}^{2}h_k\theta_z^k \\ \sum_{k=1}^{2}h_k v_k \\ 0 \end{Bmatrix} \tag{46}$$

Here, we use the correction for $e_{rr}$ as discussed in the previous section and the covariant strains are obtained by Eq. (7)

$$\bar{e}_{rr} = -k\frac{bLs}{8}\theta_z^2, \quad e_{rs} = \frac{b}{8}\left(v_2 - \frac{L}{2}(1+r)\theta_z^2\right), \quad e_{tr} = 0 \tag{47}$$

Considering a tying position $r = \bar{r}$ for $e_{rs}$ $(e_{rs} = e_{rs}(\bar{r}))$ and, using Eq. (17), the strain components in the local Cartesian coordinate system are

$$\bar{\varepsilon}_{xx} = -k\frac{bs}{2L}\theta_z^2, \quad \bar{\varepsilon}_{xy} = \frac{1}{2L}\left(v_2 - \frac{L}{2}(1+\bar{r})\theta_z^2\right), \quad \bar{\varepsilon}_{zx} = 0 \tag{48}$$

The same procedure in Eqs. (19)~(22) results in

$$\frac{v_2}{\theta_z^2} = \frac{L(1+\bar{r})}{2} \tag{49}$$

and, from Eq. (45), we obtain the tying position which gives the analytical ratio between tip deflection and rotation

$$\bar{r} = \frac{2}{\ln(a_1/a_2)} - \frac{2}{(a_1/a_2)-1} - 1 \tag{50}$$

which is presented in Fig. 7. The figure shows that $-1 < \bar{r} < 1$ and, when $a_1 = a_2$ for prismatic beams, the tying position $\bar{r}$ is 0 which corresponds to the center of the beam element[2].

As mentioned in the previous section, for general finite element models, $a_1$ and $a_2$ are the cross-sectional dimensions of each finite element at node 1 and 2 in Eq. (1). Note that the pure-bending mode can be exactly captured only when the tying position of Eq. (50) for the transverse shear strain and the correction factor for axial strain in Eq. (42) are used together.

The cantilever beam problem in Fig. 5(c) can be used to determine the geometry-dependent tying position for the strain component $e_{tr}$. However, since the response for the beam problem is not based on the beam theory when the section is rapidly varying, we do not use the geometry-dependent tying scheme for $e_{tr}$.

[2]Since, when $a_1 = a_2$, the second term in the right side of Eq. (50) becomes infinite, special care is required in the numerical implementation for the case.

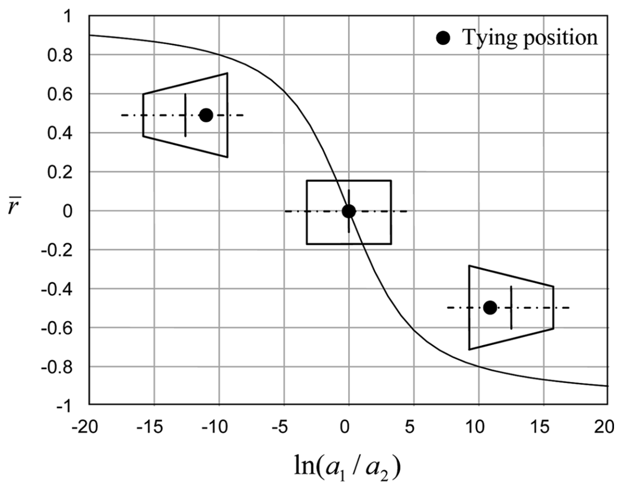


Fig. 7 Typing position for transverse shear covariant strain depending on $a_1/a_2$ ($-1 < \bar{r} < 1$)

## 4. Numerical tests

In the previous sections, we studied the role of tying position and presented the axial strain correction (ASC) scheme and the geometry-dependent tying (GDT) scheme. In this section, we analyze some beam structures with varying section using the MITC method with "GDT" or "GDT+ASC." The results are compared with the beam element using reduced integration or the equivalent MITC method with fixed tying (FT) position.

