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Buckling of fully and partially embedded non-prismatic columns using differential quadrature and differential transformation methods

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Abstract. Numerical solution to buckling analysis of beams and columns are obtained by the method of differential quadrature (DQ) and harmonic differential quadrature (HDQ) for various support conditions considering the variation of flexural rigidity. The solution technique is applied to find the buckling load of fully or partially embedded columns such as piles. A simple semi- inverse method of DQ or HDQ is proposed for determining the flexural rigidities at various sections of non-prismatic column (pile) partially and fully embedded given the buckling load , buckled shape and sub-grade reaction of the soil. The obtained results are compared with the existing solutions available from other numerical methods and analytical results. In addition, this paper also uses a recently developed technique, known as the differential transformation (DT) to determine the critical buckling load of fully or partially supported heavy prismatic piles as well as fully supported non-prismatic piles. In solving the problem, governing differential equation is converted to algebraic equations using differential transformation methods (DT) which must be solved together with applied boundary conditions. The symbolic programming package, Mathematica is ideally suitable to solve such recursive equations by considering fairly large number of terms.

Keywords: column; stability; embedded piles; differential quadrature; semi-inverse approach; differential transformation.

1. Introduction

Structurally, axially loaded columns (piles) are slender columns with lateral support from the surrounding soil. If unsupported, these columns will fail by buckling instability and not due to crushing of the material. Piles normally have ratio of length (L) to diameter (D) of 25 to 100. Slender columns (piles) passing through water or thick deposits of very weak soil (see Fig. 1(a)) need to be checked against buckling. A slender pile extending a considerable distance above the ground line (Fig. 1(b)), the unsupported length becomes critical and stability governs the design. Some of the earlier methods ignore the surrounding ground and consider a pile as a free standing column and assume that surrounding ground offers infinite resistance

Main factors affecting the column (pile) stability are

1) soil resistance

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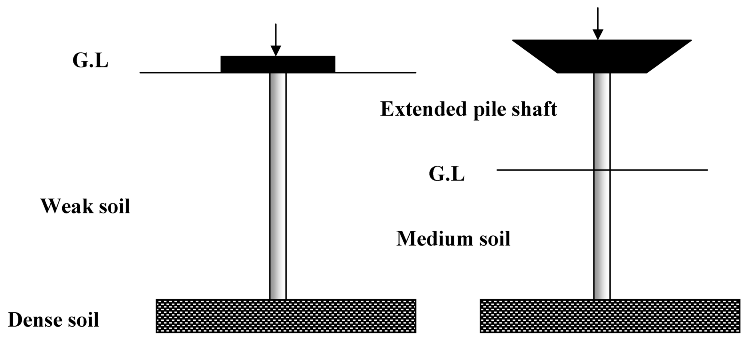

Fig. 1(a) Fully embedded pile, (b) Partially embedded pile

2) length of the column
3) Stiffness of column.

According to Terzaghi (1955), the soil modulus or sub-grade reaction can be assumed to be constant with depth for clayey soils and to increase with depth for granulated materials.

2. Differential Quadrature Method (DQ)

These problems of stability of piles either prismatic or non-prismatic, partially or fully embedded, could easily be solved using Differential Quadrature Method (DQ) which was introduced by (Bellman and Casti 1971). With the application of boundary conditions as per Wilson's method (Wilson 2002) DQ method will also be straight forward and easy to use by the engineers. Since the introduction of this method, applications of the differential quadrature method to various engineering problems have been investigated and their success has shown the potential of the method as an attractive numerical analysis tool. The basic idea of the DQ method is to quickly compute the derivatives of a function at any grid point within its bounded domain by estimating the weighted sum of the values of the functions at a small set of points related to the domain. In the originally derived DQ, Lagrangian interpolation polynomial was used (Bert and Malik 1996, Bert *et al.* 1993, 1994). A recent approach of the original differential quadrature approximation is called Harmonic Differential Quadrature (HDQ) originally proposed by (Striz *et al* 1995). Unlike DQ, HDQ uses harmonic or trigonometric functions as the test functions. As the name of the test function suggested, this method is called the HDQ method. All the problems in this paper have demonstrated that the application of the DQ and HDQ will lead to accurate results with less computational effort and that there is a potential that the method may become alternative to conventional methods such as Finite Difference, Finite Element and Boundary Element methods.

3. Governing equation for stability of a column

The fourth order governing differential equation for buckling of column with varying flexural rigidity 'D' ($D = EI$), and w (= the lateral deflection) may be written as

$$\frac{d^2}{dx^2}\left(D\frac{d^2w}{dx^2}\right) + P\frac{d^2w}{dx^2} = 0 \tag{1}$$

or

$$\left(D\frac{d^4w}{dx^4}\right) + 2\left(\frac{dD}{dx}\frac{d^3w}{dx^3}\right) + \left(\frac{d^2D}{dx^2}\right)\left(\frac{d^2w}{dx^2}\right) = -P\frac{d^2w}{dx^2} \tag{2}$$

for a function $f(\xi)$, DQ approximation of the m^{th} order derivative at the i^{th} sampling point is given by

$$\frac{d^m}{d\xi^m}\begin{Bmatrix} f(\xi_1) \\ f(\xi_2) \\ \cdot \\ f(\xi_n) \end{Bmatrix} = C_{ij}^{(m)}\begin{Bmatrix} f(\xi_1) \\ f(\xi_2) \\ \cdot \\ f(\xi_n) \end{Bmatrix} \quad \text{for} \quad i,j = 1, 2, \ldots, n \tag{3}$$

where 'n' is the number of sampling points. Assuming Lagrangian interpolation polynomial

$$f(\xi) = \frac{M(\xi)}{(\xi-\xi_i)M_1(\xi_i)} \quad \text{for} \quad i = 1, 2, \ldots, n \tag{4}$$

where

$$\begin{aligned} M(\xi) &= \textstyle\prod_{j=1}^{n}(\xi-\xi_j) \\ M_1(\xi_i) &= \textstyle\prod_{j=1, j\neq i}^{n}(\xi_i-\xi_j) \quad \text{for} \quad i = 1, 2, \ldots, n \end{aligned} \tag{5}$$

