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Multi-stage design procedure for modal controllers of multi-input defective systems

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Abstract. The modal controller of single-input system cannot stabilize the defective system with positive real part of repeated eigenvalues, because some of the generalized modes are uncontrollable. In order to stabilize the uncontrollable modes with positive real part of eigenvalues, the multi-input system should be introduced. This paper presents a recursive procedure for designing the feedback controller of the multi-input system with defective repeated eigenvalues. For a nearly defective system, we first transform it into a defective one, and apply the same method to manage. The proposed methods are based on the modal coordinate equations, to avoid the tedious mathematic manipulation. As an application of the presented procedure, two numerical examples are given at end of the paper.

Keywords: design of the feedback controller; multiple-input systems; recursive design procedure; defective systems; nearly defective systems.

1. Introduction

The vibration control theory for non-defective system which has the complete eigenvectors to span the eigenspace has been well developed, and many important achievements have been obtained. For example, the conditions that the closed-loop eigenvectors have to satisfy to obtain the output feedback gain matrices and to enable the desired eigenvalue placements have been discussed (Kimura 1977). The techniques for synthesis of output feedback gains have been developed by (Srinathkumar 1978, Maghami and Juang 1990, Andry *et al.* 1983). Dissipative output feedback gain matrices were used to assign eigenproblem (Maghami and Gupta 1997). The measures of controllability and observability of the repeated modes are discussed (Liu *et al.* 1994), but it does not deal with the corresponding design of the feedback control laws. The standard design methods for feedback control laws can be found in Meirovitch 1990. Recent papers in this field include the robust state feedback design (Smagina and Brewer 2000), robust stability and performance (Ugrinovskii and Peterson 2001), and model reduction of uncertain system (Dolgin, and Zeheb 2005).

In actual engineering problems, such as general damping systems, flutter analysis of aeroelasticity, and so on, the system called defective system (Appendix A) does not have a set of complete eigenvectors to span the eigenspace (Shi and Zhu 1989, Chen and Xu 1992, Xu and Chen 1994).

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The defective characteristics of the dynamic analysis of mobility and graspability of general manipulation systems, and the consistent task specification for manipulation systems with general kinematics are discussed (Prattichizzo and Bicchi 1997, 1998). Thus, it is desirable to develop the control theory for the defective systems. Recently, the modal optimal control procedure for nearly defective systems (Appendix B), and the quantitative measurements of modal controllability and observability of defective and nearly defective systems were developed (Chen *et al.* 2001, Chen *et al.* 2001).

The defective systems differ from nondefective ones in that the state matrix **A** cannot be diagonalized, thus the standard methods for designing the feedback controllers cannot be used to deal with the modal control problems of the defective and nearly defective systems (Appendix A, B). For this reason, an approach for designing modal controllers for the defective and nearly defective system was developed, but the procedure for determining the gain vector is limited to the single-input system (Chen 2003). However, the modal controller of single-input system cannot stabilize the defective system with repeated eigenvalues possessing positive real part, because some of the generalized modes are uncontrollable. Therefore, It is necessary to introduce the multi-input system and corresponding procedure to determine the feedback gain matrix of the multi-input system. To this end, this study presents a multi-stage procedure to determine the gain matrix of the multi-input systems. The proposed method is based on the modal control equation to avoid the tedious mathematic manipulation. For the nearly defective system, we first transform it into a defective one, and then use the same way to proceed. The theory is demonstrated by two numerical examples to show the validity.

2. Feedback control design of the single-input systems with defective eigenvalues

Consider the single-input control system indicated by the following state equation

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}z(t) \tag{1}$$

A is the state matrix. $\mathbf{x}(t) \in \mathbf{R}^{n \times 1}$ is the state vector, $z(t)$ is the input, $\mathbf{b} \in \mathbf{R}^{n \times 1}$ is called the actuator distribution matrix, indicating the locations of control forces.

For the sake of simplicity, In Eq. (1), we assumed that the n eigenvalues of **A** are defective repeated ones, i.e., $m = n$, $\lambda_1 = \lambda_2 = \ldots = \lambda_m = \lambda$. The right and left modal matrices are expressed as $\mathbf{U}_m$ and $\mathbf{V}_m$, and ξ_m is the modal coordinate corresponding to the repeated eigenvalues.