It is important to note that reduced integration (or the equivalent MITC method with FT) and the MITC method with GDT or GDT+ASC do not result in locking for beam problems of varying section. Therefore, the solution accuracy of the results presented in this section does not depend on the ratio between sectional dimension and overall length. As well known, locking deteriorates the solution accuracy when the ratio becomes smaller.

### *4.1 Cantilever beam problems*

We first analyze a cantilever beam structure of length $L$ with decreasing and increasing sections, see Figs. 8 and 9. For the decreasing case, the beam structure is clamped at point $A$ and free at point $B$. The structure is loaded with forces and moments at the free tip. As shown in Figs. 8 and 9, we are considering three cases of loading: axial force $P_x$, moment $M_z$ and transverse force $P_y$. The finite element meshes used are shown in Fig. 10.

Figs. 11-13 show the ratio between the finite element solutions obtained and the exact solutions at free tip of the cantilever structure under the three different load cases when the beam section is decreasing as shown in Fig. 8. In the figures, "FT" represents the 2-node iso-beam finite elements formulated using reduced integration or using the corresponding MITC scheme where the tying points for transverse shear strains is fixed at $r = 0$ and no correction factor is used for axial strain. "GDT" represents the beam finite element formulated by the MITC scheme using Eq. (50) and "ASC" denotes the axial strain correction in Eq. (42).

For the axial stretching case (by axial force) and the pure bending (by tip moment) case, the use

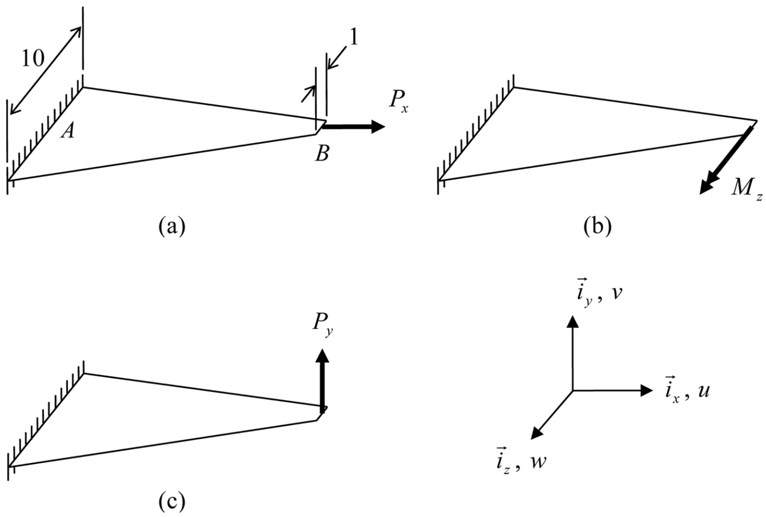


Fig. 8 Cantilever beam problems with decreasing section (Beam length = $L$): (a) axial force, (b) moment (pure bending case), (c) transverse force

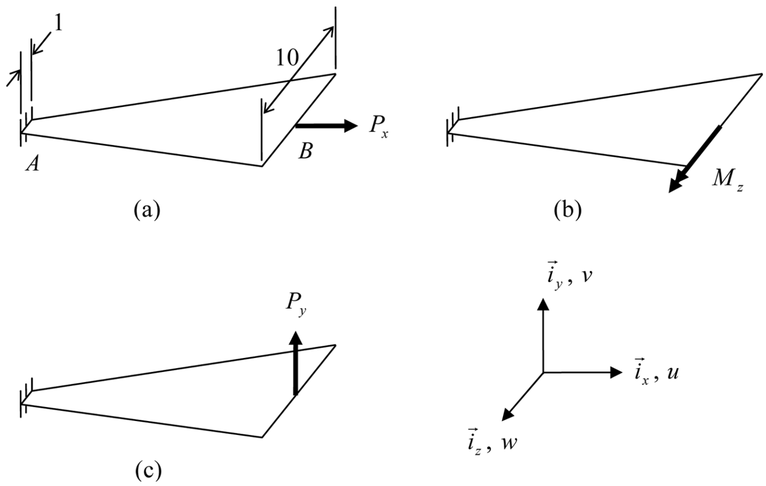


Fig. 9 Cantilever beam problems of increasing section. (Beam length = $L$): (a) axial force, (b) moment (pure bending case), (c) transverse force