Substituting Eq. (5) in Eq. (3) leads to

$$\begin{aligned} C_{ij}^{(1)} &= \frac{M_1(\xi_i)}{(\xi_i-\xi_j)M_1(\xi_j)} \quad \text{for} \quad i,j = 1, 2, \ldots, n; \; i \neq j \\ C_{ii}^{(1)} &= -\sum_{\substack{j=1 \\ j\neq i}}^{n} C_{IJ}^{(1)} \end{aligned} \tag{6}$$

The second and third and higher derivative can be calculated as

$$\begin{aligned} C_{ij}^{(2)} &= \sum_{k=1}^{n} C_{ik}^{(1)}C_{kj}^{(1)} \qquad i = j = 1, 2, \ldots, n \\ C_{ij}^{(m)} &= \sum_{k=1}^{n} C_{ik}^{(1)}C_{kj}^{(m-1)} \qquad \text{for} \quad i = j = 1, 2, \ldots, n \end{aligned} \tag{7}$$

and the number of sampling points $n > m$.

A natural and often convenient choice for sampling point is that of equally spaced points or CGL mesh distribution as given by Eq. (8). For the sampling points, we adopt well accepted Chebyshev-Gauss-Lobatto mesh distribution and its normalized form is given by Shu (2000) as

$$\xi_i = \frac{1}{2}\left[1 - \cos\frac{(i-1)}{(n-1)}\pi\right] \tag{8}$$

where

$$\xi_i = \frac{x_i}{L} \tag{9}$$

'L' is the length of the column and the column is divided into ten divisions or eleven sampling points in case of DQ method and x_i is the distance from the bottom end of the column.

4. Harmonic Differential Quadrature Method (HDQ)

The Harmonic test function $h_i(\xi)$ used in HDQ method is defined as

$$h_i(\xi) = \frac{\prod_{\substack{k=0 \\ k \neq i}}^{n} \sin[\pi(\xi - \xi_k)/2]}{\prod_{\substack{k=0 \\ k \neq i}}^{n} \sin[\pi(\xi_i - \xi_k)/2]} \tag{10}$$

According to the HDQ, the weighting coefficients of the first order derivative C_{ij}^{1} for $i \neq j$ is obtained using the form (Ulker and Civalek 2004)

$$C(i, j, 1) = \frac{\pi P(\xi_i)/2}{P(\xi_j)\sin[\pi(\xi_i - \xi_j)/2]} \quad i, j = 1, 2, \ldots, n \tag{11}$$

$$C(i, i, 1) = -\sum_{\substack{j=1 \\ j \neq i}}^{n} C(i, j, 1) \tag{12}$$

where

$$P(\xi_i) = \prod_{\substack{j=1 \\ j \neq i}}^{n} \sin[\pi(\xi_i - \xi_j)/2] \quad \text{for} \quad j = 1, 2, \ldots, n \tag{13}$$

The weighting coefficients of the second order derivative are given by

$$C(i, j, 2) = C(i, j, 1)\{2C(i, j, 1) - \pi\cot[\pi(\xi_i - \xi_j)/2]\} \quad i, j = 1, 2, \ldots, n \tag{14}$$

$$C(i, i, 2) = -\sum_{\substack{j=1 \\ j \neq i}}^{n} C(i, j, 2) \tag{15}$$

Higher order derivatives can be obtained using Eq. (7)

4.1 To find buckling load and buckled shape given variation of D=EI

In this problem, D is known and w and P are unknown which can be found by solving as an eigen value problem as explained below.

Assume $c(:,:,m)$ is the 'm^{th} derivative, i.e., C_{ij}^{m}, Eq. (2). may be written as

$$\{[K]\}\{[c(:,:,4)]/L^4 + 2[\alpha][c(:,:,3)]/L^3 + [\beta][c(:,:,2)]/L^2\}\{w\} = -\frac{P}{L^2}[c(:,:,2)]\{w\} \tag{16}$$

where

$$[K] = \begin{bmatrix} D_1 & & & \\ & D_2 & & \\ & & \cdot & \\ & & & D_n \end{bmatrix} \tag{17a}$$

where $[\alpha]$ and $[\beta]$ are diagonal matrices given as

$$[\alpha] = \begin{bmatrix} \alpha_{11} & 0 & 0 & 0 \\ 0 & \alpha_{22} & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \alpha_{nn} \end{bmatrix}; \quad [\beta] = \begin{bmatrix} \beta_{11} & 0 & 0 & 0 \\ 0 & \beta_{22} & 0 & 0 \\ 0 & 0 & \cdot & 0 \\ 0 & 0 & 0 & \beta_{nn} \end{bmatrix} \tag{17b}$$

and

$$\{\alpha_{ii}\} = [c(i, 1:n, 1)\{D\}]/L = \frac{dD}{dx} \tag{17c}$$

$$\{\beta_{ii}\} = [c(i, 1:n, 2)\{D\}]/L^2 = \frac{d^2D}{dx^2} \tag{17d}$$

$$E = -c(;,;,2)/L^2 \tag{17e}$$

Eq. (16) may be written as

$$\frac{1}{P}\underset{n \times n}{[G]}\{w\} = \underset{n \times n}{[E]}\{w\} \tag{18}$$

4.2 Boundary conditions

Since it is a fourth order differential equation, four boundary conditions should be given. The boundary conditions will be applied as follows.