Using the modal transformation, we obtain the modal control equations

$$\dot{\xi}_m = \mathbf{J}_m\xi_m + \mathbf{V}_m^H\mathbf{b}z_m(t) = \mathbf{J}_m\xi_m + \mathbf{p}_m z_m(t) \tag{2}$$

where

$$\mathbf{p}_m = \mathbf{V}_m^H\mathbf{b} \tag{3}$$

If the $m(=n)$ repeated eigenvalues are defective, the standard method for determining gain vector cannot be used for this special case (Meirovitch 1990).

If the direct state feedback control is used, the modal control force is given as follows

$$\mathbf{p}_m z_m(t) = \mathbf{V}_m^H \mathbf{b} \mathbf{g}_m^T \xi_m \tag{4}$$

where $\mathbf{g}_m = [GM_1, GM_2, \ldots, GM_m]^T$ is the gain vector.

Substituting Eq. (4) into Eq. (2), yields

$$\dot{\xi}_m = (\mathbf{J}_m + \mathbf{p}_m \mathbf{g}_m^T)\xi_m \tag{5}$$

Eq. (5) indicates that the effect of the input variables given by Eq. (4) is to change the Jordan matrix $\mathbf{J}_m$ into a new matrix $\mathbf{H}_m$, that is

$$\mathbf{H}_m = \mathbf{J}_m + \mathbf{p}_m \mathbf{g}_m^T \tag{6}$$

Denote the assigned new distinct eigenvalues as $\rho_j (j = 1, 2, \ldots, m)$ and corresponding eigenvectors as $\mathbf{w}_j$, we have the following eigenvalue problem

$$(\mathbf{J}_m + \mathbf{p}_m \mathbf{g}_m^T)\mathbf{w}_j = \rho_j \mathbf{w}_j, \quad (j = 1, 2, \ldots, m) \tag{7}$$

Since $\mathbf{w}_j \neq 0$, the eigen-determinant of the matrix is zero

$$\det(\mathbf{J}_m + \mathbf{p}_m \mathbf{g}_m^T - \rho_j \mathbf{I}) = 0 \tag{8}$$

Considering $\mathbf{p}_m = [p_1, p_2, \ldots, p_m]^T$, we have

$$\mathbf{p}_m \mathbf{g}_m^T = \begin{bmatrix} p_1 GM_1 & p_1 GM_2 & \ldots & p_1 GM_m \\ p_2 GM_1 & p_2 GM_2 & \ldots & p_2 GM_m \\ \vdots & \vdots & \ddots & \vdots \\ p_m GM_1 & p_m GM_2 & \ldots & p_m GM_m \end{bmatrix} \tag{9}$$

After introduction of Eq. (9) into Eq. (8), we have

$$\det \begin{bmatrix} \lambda - \rho_j + p_1 GM_1 & 1 + p_1 GM_2 & \ldots & p_1 GM_m \\ p_2 GM_1 & \lambda - \rho_j + p_2 GM_2 & \ldots & p_2 GM_m \\ \vdots & \vdots & \ddots & \vdots \\ p_m GM_1 & p_m GM_2 & \ldots & \lambda - \rho_j + p_m GM_m \end{bmatrix} = 0, \quad (j = 1, 2, \ldots, m) \tag{10}$$

Expanding Eq. (10), yields

$$(\lambda - \rho_j)^m \left[1 + \sum_{l=0}^{m-1} \sum_{s=1}^{m} (-1)^l \frac{p_{l+s} GM_s}{(\lambda - \rho_j)^{l+1}} \right] = 0, \quad (j = 1, 2, \ldots, m) \tag{11}$$

If $\rho_j \neq \lambda$, we have

$$-\sum_{l=0}^{m-1} \sum_{s=1}^{m} (-1)^l \frac{p_{l+s} GM_s}{(\lambda - \rho_j)^{l+1}} = 1, \quad (j = 1, 2, \ldots, m) \tag{12}$$