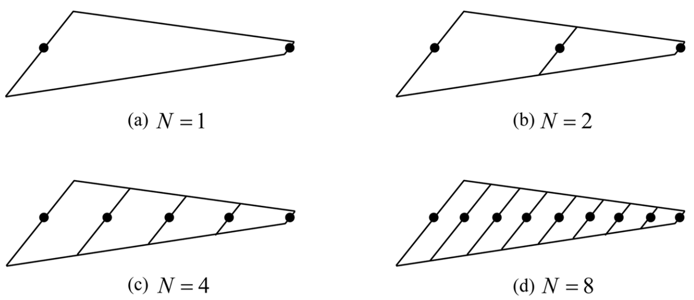


Fig. 10 Meshes used (a) 1 element, (b) 2 elements, (c) 4 elements, (d) 8 elements ($N$ = the number of elements used)

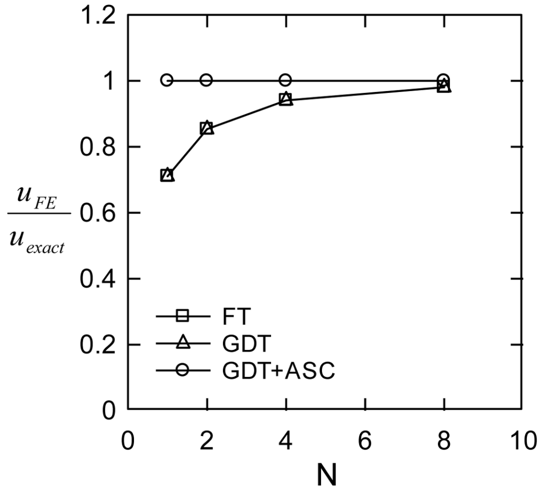


Fig. 11 Axial displacements at the free tip under axial force (decreasing section)