Clamped - Pinned

$$w = 0 \text{ at } x = 0; \; G[n+1, 1] = 1.0 \tag{19a}$$

$$w' = 0 \text{ at } x = 0; \; G[n+2, 1:n] = c(1, 1:n, 1)/L \tag{19b}$$

$$w = 0 \text{ at } x = L; \; G[n+3, n] = 1 \tag{19c}$$

$$w'' = 0 \text{ at } x = L; \; G[n+4, 1:n] = c(n, 1:n, 2)/L^2 \tag{19d}$$

Clamped – Clamped

$$w = 0 \text{ at } x = 0; \; G[n+1, 1] = 1.0 \tag{20a}$$

$$w' = 0 \text{ at } x = 0; \; G[n+2, 1:n] = c(1, 1:n, 1)/L \tag{20b}$$

$$w = 0 \text{ at } x = L;\ G[n+3, n] = 1 \tag{20c}$$

$$w' = 0 \text{ at } x = L;\ G[n+4, 1:n] = c(n, 1:n, 2)/L \tag{20d}$$

Pinned – Pinned

$$w = 0 \text{ at } x = 0;\ G[n+1, 1] = 1.0 \tag{21a}$$

$$w'' = 0 \text{ at } x = 0;\ G[n+2, 1:n] = c(1, 1:n, 2/L^2) \tag{21b}$$

$$w = 0 \text{ at } x = L;\ G[n+3, n] = 1 \tag{21c}$$

$$w'' = 0 \text{ at } x = L;\ G[n+4, 1:n] = c(n, 1:n, 2/L^2) \tag{21d}$$

Clamped – Free

$$w = 0 \text{ at } x = 0;\ G[n+1, 1] = 1.0 \tag{22a}$$

$$w' = 0 \text{ at } x = 0;\ G[n+2, 1:n] = c(1, 1:n, 1)/L \tag{22b}$$

$$w'' = 0 \text{ at } x = L;\ G[n+3, n] = c(n, 1:n, 2)/L \tag{22c}$$

$$\frac{dD}{dx}w'' + Dw''' = -Pw' \text{ at } x = L;\ G[n+4, 1:n] = \alpha_{nn}c(n, 1:n, 2)/L^2 + D_{nn}c(n, 1:n, 3)/L^3$$

$$= -Pc(n, 1:n, 1)/L \tag{22d}$$

4.3 Wilson's method of applying boundary conditions (Wilson 2002)

In general, the boundary conditions are given by

$$\underset{4\times n}{[G]_1}\ \underset{n\times 1}{\{w\}} = \underset{4\times 1}{[E]_1\{w\}} \tag{23}$$

Combining governing equations and boundary conditions we get

$$\frac{1}{P}\begin{bmatrix} [G]_0 \\ n\times n \\ [G]_1 \\ 4\times n \end{bmatrix}\{w\} = \begin{bmatrix} [E] \\ n\times n \\ [E]_1 \\ 4\times n \end{bmatrix}\{w\} \tag{24}$$

Using Lagrange multiplier approach as recommended by Wilson (2002), Eq. (24) can be modified to square matrix as

$$\frac{1}{P}\begin{bmatrix} [G]_0 & [G]_1^T \\ [G]_1 & [0] \end{bmatrix}\begin{Bmatrix} \{w\} \\ \{\lambda\} \end{Bmatrix} = \begin{bmatrix} [E] & [E]_1^T \\ [E]_1 & [0] \end{bmatrix}\begin{Bmatrix} w \\ \lambda \end{Bmatrix} \tag{25}$$

The above equation has both equilibrium and equation of geometry. Solving Eq. (25) as an eigen value problem, one will be able to get the buckling load.

4.4 Buckling of fully and partially embedded uniform piles

Bowles (1996) used the method of Wang (1970) developed a procedure that can be used to obtain the buckling load for pile either fully or partially embedded.

Example 1

To illustrate pile buckling and the effect of soil on buckling of piles, the example of Bowles (1996) is presented. A 254 mm dia × 6.35 mm wall pipe pile is 12 m length. It is embedded 5 m in an extremely soft soil. Assume sub-grade reaction S_r = 3100 kN/m^3 and hence spring constant k_s = $S_r(dia)$ = 787.4 kN/m^2 may be assumed. E = 200e6 kN/m^2 and I = 0.000038 m^4. The governing differential equation for beam on elastic foundation s given by

$$\frac{d^2}{dx^2}\left(D\frac{d^2w}{dx^2}\right) + K_s w = -P\frac{d^2w}{dx^2} \tag{26}$$

Comparing with Eq. (1), there is one additional term due to sub grade reaction and this can be considered by adding the contribution of sub grade reaction matrix with [G] in Eq. (18) as

$$\frac{1}{P}\underset{n \times n}{[\bar{G}]}\{w\} = \underset{n \times n}{[E]}\{w\} \tag{27}$$

where

$$[\bar{G}] = [G] + [K_s] \tag{28}$$

$[K_s]$ is a diagonal matrix containing spring constants and the boundary conditions remain same as discussed before. Eleven sampling points are assumed as per (CGL) mesh distribution. The buckling load in DQ method is obtained as 196.15 kN as compared to 198 kN by Bowles (1996). Same value is obtained by HDQ method as well with 31 sampling points. The buckling load is larger than the buckling of cantilever pile of 12 m length (BL = 130 kN) and less than the for the pile of 7 m length (BL = 382 kN).

4.5 Buckling load of non-prismatic column

Recently Elishakoff (Elishakoff *et al.* 2006) obtained the buckling load from the closed form solution of second order differential equation for a non-prismatic column

Example 2. Buckling of Clamped – Pinned non-prismatic column

For the case of clamped and pinned column and where the flexural rigidity varies as

$$D = \left(\frac{5}{16} + \frac{5}{4}\xi - \xi^2\right); \quad \text{where} \quad \xi = \frac{x}{L} \tag{29}$$

the buckling load using Eq. (16) with boundary conditions for both DQ and HDQ is obtained as 0.12 which agrees with Elishakoff *et al.* (2006) ($P = 12b_2/L^2 = 0.12$) (where $b_2 = 1$ and $L = 10$).