In order to obtain a convenient form, we introduce the following notations

$$\mathbf{F} = \begin{bmatrix} \mathbf{f}_1 \\ \mathbf{f}_2 \\ \vdots \\ \mathbf{f}_m \end{bmatrix} \tag{13}$$

where

$$\mathbf{f}_j = \left[\frac{(-1)^0}{(\rho_j - \lambda)}, \frac{(-1)^1}{(\rho_j - \lambda)^2}, \ldots, \frac{(-1)^{m-1}}{(\rho_j - \lambda)^m} \right], \quad (j = 1, 2, \ldots, m) \tag{14}$$

and

$$\mathbf{P} = \begin{bmatrix} p_1 & p_2 & \cdots & p_m \\ p_2 & p_3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ p_m & 0 & \cdots & 0 \end{bmatrix} \tag{15}$$

$$\mathbf{e} = \underbrace{[1, 1, \ldots, 1]}_{m}{}^T \tag{16}$$

With above notations, and considering the gain vector $\mathbf{g}_m = [GM_1, GM_2, \ldots, GM_m]^T$, we can write Eq. (12) in a compact matrix form

$$\mathbf{FPg}_m = \mathbf{e} \tag{17}$$

Hence, the gain vector $\mathbf{g}_m$ can be obtained by

$$\mathbf{g}_m = \mathbf{P}^{-1}\mathbf{F}^{-1}\mathbf{e} \tag{18}$$

This is the solution for the gain vector of the defective systems with repeated eigenvalues. The control law of the defective system is given by

$$z(t) = \mathbf{g}_m^T \xi_m \tag{19}$$

Using the modal transformation

$$\mathbf{x}(t) = \mathbf{U}_m \xi_m(t) \tag{20}$$

one has

$$\xi_m(t) = \mathbf{V}_m^H \mathbf{x}(t) \tag{21}$$

thus, Eq. (19) becomes

$$z(t) = \mathbf{g}_m^T \mathbf{V}_m^H \mathbf{x}(t) \tag{22}$$

When the eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ are distinct, the gain vector is denoted by $\mathbf{g}_d = [GD_1, GD_2, \ldots, GD_n]$, which can be computed by

$$GD_j = \prod_{k=1}^{n} (\rho_k - \lambda_j) \Big/ p_j \prod_{\substack{k \neq j \\ j=1}}^{n} (\lambda_k - \lambda_j) \qquad j = 1, 2, \ldots, n \tag{23}$$

where $\rho_k (k = 1, 2, \ldots, n)$ are assigned new eigenvalues, and $\lambda_j (j = 1, 2, \ldots, n)$ are eigenvalues associated with controllable modes. The eigenvalue diagonal matrix $\mathbf{\Lambda}_d$ is changed into $\mathbf{H}_d$

$$\mathbf{H}_d = \mathbf{\Lambda}_d + \mathbf{p}_d \mathbf{g}_d^T \tag{24}$$

where $\mathbf{p}_d = \mathbf{V}_d^T \mathbf{b}$. (Meirovitch 1990).

3. Recursive design procedure for modal controller of multi-input defective systems

The modal controller of single-input system discussed in the above section cannot stabilize the defective systems with positive real part of the repeated eigenvalues, if the generalized modes are uncontrollable. In order to stabilize the uncontrollable modes with positive real part of eigenvalues, the multi-input control system should be introduced. In this section, we deal with the design of the modal controller of multi-input systems given by state equations of the form

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{z}(t) \tag{25}$$

where $\mathbf{x}(t)$ is the $n \times 1$ state vector, $\mathbf{A}$ is the $n \times n$ state matrix, $\mathbf{B}$ is the $n \times r$ input matrix, and $\mathbf{z}(t)$ is the $r \times 1$ input vector.