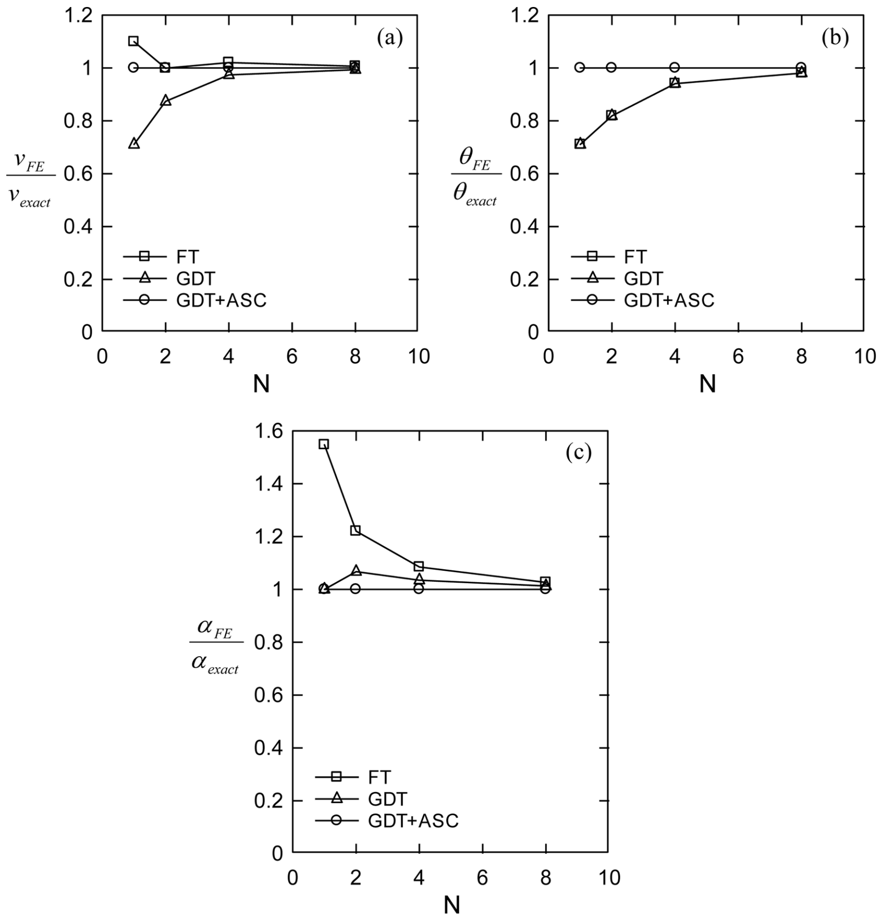


Fig. 12 Displacements at the free tip under tip moment (decreasing section): (a) deflection, (b) rotation, (c) rotation/deflection ($\alpha = \theta/v$)

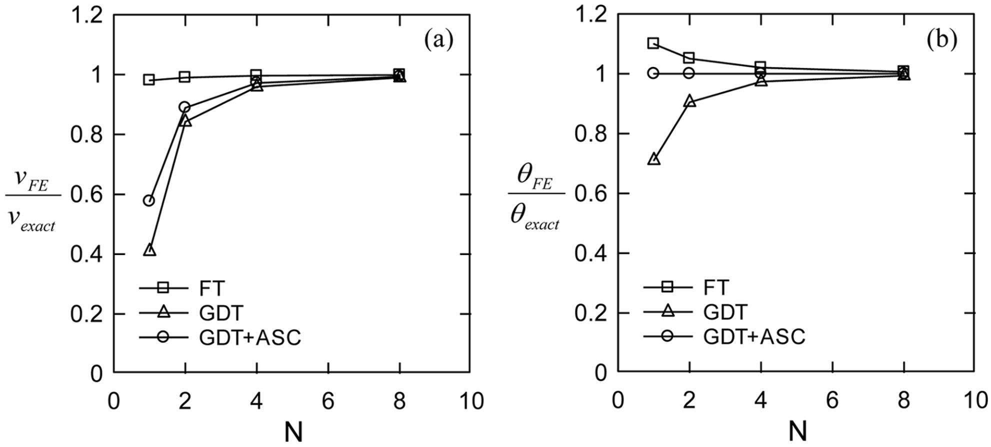


Fig. 13 Displacements at the free tip under transverse force (decreasing section): (a) deflection, (b) rotation