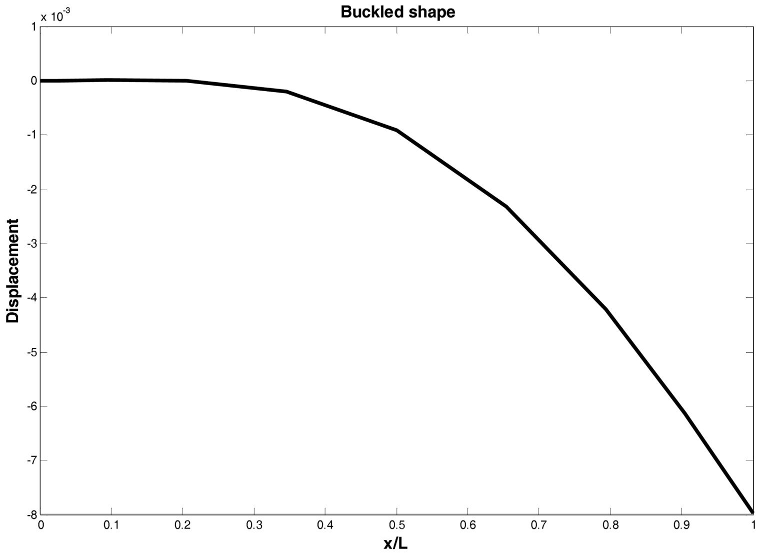

Fig. 2 Buckled shape for partially embedded Pile (Example. 4)

Example 3. Buckling of Clamped – Clamped non-prismatic column

For the same variation of D, for clamped and clamped condition, the buckling load is obtained as 0.2262

When D varies as

$$D = \left(\frac{1}{6} + \xi - \xi^2\right) \tag{30}$$

as given by Elishakoff *et al.* (2006), for clamped condition the buckling load is 0.12 for both DQ and HDQ which agrees with that of Elishakoff *et al.* (2006). Hence in practice, we get problems of varying flexural rigidity for which the DQ and HDQ can easily be applied incorporating the boundary conditions to obtain the buckling load and buckled shape.

Example 4. Buckling of non-prismatic partially embedded pile.

For Example 1 if we assume the diameter of the pile varies as 0.254 m at the base and 0.127 m at the top, DQ method is applied to obtain buckling load as 97.08 kN with 11 sampling points. For the same problem if 21 sampling points are used in HDQ to obtain the buckling load as 94.3 kN. The buckled shape is shown in Fig. 2.

4.6 To find the variation of D = EI given buckling load and buckled shape (Semi-Inverse Method)

Eq. (2) is rewritten as

$$\left(\frac{d^4w}{dx^4}D\right) + 2\left(\frac{d^3w}{dx^3}\frac{dD}{dx}\right) + \left(\frac{d^2w}{dx^2}\right)\left(\frac{d^2D}{dx^2}\right) = -P\frac{d^2w}{dx^2} \tag{31}$$

In Eq. (31), P and w are given and it is required to find the flexural rigidity D along the length of the column. Applying DQM, Eq. (31) is written as

$$\{[c(:,:,4)/L^4] + 2[\alpha][c(:,:,1)/L] + [\beta][c(:,:,2)/L^2]\}\{D\} = -P[(:,:,2)/L^2]\{w\} = \{F\} \quad (32)$$

where $[\alpha]$ and $[\beta]$ are diagonal matrices and are explicitly given by

$$\{\alpha_{ii}\} = \left[\frac{1}{L^3}c(i,1:n,3)\{w\}\right] = \frac{d^3w}{dx^3} \quad (33a)$$

$$\{\beta_{ii}\} = \left[\frac{1}{L^2}c(i,1:n,2)/\{w\}\right] = \frac{d^2w}{dx^2} \quad (33b)$$

Simplifying Eq. (32), we get

$$[G]\{D\} = \{F\} \quad (34)$$

Solving the above equation, one will be able to get $D = EI$ along the column.

Example 5. To find D for Clamped – Pinned column if P and buckled shape are known

For the case of Clamped - Pinned column ($P = 12b_2/L^2 = 0.12$) ($b2 = 1$; $P = 0.12$, $L = 10$) the buckled shape $\{w\}$ is given from the following equation

$$w(\xi) = 3\xi^2 - 5\xi^3 + 2\xi^4 \quad (35)$$

D value obtained from the closed form solution of Elishakoff *et al.* (2006) is

$$D(\xi) = |b_2|\left(\frac{5}{16} + \frac{5}{4}\xi - \xi^2\right) \quad (36)$$

D values are obtained using DQ at the sampling points and plotted in Fig. 3 and compared with those of Elishakoff *et al.* (2006) and the comparison is quite good.

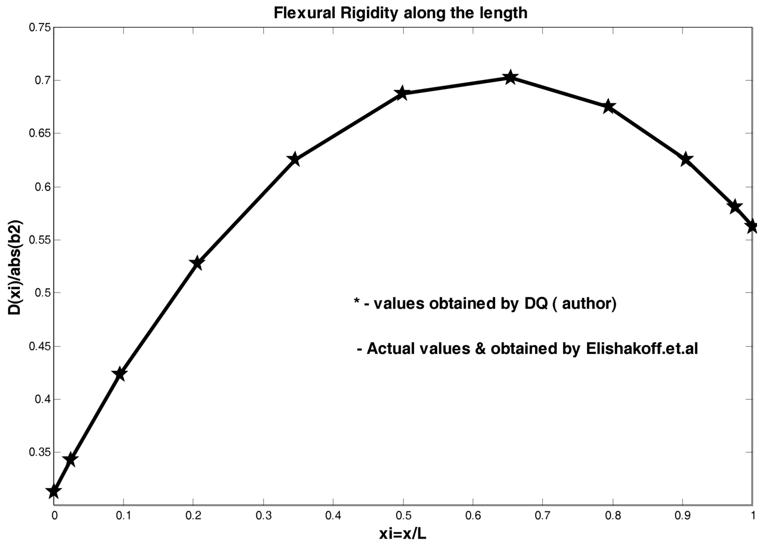

Fig. 3 Variation of $D(xi)$ by DQM compared with the authors (clamped-pinned) (Example.5)