If the input matrix $\mathbf{B}$ has the partitioned form

$$\mathbf{B} = [\mathbf{b}_1, \mathbf{b}_2, \ldots, \mathbf{b}_r] \tag{26}$$

then Eq. (25) can be written as

$$\dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \sum_{i=1}^{r} \mathbf{b}_i z_i(t) \tag{27}$$

If Eq. (27) is compared with Eq. (1), it is evident that each entry $z_i(t) (i = 1, 2, \ldots, r)$ of the input vector $\mathbf{z}(t)$ in Eq. (25) can be generated in the same manner as $z(t)$ in Eq. (1) to determinate the desired eigenvalue changes.

If $z_i(t)$ in Eq. (27) are written in the following form

$$z_i(t) = \mathbf{g}_{im}^T \mathbf{V}_{im}^H \mathbf{x}(t), \quad (i = 1, 2, \ldots, r) \tag{28}$$

where $\mathbf{g}_{im}$ are the gain vectors associated with the ith stage design.

According to Eq. (17), we can obtain the gain vector for the defective eigenvalues by setting the elements of $\mathbf{g}_{im}$ corresponding to the uncontrollable modes at zero in the following equation

$$\mathbf{F}_i \mathbf{P}_i \mathbf{g}_{im} = \mathbf{e} \tag{29}$$

where $\mathbf{P}_i$ is composed by

$$\mathbf{p}_{im} = \mathbf{V}_{im}^H \mathbf{b}_i \tag{30}$$

and $\mathbf{F}_{ij}$ has the form

$$\mathbf{F}_{ij} = \left[\frac{(-1)^0}{(\rho_{ij} - \lambda_i)}, \frac{(-1)^1}{(\rho_{ij} - \lambda_i)^2}, \ldots, \frac{(-1)^{m-1}}{(\rho_{ij} - \lambda_i)^m} \right] \tag{31}$$

where ρ_{ij} is used to denote the jth assigned new eigenvalue at the ith stage design.

If the eigenvalues are distinct, the elements of the corresponding gain vector $\mathbf{g}_{id}$ can be obtained by Eq. (23), that is

$$GD_j = \prod_{k=1}^{n} (\rho_{ik} - \lambda_{ij}) \Big/ p_{ij} \prod_{\substack{k \neq j \\ j=1}}^{n} (\lambda_{ik} - \lambda_{ij}) \qquad j = 1, 2, \ldots, n \tag{32}$$

where p_{ij} are the corresponding elements of $\mathbf{p}_{id}$.

$$\mathbf{p}_{id} = \mathbf{V}_{id}^H \mathbf{b}_i \tag{33}$$

and

$$z_i(t) = \mathbf{g}_{id}^T \mathbf{V}_{id}^H \mathbf{x}(t), \quad (i = 1, 2, \ldots, r) \tag{34}$$

In Eqs. (28), (30), and (33), $\mathbf{V}_{im}$ and $\mathbf{V}_{id}$ are the left modal matrix of the matrix $(\mathbf{A})_i$, which is given by the following recurrence relation

$$(\mathbf{A})_{i+1} = \mathbf{U}_i (\mathbf{A})_i \mathbf{V}_i^H, \quad i = 1, 2, \ldots, r-1 \tag{35}$$

where

$$(\mathbf{A})_i = \mathbf{A} \tag{36}$$

$$(\mathbf{A})_{i+1} = \mathbf{U}_i \begin{bmatrix} \mathbf{J}_i + \mathbf{p}_{im} \mathbf{g}_{im}^T & \vdots & \mathbf{0} \\ \ldots & \ldots & \ldots \\ \mathbf{0} & \vdots & \mathbf{\Lambda}_{id} + \mathbf{p}_{id} \mathbf{g}_{id}^T \end{bmatrix} \mathbf{V}_i^H \tag{37}$$

The details of the computations of recursive design procedure for modal controller of multi-input defective systems are shown in the next section's numerical examples.

4. Numerical examples

In order to illustrate the applications of the present procedure, two numerical examples of the defective system are given as follows.