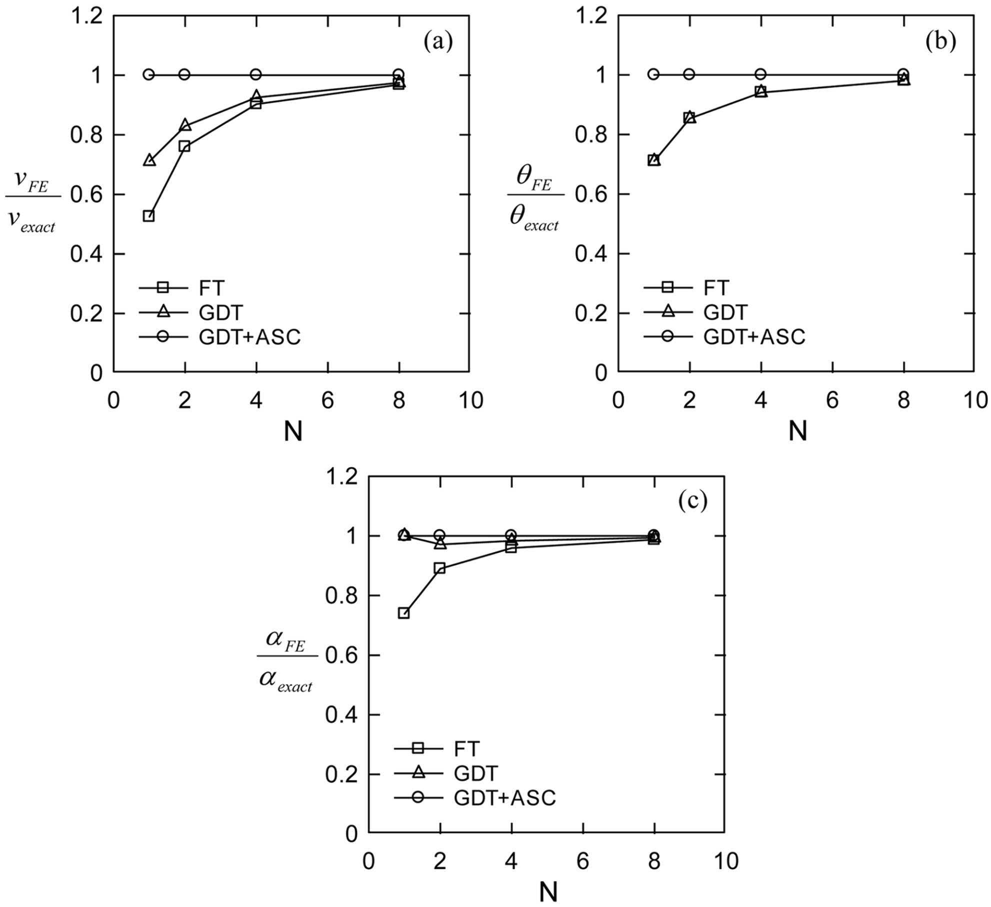


Fig. 14 Displacements at the free tip under moment (increasing section): (a) deflection, (b) rotation, (c) deflection /translation

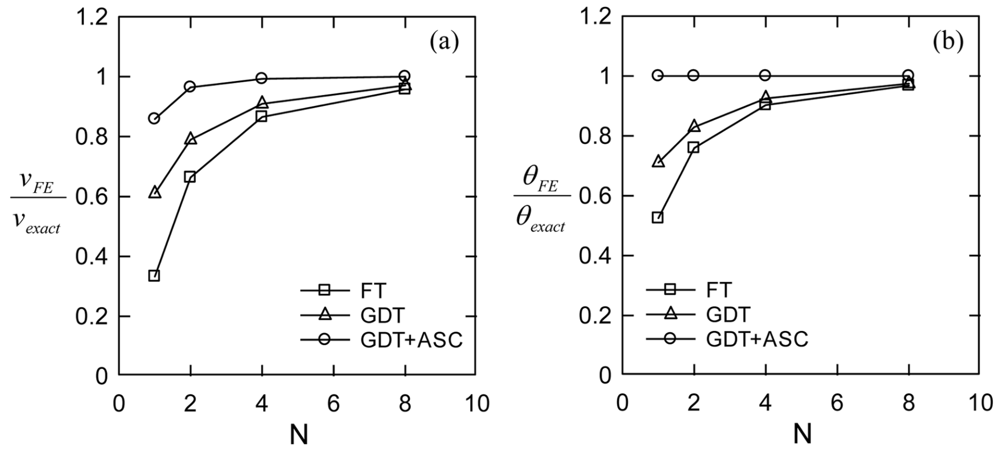


Fig. 15 Displacements at the free tip under transverse force (increasing section): (a) deflection, (b) rotation

of both GDT and ASC gives the exact solutions regardless of the number of elements used. In the case of a transverse force at free tip, the rotation is exactly predicted by GDT+ASC but deflection converges from below. The fixed tying scheme (FT) gives better convergence than GDT+ASC but the fixed tying scheme overestimates the displacements as shown in Figs. 12(a) and 13(b), that is, the element is too flexible when the number of elements used is few[3]. Note that the geometry-dependent tying scheme does not overestimate the displacements.

Considering the increasing section, Figs. 14 and 15 present that the geometry-dependent tying scheme gives much better solution accuracy and the exact solutions in most cases. We here do not show the axial behavior for the increasing section because the solutions are exactly the same to the graphs in Fig. 11.

### *4.2 Simple beam problems*

We next consider simple beam problems of length $L$ as shown in Fig. 16. The beams are simply supported and subjected to the moments at both ends (pure bending case) and the distributed transverse force, see Figs. 16(a) and (b).