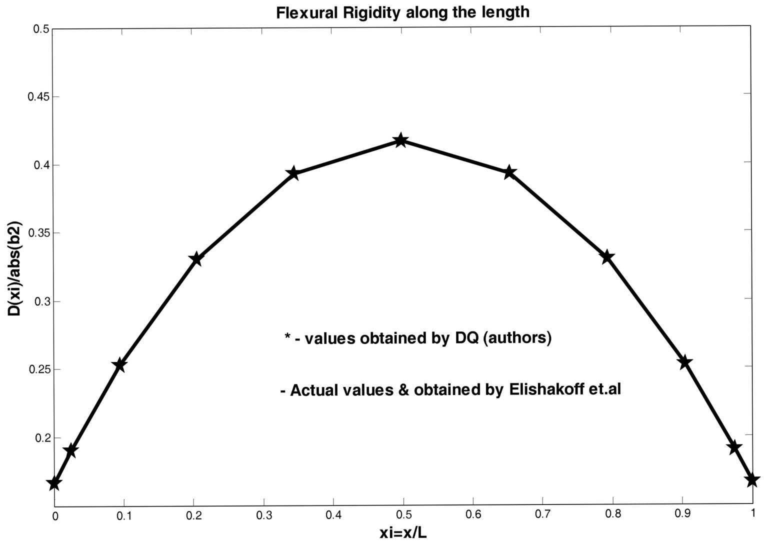

Fig. 4 Variation of D(xi) by DQM compared with the authors (clamped-clamped) (Example.7)

Example 6. To find D for Pinned – Pinned column if P and buckled shape are known

For the case of Pinned – Pinned column (P = 0.0986, L = 10) if the buckled shape $\{w\}$ is given as $w = \sin(\pi x/L)$ we get $D = 1$ throughout the length by applying DQ method.

Example 7. To find D for Clamped Clamped column if P and buckled shape are known

For the case of Clamped – Clamped column (P = 0.12, $b2$ = 1, L = 10) the buckled shape is given from the following equation

$$w = \xi^2 - 2\xi^3 + \xi^4 \tag{37}$$

D value obtained from the closed form solution of Elishakoff *et al.* (2006) is

$$D(\xi) = |b_2|\left(\frac{1}{6} + \xi - \xi^2\right) \tag{38}$$

D values are obtained at the sampling points and plotted in Fig. 4 and compared with those of the Elishakoff *et al.* (2006) and the comparison is quite good.

4.7 To find D = EI given buckling load and buckled shape of partially embedded pile.

Eq. (31) is modified for beam on partial elastic foundation as

$$\left(\frac{d^4w}{dx^4}D\right) + 2\left(\frac{d^3w}{dx^3}\frac{dD}{dx}\right) + \left(\frac{d^2w}{dx^2}\right)\left(\frac{d^2D}{dx^2}\right) = -P\frac{d^2w}{dx^2} - K_s w \tag{39a}$$

Applying DQ method Eq. (39a) is written as

$$\begin{aligned}&\{[c(:,:,4)/L^4] + 2[\alpha][c(:,:,1)/L] + [\beta][c(:,:,2)/L^2]\}[D]\\ &= -P[c(:,:,2)/L^2]\{w\} - [K_s]\{w\} = \{\bar{F}\}\end{aligned} \tag{39b}$$

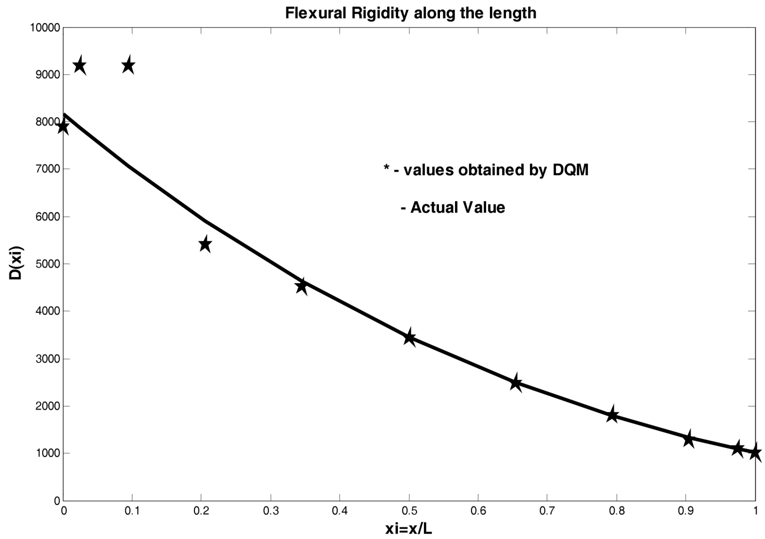

Fig. 5 Variation of $D(xi)$ by DQM compared with actual values (partially embedded pile- Example. 8)

As $K_s = S_r$ (dia), also depends on diameter of the pile at any sampling point ($dia = (8D/E\pi t)^{1/3}$) Eq. (38) is a non-linear equation which can be solved by iteration. Initially diameter of embedded pile at sampling points are assumed and K_s is obtained and solving the Eq. (39b) will give D and hence diameter at any sampling point. Two or three iterations will yield fairly accurate results.

Example 8. To find D = EI for a non-prismatic pile for Example 4

Here buckled shape in Fig. 2 as well as buckling load and K_s are input to the program and $D = EI$ is calculated at the sampling points. Fig. 5 shows the comparison of D obtained by DQ method with the actual values. Except nearer to the base the comparison is quite good. Since it is very sensitive to buckled shape hence this numerical error.