Example 1.
Consider a system with the state matrix

$$\mathbf{A} = \begin{bmatrix} 17 & 0 & -25 \\ 0 & 3 & 0 \\ 9 & 0 & -13 \end{bmatrix}$$

The control matrix **B** in Eq. (25) for the multi-input control force vector $\mathbf{z}(t)$ is

$$\mathbf{B} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{bmatrix}$$

The eigenvalues of **A** are

$$\lambda_1 = \lambda_2 = 2.0, \; \lambda_3 = 3.0$$

The Jordan form matrix is

$$\mathbf{J} = \begin{bmatrix} 2 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

This system is defective (Appendix A). By the invariant subspace recursive procedure (Chen *et al.* 2001), the right and left modal matrices $\mathbf{U}_1$ and $\mathbf{V}_1$, can be obtained as follows

$$\mathbf{U}_1 = \begin{bmatrix} 0.857493 & -0.514496 & 0.000000 \\ 0.000000 & 0.000000 & 1.000000 \\ 0.514496 & 0.857493 & 0.000000 \end{bmatrix}$$

and

$$\mathbf{V}_1 = \begin{bmatrix} 0.857493 & -0.514496 & 0.000000 \\ 0.000000 & 0.000000 & 1.000000 \\ 0.514496 & 0.857493 & 0.000000 \end{bmatrix}$$

With the singular-value decomposition to obtain the modal controllability (Chen *et al.* 2001), it can be seen that the first and third modes of the system are controllable, and the second mode is uncontrollable.

In order to improve the defective characteristics of the original uncontrolled system, the new eigenvalues can be assigned as $\rho_1 = -2$ and the third eigenvalue is unchanged, such that the system is changed into nondefective one with distinct eigenvalues.

Because the second mode is uncontrollable, the modal control force is given by

$$\mathbf{p}_{1m} z_{1m}(t) = \mathbf{V}_{1m}^H \mathbf{b}_1 \mathbf{g}_{1m}^T \xi_m$$

where $\mathbf{V}_{1m}$ contains only the first 2 columns, $\mathbf{g}_{1m} = [GM_1, 0]^T$, $\xi_m = [\xi_1, \xi_2]^T$. Using Eq. (9), one has

$$\mathbf{p}_{1m}\mathbf{g}_{1m}^T = \begin{bmatrix} p_1 GM_1 & 0 \\ p_2 GM_1 & 0 \end{bmatrix}$$

The eigendeterminate (10) becomes

$$\det\begin{bmatrix} \lambda_1 - \rho_1 + p_1 GM_1 & 1 \\ p_2 GM_1 & \lambda_1 - \rho_1 \end{bmatrix} = 0$$

Expanding this equation, yields

$$(\lambda_1 - \rho_1 + p_1 GM_1)(\lambda_1 - \rho_1) - p_2 GM_1 = 0$$

or

$$(\lambda_1 - \rho_1)^2\left[1 - \frac{p_1 GM_1}{\rho_1 - \lambda_1} - \frac{p_2 GM_1}{(\rho_1 - \lambda_1)^2}\right] = 0$$

If $\rho_1 \neq \lambda_1$, we have

$$\frac{p_1 GM_1}{\rho_1 - \lambda_1} + \frac{p_2 GM_1}{(\rho_1 - \lambda_1)^2} = 1$$

It follows that the entry GM_1 of the gain vector $\mathbf{g}_{1m} = [GM_1, 0]^T$ is

$$GM_1 = \frac{(\rho_1 - \lambda_1)^2}{p_1(\rho_1 - \lambda_1) + p_2} = -3.109841$$

where $\rho_1 = -2$.

Since the third eigenvalue is unchanged, we can set $\mathbf{g}_{1d} = 0$, then the required control law can be obtained

$$z_1(t) = [\mathbf{g}_{1m}^T \vdots \mathbf{g}_{1d}^T]\mathbf{V}_1^H \mathbf{x}(t)$$

$$= \begin{bmatrix} -2.66667 \\ 0.00000 \\ -1.60000 \end{bmatrix}^T \mathbf{x}(t)$$

It may readily be verified that the state matrix of the close-loop system defined by

$$\mathbf{H}_1 = \begin{bmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix} + \begin{bmatrix} \mathbf{p}_{1m}\mathbf{g}_{1m}^T & \vdots & 0 \\ \cdots & \vdots & \cdots \\ 0 & \vdots & \mathbf{p}_{1d}\mathbf{g}_{1d}^T \end{bmatrix}$$

$$= \begin{bmatrix} -2.266667 & 1.000000 & 0.000000 \\ -1.066667 & 2.000000 & 0.000000 \\ 0.000000 & 0.000000 & 3.000000 \end{bmatrix}$$