Figs. 17 and 18 display the ratio between the finite element solutions obtained and the exact solutions. In the pure bending case by tip moments, the geometry-dependent tying scheme (GDT) used with the axial strain correction scheme (ASC) gives the exact solution regardless of the number of the beam elements used. When the distributed transverse force is applied along the beam, the displacements obtained using FT and GDT+ASC converge to the exact solutions and their solution accuracy is similar.

In the numerical tests presented, we only consider the beam problems of varying width from 10 to 1. Note that, as the beam width is more slowly varying, the numerical results of FT and GDT+ASC get closer and finally become the same when the beam width is constant along the beam.

[3]This is not a good property of finite element solutions when used in engineering applications because, in general, finite element models of coarser mesh are stiffer.

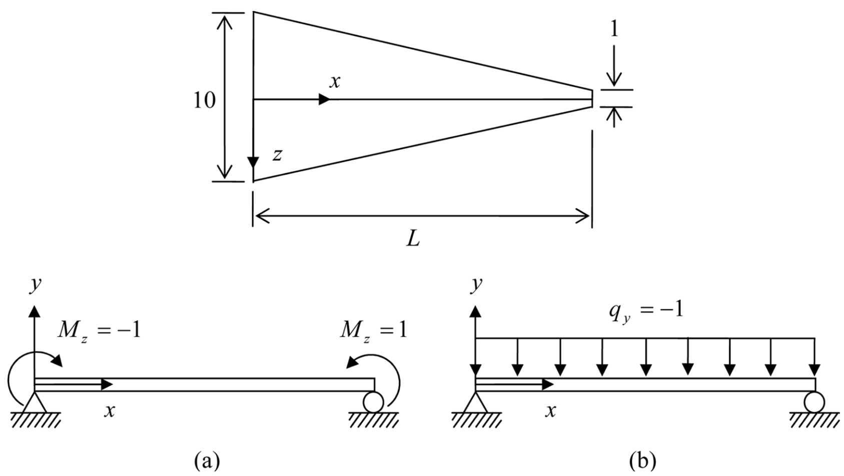


Fig. 16 Simple beam problems: (a) moments at both ends (pure bending case), (b) uniformly distributed transverse force ($q_y = -1$ per area)

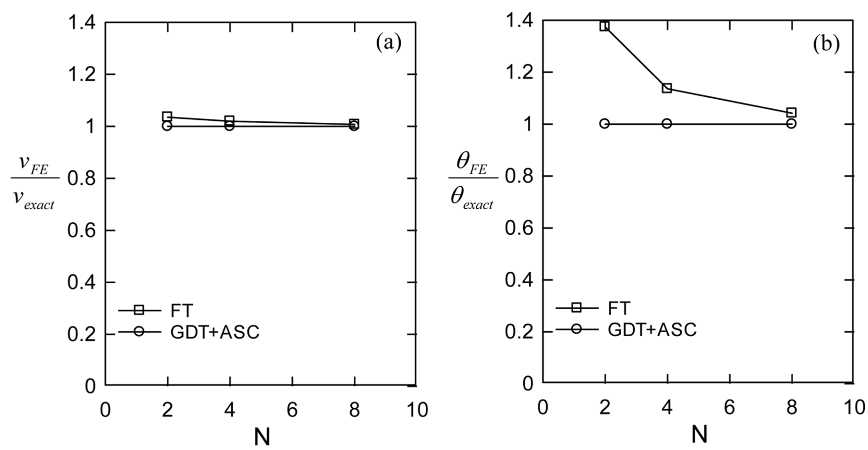


Fig. 17 Displacements at the beam center in the pure bending case in Fig. 15(a): (a) deflection, (b) rotation

