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Technical Note

A general fourth order ordinary differential equation with solution in terms of Bessel functions: theory and engineering applications

Reza Attarnejad† and Amir K. Ghorbani-Tanha‡

School of Civil Engineering, University of Tehran, P.O. Box 11365-4563, Tehran, Iran

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1. Introduction

Bessel functions which are the solutions of Bessel's differential equation arise in numerous problems of mathematical physics and engineering, especially in cylindrical and spherical coordinates. Bessel functions and closely related functions form a rich area of mathematical analysis (Arfken and Weber 2001).

The problem of transverse vibration of non-uniform beams and determination of their natural frequencies has been extensively studied by engineers and mathematicians. This problem leads to a fourth order ordinary differential equation which generally has not a closed-form solution. However, for some special cases, this differential equation can be reduced to another form that can then be solved in terms of Bessel functions. Although these special cases are considered by the researchers (Mabie and Rogers 1964, 1968, 1974, Goel 1976, Yang 1990, Auciello and Ercolano 1997), the general form of the aforementioned differential equation is not yet presented.

In this paper the general form of a fourth order ordinary differential equation is derived whose solution, analogous to Bessel's differential equation, may be expressed in terms of Bessel functions. In order to confirm the exactness of derivation, two available particular cases are recovered. This simple form facilitates its application to other similar cases which may appear in engineering or mathematical problems.

2. General form of fourth order ordinary differential equation which admits solution in terms of Bessel functions

Performing some mathematical manipulations, from Watson (1958), the general form of a fourth order ordinary differential equation whose solution may be expressed in terms of Bessel functions is derived as

† Assistant Professor
‡ Ph.D. Candidate, Corresponding author, E-mail: ghtanha@ut.ac.ir

$$z^4u^{IV}(z) + (\Theta + 6)z^3u'''(z) + (3\Theta + \Psi + 7)z^2u''(z) \\ + (\Theta + \Psi + \Phi + 1)zu'(z) + \Omega u(z) = q^4c^4z^{4q}u(z) \tag{1}$$

in which

$$\Theta = A + B + C + D, \ \Psi = AB + AC + AD + BC + BD + CD \\ \Phi = ABC + ABD + ACD + BCD, \ \Omega = ABCD \tag{2}$$

where

$$A = p, \quad B = p - 2q, \quad C = p - 2q\nu, \quad D = p - 2q\nu - 2q \tag{3}$$

and a prime denotes a derivative with respect to z.

The differential equation (1) admits a general solution of the form

$$u(z) = z^{q\nu - p}[\lambda_1 J_\nu(cz^q) + \lambda_2 Y_\nu(cz^q) + \lambda_3 I_\nu(cz^q) + \lambda_4 K_\nu(cz^q)] \tag{4}$$

where J_ν and Y_ν are Bessel functions of the first and second kind and of order ν respectively, and I_ν and K_ν are modified Bessel functions of the first and second kind and of order ν respectively. λ_1, λ_2, λ_3 and λ_4 are to be determined by imposing the boundary conditions. The details of derivation of Eqs. (1) and (4) are tedious and not presented herein.

3. Transverse vibration of non-uniform beams

The governing differential equation of motion for a non-uniform Euler-Bernoulli beam, regardless of shear deformations and rotational inertia, is as follows (Humar 1990)

$$\frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y(x,t)}{\partial x^2}\right) + \rho A(x)\frac{\partial^2 y(x,t)}{\partial t^2} = 0 \tag{5}$$

where E is Young's modulus; $I(x)$ is the moment of inertia; $y(x, t)$ is the transverse displacement; ρ is the mass density; $A(x)$ is the cross-sectional area; x is axial coordinate, and t is time. For the normal modes of vibration, the transverse displacement $y(x, t)$ may be written as

$$y(x,t) = f(x)(M\sin\omega t + N\cos\omega t) \tag{6}$$

where $f(x)$ is a function of x alone, defining the modal displacement under consideration, and ω is the natural frequency. Substituting Eq. (6) into Eq. (5), we obtain

$$\frac{d^2}{dx^2}\left(EI(x)\frac{d^2f(x)}{dx^2}\right) - \omega^2\rho A(x)f(x) = 0 \tag{7}$$

This fourth order differential equation along with boundary conditions represents an eigenvalue problem. A general closed-form solution of Eq. (7) is not known. However, for some particular cases this equation may be converted to the form of Eq. (1) which yields closed-form solution in terms of Bessel functions. These cases are discussed below.