5. Differential transformation method (DT)

The concept of DT was first introduced some thirty years ago by Pukhov (Chai and Wang 2006). Since then, DT has been used with success in structural mechanics (Bert and Zeng 2000, Chen and Ho 1999, Li 2000, Malik and Allali 2000). The concept of DT is readily available in (Chai and Wang 2006). For a function $w(x)$, differential transformation exists as

$$W[k] = \frac{1}{k!}\left[\frac{d^k w(x)}{dx^k}\right]_{k=0}; \quad 0 \le x \le 1 \tag{40a}$$

where $w(x)$ can be regarded as buckled shape of the piles. By inverse transformation, one can also obtain $w(x)$ as

$$w(x) = \sum_{k=0}^{\infty} x^k W[k] \tag{40b}$$

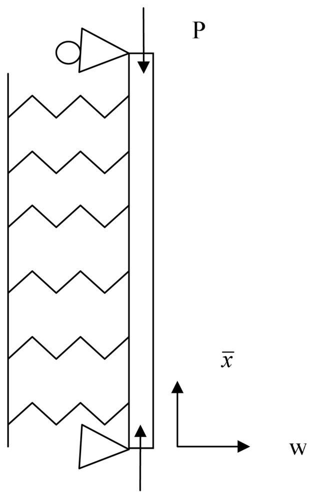

Fig. 6 Mathematical model for fully supported pile

or

$$w(x) = \sum_{k=0}^{\infty} \frac{x^k}{k!}\left[\frac{d^k w(x)}{dx^k}\right]_{x=0} \tag{41}$$

Eq. (41) is obviously a Taylor series expansion of the function $w(x)$ about $x = 0$. The differential technique essentially converts a differential equation into an algebraic equation, similar to integral transform methods such as Laplace and Fourier Transform methods. Final resulting algebraic equations are solved together with boundary conditions.

Example 1. Application to Stability of Fully supported Heavy Pile

The governing differential equation for a fully supported heavy uniform pile shown in Fig. 6 is given by

$$EI\frac{d^4 w}{d\bar{x}^4} + P\frac{d^2 w}{d\bar{x}^2} + sw - \rho\bar{x}\frac{d^2 w}{d\bar{x}^2} - \rho\frac{dw}{d\bar{x}} = 0 \quad 0 \le \bar{x} \le L \tag{42}$$

where EI- flexural rigidigy, P- axial compressive load, s = sub grade reaction of the soil, ρ- self weight of the pile/unit length and w- buckled shape of the pile.

Substituting $y = \frac{w}{L}$; $x = \frac{\bar{x}}{L}$ Eq. (42) may be written as

$$\frac{EI}{L^3}\frac{d^4 y}{dx^4} + \frac{P}{L}\frac{d^2 y}{dx^2} + SLy - \rho x\frac{d^2 y}{dx^2} - \rho\frac{dy}{dx} = 0 \qquad 0 \le x \le 1 \tag{43}$$

Substituting
$$\alpha = \frac{PL^2}{EI}; \quad \beta = \frac{\rho L^3}{EI}; \quad S_p = \frac{sL^4}{EI} \tag{44}$$

Eq. (43) simplifies to

$$y^{IV} + \alpha y'' + S_p y - \beta x y'' - \beta y' = 0 \tag{45}$$

Using the following definitions of DT

$$y^{IV} = (k+1)(k+2)(k+3)(k+4)Y[k+4] \tag{46a}$$

$$y'' = (k+1)(k+2)Y[k+2] \tag{46b}$$

$$y = Y[k] \tag{46c}$$

$$xy'' = k(k+1)Y[k+1] \tag{46d}$$

$$y' = (k+1)Y[k+1] \tag{46e}$$

Substituting Eq. (46) in Eq. (45) yields

$$\frac{Y[k+4] = \{\beta(k+1)Y[k+1] - \alpha(k+2)Y[k+2] - S_p Y[k]/(k+1)\}}{\{(k+2)(k+3)(k+4)\}} \tag{47}$$

Case-1 Pin roller support

Boundary conditions are

$$y(0) = y''(0) = y(1) = y''(1) = 0 \tag{48}$$

This can be interpreted in terms of DT as

$$Y[0] = 0; \quad Y[2] = 0; \quad Y[1] = c; \quad Y[3] = d \tag{49a}$$

and

$$y(1) = 0 \quad i.e \quad \sum_{k=0}^{\infty} Y[k] = 0 \tag{49b}$$

$$y''(1) = 0 \quad i.e \quad \sum_{k=0}^{\infty} k(k-1)Y[k] = 0 \tag{49c}$$

All other DT coefficients such as $Y[4]$, $Y[5]$.... can be written in terms of c and d.

Eqs. (49b) and (49c) simplify to

$$\begin{bmatrix} aa & bb \\ cc & dd \end{bmatrix} \begin{Bmatrix} c \\ d \end{Bmatrix} = \{0\} \quad i.e \quad [A]\begin{Bmatrix} c \\ d \end{Bmatrix} = \{0\} \tag{50}$$

where aa, bb are the coefficients of c and d in Eq. (49b) and cc and dd are the coefficients of c and d in Eq. (49c). For numerical solution to exist, the determinant $\|A\|$ has to be zero and hence α value or P value can be found out once β and S_p are given. One has to include more terms say more than 35 in Eq. (49) for accuracy.

Table 1 shows the α values obtained for various values of β and S_p. It is seen that the values obtained by DT compare very well with Differential quadrature method. Buckled shape could also be obtained by

Table 1 Buckling coefficient for various values of β and S_p
(values in brackets obtained from Differential Quadrature method)

β \ S_p	0	10	50	100
0	9.8696 (9.8696)	10.8826	14.9357	20.0017
10	14.6983 (14.698)	15.7052	19.7395	24.7723
50	30.9207 (30.908)	31.8489	35.5258	40.0248
100	46.3778 (46.379)	48.0752	50.0696	53.9468

$$y(x) = \sum_{k=0}^{\infty} x^k Y[k] \tag{51}$$

assuming $c = 1$ find $d = -aa/bb$.

Similarly buckling load can be calculated for a fully supported uniform pile with different boundary conditions.