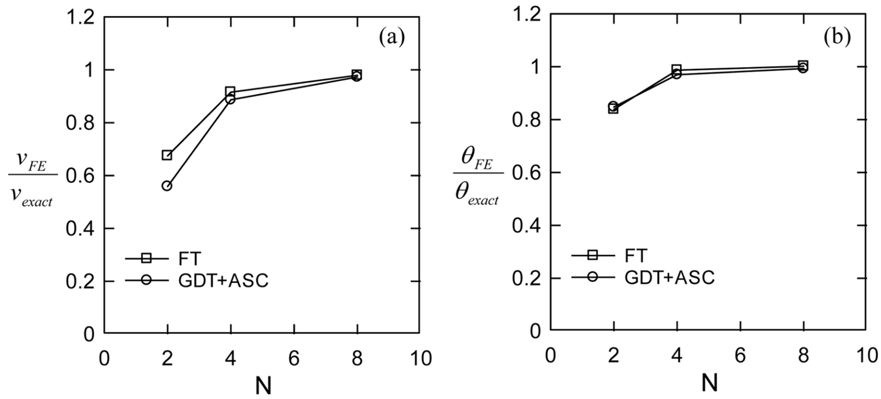


Fig. 18 Displacements in the distributed transverse loading case in Fig. 16(b): (a) deflection at center, (b) rotation at right end

## 5. Conclusions

We introduced the concept of the geometry-dependent MITC method applying it to improve a 2-node iso-beam finite element of varying section. We first studied the pure bending behavior of a 2-node iso-beam finite element of prismatic section. Investigating the detailed behaviors of the cantilever beams of varying section modeled by a single beam element and connecting them with analytical solutions, the geometry-dependent correction factor $k$ for axial strain and the geometry-dependent tying positions for transverse shear strains were proposed. The resulting Eqs. (42) and (50) were implemented and some numerical analyses were performed to show the effectiveness of the geometry-dependent MITC method.

In this study, we obtained the following observations:

- In 2-node iso-beam finite elements formulated with the MITC method, the tying position determines the ratio between transverse translational displacement and rotation.
- When the two proposed schemes (GDT+ASC) are used together, the exact displacements of the beam elements are predicted regardless of the number of elements used in pure bending and axial stretching cases.
- Considering the transverse shearing behaviors, the fixed tying scheme gives more flexible responses than GDT+ASC. However, in some cases (as observed in this study), the scheme results in more flexible responses in coarser mesh models.

Since the geometry-dependent MITC method is more general than the original MITC method, it can be extended to improve other structural finite elements suffering from locking. However, it is not easy to find the optimal tying positions for general finite elements of arbitrary geometries.

## References

Baker, G. (1996), "Exact deflections in nonprismatic members", *Comput. Struct.*, **61**, 515-528.

Bathe, K.J. (1996), *Finite Element Procedures*, Prentice Hall: New Jersey.

Bathe, K.J. and Bolourchi, S. (1979), "Large displacement analysis of three-dimensional beam structures", *Int. J. Numer. Meth. Eng.*, **14**, 961-986.

Bucalem, M. and Bathe, K.J. (1993), "Higher-order MITC general shell elements", *Int. J. Numer. Meth. Eng.*, **36**, 3729-3754.

Choi, C.K. and Paik, J.G. (1994), "An efficient four node degenerated shell element based on the assumed covariant strain", *Struct. Eng. Mech.*, **2**(1), 17-34.

Choi, C.K., Lee, P.S. and Park, Y.M. (1999), "Defect-free 4-node flat shell element: NMS-4F element", *Struct. Eng. Mech.*, **8**(2), 207-231.

Dvorkin, E.N. and Bathe, K.J. (1984), "A continuum mechanics based four-node shell element for general nonlinear analysis", *Eng. Comput.*, **1**, 77-88.

Hong, H.S., Kim, K.H. and Choi, C.K. (2004), "Assumed strain finite strip method using the non-periodic B-spline", *Struct. Eng. Mech.*, **18**(5), 671-690.

Lee, P.S. and Bathe, K.J. (2002), "On the asymptotic behavior of shell structures and the evaluation in finite element solutions", *Comput. Struct.*, **80**, 235-255.

Lee, P.S. and Bathe, K.J. (2004), "Development of MITC isotropic triangular shell finite elements", *Comput. Struct.*, **82**, 945-962.

Lee, P.S. and McClure, G. (2006), "A general three-dimensional L-section beam finite element for elastoplastic large deformation analysis", *Comput. Struct.*, **84**, 215-229.