Using Eq. (37), the state matrix, $(\mathbf{A})_2$, can be obtained

$$(\mathbf{A})_2 = \mathbf{U}_1\mathbf{H}_1\mathbf{V}_1^H$$

$$= \begin{bmatrix} -1.10784 & 0.00000 & -0.864707 \\ 0.00000 & 3.00000 & 0.000000 \\ 2.93138 & 0.00000 & 0.841174 \end{bmatrix}$$

The eigenvalues and modal matrices of $(\mathbf{A})_2$ are as follows

$$\lambda_{21} = 1.73333, \quad \lambda_{22} = 3.0, \quad \lambda_{23} = -2.0$$

$$\mathbf{U}_2 = \begin{bmatrix} 0.291162 & 0.000000 & -0.695973 \\ 0.000000 & 1.000000 & 0.000000 \\ -0.956674 & 0.000000 & -0.718068 \end{bmatrix}$$

$$\mathbf{V}_2 = \begin{bmatrix} 0.718068 & 0.000000 & -0.956674 \\ 0.000000 & 1.000000 & 0.000000 \\ -0.695973 & 0.000000 & -0.291162 \end{bmatrix}$$

The above results show that the eigenvalues of $(\mathbf{A})_2$ are distinct after the first state design.

In order to stabilized the first mode and the second mode of $(\mathbf{A})_2$, the first and the second eigenvalues can be assigned as $\rho_{21} = -4.0, \rho_{22} = -3.0$, and the λ_{23} is unchanged.

Using Eq. (23), the gain vector $\mathbf{g}_{2d}$ can be obtained

$$\mathbf{p}_{2d} = \mathbf{V}_{21}^H\mathbf{b}_2 = [0.022095 \quad 1.0 \quad -1.24784]^T$$

$$\mathbf{g}_{2d} = [969.653 \quad -33.1578 \quad 0.0]^T$$

The state matrix of the closed-loop system obtained by the second stage design is

$$\mathbf{H}_2 = \begin{bmatrix} \lambda_{21} & 0 & 0 \\ 0 & \lambda_{22} & 0 \\ 0 & 0 & \lambda_{23} \end{bmatrix} + \mathbf{p}_{2d}\mathbf{g}_{2d}^T$$

It may be verified that the eigenvalues of $\mathbf{H}_2$ are as required $\lambda_{31} = -4.0, \lambda_{32} = -3.0, \lambda_{33} = -2.0$.

The required control law can be obtained for the second stage

$$z_2(t) = \mathbf{g}_{2d}^T\mathbf{V}_2^H\mathbf{x}(t)$$

$$= \begin{bmatrix} 696.277 \\ -33.1578 \\ -678.852 \end{bmatrix}^T \mathbf{x}(t)$$

The multi-input vector is

$$\mathbf{z}(t) = \begin{Bmatrix} z_1 \\ z_2 \end{Bmatrix} = \begin{bmatrix} -2.666667 & 0.0 & -1.6 \\ 696.277 & -33.1578 & -674.852 \end{bmatrix} \mathbf{x}(t)$$

Example 2.

Assume that the state matrix presented in Example 1 is perturbed into

$$\mathbf{A} = \begin{bmatrix} 17 & 0 & -24.99999 \\ 0 & 3 & 0 \\ 9 & 0 & -13 \end{bmatrix}$$

The system has two close eigenvalues, i.e., $\mathbf{J} = \mathrm{diag}(\lambda_1, \lambda_2, \lambda_3)$, where

$$\lambda_1 = 2.009487\,, \quad \lambda_2 = 1.990513\,, \quad \lambda_3 = 3.0$$

Therefore, the system is nearly defective (Appendix B).