For Fixed – Roller support (bottom roller and top-fixed)

$$Y[0] = 0;\ Y[1] = c;\ Y[2] = 0;\ Y[3] = d \quad \text{at roller support} \tag{52}$$

$$y(1) = \sum_{k=0}^{\infty} Y[k] = 0;\ y'(1) = \sum_{k=0}^{\infty} kY[k] = 0 \quad \text{at fixed support} \tag{53}$$

Example 2. Application of stability of non-prismatic pile

Consider non prismatic pile where moment of inertia varies as

$$I = I_1\{1 + x(r-1)\};\quad 0 \le x \le 1 \tag{54}$$

where r is the ratio of moment of inertia at the top of the pile I_2 to the bottom of the pile I_1. All the terms in the governing equation Eq. (43) remain the same except the first term because of non-prismatic nature of the pile. Now Eq. (43) is rewritten as (neglecting self of the pile)

$$\frac{E}{L^3}\frac{d^2}{dx^2}\left(I_1\frac{d^2y}{dx^2}\right) + \frac{P}{L}\frac{d^2y}{dx^2} + sLy = 0 \tag{55}$$

Eq. (55) simplifies to

$$\frac{EI_1}{L^3} + \left\{\frac{d^4y}{dx^4} + 2(r-1)\frac{d^3y}{dx^3} + (r-1)x\frac{d^4y}{dx^4} + \frac{P}{L}\frac{d^2y}{dx^2} + sLy = 0\right. \tag{56}$$

It can be proved that if $w(x) = xy^{iv}$; $W[k] = DT[w(x)]$ (57)

$$W[k] = k(k+1)(k+3)(k+3)Y[k+3] \tag{58}$$

Table 2 Buckling coefficient for various values of r and S_p (Values in brackets obtained from Differential Quadrature method)

S_p \ r	1	1.2	1.4	1.6	1.8	2.0
0	9.8696 (9.869)	10.8409 (10.849)	11.7859 (11.816)	12.7102 (12.77)	13.6244 (13.72)	14.6 (14.66)
10	10.8826	11.8538	12.7979	13.721	14.6071	Not obtained

Simplifying Eq. (56), we get

$$Y[k+4] = (1-r)(2+k)Y[k+3]/(k+4) - \alpha Y[k+2]/\{(k+3)(k+4)\}$$
$$- S_p Y[k]/(k+1)(k+2)(k+3)(k+4) \tag{59}$$

Table 2 shows the values of α for $S_p = 0$; and $S_p = 10$ (self weight of the pile is neglected). The values are compared with the results obtained by Differential quadrature methods and very good comparison is obtained.

Example 3. Partially restrained pile

Consider an axially loaded pinned – pinned pile (shown in Fig. 6) which is supported partially to half the height. The differential equation for bottom half of the pile is written as

$$y^{iv} + \alpha y'' + S_p y = 0 \quad \text{for} \quad 0 \le x \le 1 \tag{60}$$

where

$$\alpha = \frac{PL^2}{4EI}; \quad S_p = \frac{SL^4}{EI}; \quad y = \frac{2\bar{y}}{L}; \quad x = \frac{2\bar{x}}{L} \tag{61}$$

and L is the length of the pile with boundary condition at the bottom as

$$y(0) = y''(0) = 0 \quad \text{at} \quad x = 0 \tag{62}$$

The differential equation for the top half of the pile is written as

$$z^{iv} + \alpha z'' = 0 \quad \text{for} \quad 0 \le x \le 1 \tag{63}$$

where
$$z = \frac{2\bar{z}}{L}; \quad x = \frac{2(x - L/2)}{L} \tag{64}$$

with boundary conditions at the top of the pile as

$$z(1) = z''(1) = 0 \quad \text{at} \quad x = 1 \tag{65}$$

At half the height of the pile the compatibility condition requires

$$z(0) = y(1); \quad z'(0) = y'(1); \quad z''(0) = y''(1); \quad z'''(0) = y'''(1) \tag{66}$$

In view of Eq. (62), the differential transformation of the boundary condition $y(0)$ gives rise to

$$Y[0] = 0 \tag{67}$$

and the boundary condition $y''(0)$ gives rise to

$$Y[2] = 0 \tag{68}$$

$Y[1]$ and $Y[3]$ are assumed as c and d respectively.

The given differential equation Eq. (60) is converted into the following recursive equation as

$$Y[k+4] = -\frac{\alpha Y[k+2]}{(k+3)(k+4)} - \frac{S_p Y[k]}{(k+1)(k+2)(k+3)(k+4)} \tag{69}$$

for bottom half of the pile.

Using Eq. (69) $Y[k]$ ($k = 4 \ldots\ldots 35$) are calculated. At half the length of the pile

$$z[0] = y(1) = \sum_{k=0}^{35} Y[k]; \quad Z[1] = y'(1) = \sum_{k=0}^{35} kY[k]; \quad Z[2] = y''(1) = \sum_{k=0}^{35} k(k-1)Y[k]$$

$$Z[3] = y'''(1) = \sum_{k=0}^{35} k(k-1)(k-2)Y[k] \tag{70}$$

Knowing the values of $Z[0]$, $Z[1]$, $Z[2]$ and $Z[3]$ and using the following recursive equation at the top half of the pile as

$$Z[k+4] = -\frac{\alpha Y[k+2]}{(k+3)(k+4)} \tag{71}$$

$Z[4] \ldots\ldots Z[N]$ can be obtained.

The boundary condition $z(1) = 0$ gives rise to

$$\sum_{k=0}^{35} Z[k] = 0 \tag{72}$$

and the boundary condition $z''(0)$ gives rise to

$$\sum_{k=0}^{35} k(k-1)Z[k] = 0 \tag{73}$$

The substitution of these recursive terms into boundary conditions in Eqs. (72) and (73) give two simultaneous equations as Eq. (50) and the α value is found out such that the determinant is zero. Buckling load P can be found out once S_p is given and one has to include at least 35 terms. Table 3 shows the value of the buckling load coefficient for various values of S_p for the column partially restrained to half the height.

Mathematica, a symbolic programming package is ideally suitable to solve the problems by using Differential transform methods. The program for solving fully supported non-prismatic pile is shown in Fig. 7.