3.1 Particular cases

Consider the transverse vibration of a non-uniform beam with linear variations of width and thickness. Supposing $b(x) = b_1 + (b_2 - b_1)x/l$ and $h(x) = h_1 + (h_2 - h_1)x/l$ where at $x = 0$, $b = b_1$ and $h = h_1$ and at $x = l$, $b = b_2$ and $h = h_2$

$$I(x) = \frac{b(x)h^3(x)}{12} = \frac{1}{12}\left[b_1 + \frac{b_2 - b_1}{l}x\right]\left[h_1 + \frac{h_2 - h_1}{l}x\right]^3 \tag{8a}$$

$$A(x) = b(x)h(x) = \left[b_1 + \frac{b_2 - b_1}{l}x\right]\left[h_1 + \frac{h_2 - h_1}{l}x\right] \tag{8b}$$

Substituting these values into Eq. (7) and letting $\xi = x/l$, yields (Mabie and Rogers 1972, 1974, Goel 1976, Auciello and Ercolano 1997)

$$\frac{d^4f}{d\xi^4} + 2\frac{d^3f}{d\xi^3}\left[\frac{3(\alpha - 1)}{1 + (\alpha - 1)\xi} + \frac{\beta - 1}{1 + (\beta - 1)\xi}\right] +$$

$$6\frac{d^2f}{d\xi^2}\left[\frac{(\alpha - 1)(\beta - 1)}{[1 + (\alpha - 1)\xi][1 + (\beta - 1)\xi]} + \frac{(\alpha - 1)^2}{[1 + (\alpha - 1)\xi]^2}\right] = \frac{(lk)^4 f}{[1 + (\alpha - 1)\xi]^2} \tag{9}$$

in which $\alpha = h_2/h_1$, $\beta = b_2/b_1$ and $k^4 = 12\rho\omega^2/Eh_1^2$.

For a beam of constant width and linearly varying thickness $(\beta = 1)$, by letting $1 + (\alpha - 1)\xi = \varphi$, Eq. (9) reduces to (Goel 1976)

$$\varphi^4\frac{d^4f}{d\varphi^4} + 6\varphi^3\frac{d^3f}{d\varphi^3} + 6\varphi^2\frac{d^2f}{d\varphi^2} = \left(\frac{lk}{\alpha - 1}\right)^4\varphi^2 f \tag{10}$$

which is a particular case of the differential Eq. (1) for $p = 1, q = 1/2$, $\nu = 1$ and $c = 2lk/(\alpha - 1)$. This equation has a general solution as

$$f = \varphi^{-1/2}\left[\lambda_1 J_1\left(\frac{2lk}{\alpha - 1}\sqrt{\varphi}\right) + \lambda_2 Y_1\left(\frac{2lk}{\alpha - 1}\sqrt{\varphi}\right) + \lambda_3 I_1\left(\frac{2lk}{\alpha - 1}\sqrt{\varphi}\right) + \lambda_4 K_1\left(\frac{2lk}{\alpha - 1}\sqrt{\varphi}\right)\right] \tag{11}$$

For a beam of equal taper ratios for the width and thickness ($\alpha = \beta$), by letting $1 + (\alpha - 1)\xi = \varphi$, Eq. (9) reduces to (Mabie and Rogers 1974)

$$\varphi^4\frac{d^4f}{d\varphi^4} + 8\varphi^3\frac{d^3f}{d\varphi^3} + 12\varphi^2\frac{d^2f}{d\varphi^2} = \left(\frac{lk}{\alpha - 1}\right)^4\varphi^2 f \tag{12}$$

which is another particular case of the differential equation (1) for $p = 2, q = 1/2$, $\nu = 2$ and $c = 2lk/(\alpha - 1)$. This equation admits a general solution as

$$f = \varphi^{-1}\left[\lambda_1 J_2\left(\frac{2lk}{\alpha - 1}\sqrt{\varphi}\right) + \lambda_2 Y_2\left(\frac{2lk}{\alpha - 1}\sqrt{\varphi}\right) + \lambda_3 I_2\left(\frac{2lk}{\alpha - 1}\sqrt{\varphi}\right) + \lambda_4 K_2\left(\frac{2lk}{\alpha - 1}\sqrt{\varphi}\right)\right] \tag{13}$$

These particular cases have been previously considered by Mabie and Rogers (1968, 1974) and have been restated by others (Goel 1976, Yang 1990, Auciello and Ercolano 1997). Here, they are recovered again as two particular cases of differential Eq. (1) to confirm that this equation together with its solution, Eq. (4), have been derived correctly.

4. Conclusions

A general form for a fourth order ordinary differential equation which admits a solution in terms of Bessel function is presented together with its solution. The application of this differential equation in transverse vibration of some particular cases of non-unifrom Euler-Bernoulli beams is demonstrated by recovering available solutions to particular cases. The derived simple form of this differential equation facilitates its application in these cases and other possible engineering or mathematical problems.

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