Table 3 Buckling coefficient for partially restrained column

	0	1.0	2.0	3.0	4.0	5.0
Buckling koad coefft	1	1.184	1.359	1.552	1.758	1.983

```
..................................................................
Y[0]=0
Y[1]=C
Y[2]=0
Y[3]=D
* RATIO OF MI OF TOP TO BOTTOM OF THE PILE*
be=2
* no of terms to be taken*
nt=30
sp=0
For[k=0,k<nt,k++,{Y[k+4]=-a*Y[k+2]/(k+3)*(k+4))
+(1-be)*(2+k)*Y[k+3]/(k+4}-sp*Y[k]/((k+1)*(k+2)*(k+3)*(k+4))}]
s=0
For[k=0,k<nt,k++,{s=s+Y[k])]
aa=Coefficient[s,c]
bb=Coefficients[s,d]
t=0
For[k=0,k<nt,k++,{t=t+be*k*(k-1)*Y[k]}]
cc=Coefficient[t,c]
dd=Coefficient[t,d]
de=aa*dd-cc*bb
FindRoot[de==0,{a,{0,20}]
************************************************************************
```

Fig. 7 Program in Mathematica for buckling of non-prismatic pile

6. Conclusions

The DQ and HDQ are applied to solve for buckling load given EI and using similar approach to solve for EI given P and buckled shape for columns as well as piles partially or fully embedded in soil. DQ and HDQ approaches can very easily be extended to find the variation in cross section of a column, pile, plate or shell for a given buckling load and buckled shape or for a given frequency and mode shape. For finding the buckling loads or natural frequencies, unlike Rayleigh-Ritz methods DQ and HDQ do not need the construction of an admissible function that satisfies the boundary conditions a priori Accurate results are obtained for the problems even with a small number of discrete points used to discretize the domain. This approach is convenient for solving problems governed by the higher order differential equations and matrix operations could be performed using MATLAB with ease. It is also easy to write algebraic equations in the place of differential equations and application of boundary conditions is also an easy task. It is also explained in this paper how Lagrange multiplier method is used to convert rectangular matrix to square matrix by incorporating boundary conditions using Wilson's method. Results with high accuracy are obtained in all study cases and DQ and HDQ are computationally efficient. DQ and HDQ is straight forward that the same procedures can be easily employed for handling problems with the other boundary conditions.

In this paper, DT method is also highlighted and the usefulness of the method is demonstrated by solving stability analysis of fully supported prismatic and non-prismatic piles. It is also shown in this paper, how DT can be used to convert differential equation to a set of algebraic equations of recursive nature. It is also shown that together with boundary conditions these equations are solved for buckling load. Fairly large number of terms says 35 to 40 are required for convergence. DT is efficient and easy to implement particularly in symbolic program packages like Mathematica. Buckled shape also could be obtained using Eq. (51). It is expected that DQ, HDQ and DT will be more promising for further development into an efficient and flexible numerical techniques for solving practical engineering problems in future (Rajasekaran 2007).

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References

Bellman, R. and Casti, J. (1971), "Differential quadrature and long term Integration", *J. Math. Anal. Appl.*, **34**, 235-238.
Bert, C.W., Wang, X. and Striz, A.G. (1993), "Differential quadrature for static and free vibration analysis of anisotropic plates", *Int. J. Solids Struct.*, **30**, 1737-1744.
Bert, C.W., Wang, X. and Striz, A.G. (1994), "Static and free vibration analysis of beams and plates by differential quadrature method", *Acta Mech.*, **102**, 11-24.
Bert, C.W. and Malik, M. (1996), "Differential quadrature method in computational mechanics", *Appl. Mech. Rev.*, **49**(1), 1-28.
Bert, C.W. and Zeng, H. (2000), "Analysis of axial vibration of compound rods by differential transformation method", *J. Sound Vib.*, **275**, 641-647.
Bowles, J.E. (1996), *Foundation Analysis and Design*, McGraw-Hill Book Co., New York.
Chai., Y.H. and Wang, C.M. (2006), "An application of differential transformation to stability analysis of heavy columns", *Int. J. Struct. Stab. D.*, **6**(3), 317-332.
Chen, C.K. and Ho. S.H. (1999), "Transverse vibration of rotating and twisted Timoshenko beams under axial loading using differential transform", *Int. J. Mech. Sci.*, **41**, 1339-1356.
Elishakoff, I., Gentilini, C. and Santoro, R. (2006), "Some conventional and unconventional educational column stability problems", *Int. J. Struct. Stab. D.*, **6**(1), 139-151.
Li, Q.S. (2000), "Exact solutions for free longitudinal vibrations of non-uniform rods", *J. Sound. Vib.,* **234**(17), 1-19.
Malik, M. and Allali, M. (2000), "Characteristic equations of rectangular plates by differential transformation", *J. Sound. Vib.*, **233**(2), 359-366.
Rajasekaran, S. (2007), "Symbolic computation and differential quadrature method – A boon to engineering analysis", *Struct. Eng.Mech.*, **27**(6), 713-739.
Shu, C. (2000), *Differential Quadrature and Its Application in Engineering*, Berlin, Springer.
Striz, A.G., Wang, X. and Bert, C.W. (1995), "Harmonic diferential quadrature method and applications to analysis of structural components", *Acta Mech.*, **111**, 85-94.
Ulker, M. and Civalek, O. (2004), "Application of Harmonic Differential Quadrature (HDQ) to deflection and bending analysis of beams and plates", F.U. Fen v Muhendislik Bilimleri Dergisi, **16**(2), 221-231.
Wang, C.K. (1970), *Matrix Methods of Structural Analysis*, 2nd Edition, Intext Educational publishers, Scranton, PA.
Wilson, E.L. (2002), *Three Dimensional Static and Dynamic Analysis of Structures*, Computers and Structures, Inc, Berkeley, California.
Terzaghi, K. (1955), "Evaluation of coefficient of sub-grade reaction", *Geotechnique,* **5**(4), 297-326.