The algebra average of λ_1 and λ_2 is

$$\lambda_0 = \frac{1}{2}\sum_{i=1}^{2} \lambda_i = 2.0$$

The Jordan form matrix is

$$\mathbf{J} = \mathbf{J}_0 + \delta \mathbf{J}_0 = \begin{bmatrix} \lambda_0 & 1 \\ & \lambda_0 \end{bmatrix} + \begin{bmatrix} \lambda_1 - \lambda_0 & -1 \\ & \lambda_2 - \lambda_0 \end{bmatrix}$$

where $\delta = \max|\lambda_i - \lambda_0| = 0.009487,\ (i = 1, 2)$. Therefore, the control problem of a nearly defective system can be approximated by a defective one.

With recursive procedure, the right and left generalized modes $\mathbf{U}_1$ and $\mathbf{V}_1$, corresponding to λ_0 and λ_3 can be obtained as follows

$$\mathbf{U}_1 = \begin{bmatrix} 0.857493 & -5.14496 & 0.0000000 \\ 0.000000 & 0.000000 & 1.0000000 \\ 0.514496 & 0.857493 & 0.0000000 \end{bmatrix}$$

$$\mathbf{V}_1 = \begin{bmatrix} 0.857493 & -5.14496 & 0.0000000 \\ 0.000000 & 0.000000 & 1.0000000 \\ 0.514496 & 0.857493 & 0.0000000 \end{bmatrix}$$

It can be shown that the nearly defective system has been transformed into a defective one, which is the same as one of the Example 1. Thus, the required control law for this nearly defective system is the same as those obtained by Example 1.

5. Conclusions

The vibration control of the systems with repeated or close eigenvalues is an important problem in engineering. This paper focuses on the case of defective repeated eigenvalues and presents the design methods of the modal controller based on the generalized modal coordinates, to avoid the tedious mathematic manipulation. Because some of the generalized modes are uncontrollable, the multi-input vector is required to stabilize the modes of the defective eigenvalues with positive real part. For such case, the present recursive procedure is effective to compute the gain matrix. The conclusions are supported by two given numerical examples.

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Appendix A: The eigenvalue problem for defective system

This section presents a brief review on eigenvalue problem for defective system given by Gantmacher (2000) and Dief (1982).

Consider a matrix eigenvalue problem given by

$$\mathbf{A}\mathbf{u} = \lambda\mathbf{u} \tag{A.1}$$

Assume that the first m eigenvalues are repeated, and the rest $n-m$ eigenvalues are distinct. It is assumed that AM is used to denote the algebra multiplicity of the eigenvalue λ in Eq. (A.1), and GM is used to denote the number of the linear independent eigenvectors corresponding to λ.

If $AM = GM$, the system with distinct and repeated eigenvalues is non-defective, and if $AM > GM$, the system with repeated eigenvalues is defective (Gantmacher 2000, Dief 1982).

From the algebra theory for the defective matrix **A**, there exists a non-singular matrix **U** to produce

$$\mathbf{A}\mathbf{U} = \mathbf{U}\mathbf{J} \tag{A.2}$$

where **U** is the generalized modal matrix of **A**, **J** is the Jordan block of **A** given by

$$\mathbf{J} = \begin{bmatrix} \lambda & 1 & & & & & \\ & \lambda & 1 & & 0 & & \\ & & \ddots & 1 & & & \\ & & & \lambda & & & \\ & & & & \lambda_{m+1} & & \\ & 0 & & & & \ddots & \\ & & & & & & \lambda_n \end{bmatrix} \tag{A.3}$$

Eq. (A.3) can be written in the following form

$$\left.\begin{aligned} (\mathbf{A}-\lambda\mathbf{I})\mathbf{u}_1 &= 0 \\ (\mathbf{A}-\lambda\mathbf{I})\mathbf{u}_i &= \mathbf{u}_{i-1}, \quad i = 2, 3, \ldots, m \\ (\mathbf{A}-\lambda_j\mathbf{I})\mathbf{u}_j &= 0, \quad j = m+1, \ldots, n \end{aligned}\right\} \tag{A.4}$$

The conjugate and transpose of **A**, i.e., $\mathbf{A}^H$, is called adjoint system, the generalized modes satisfy the following equation

$$\mathbf{A}^H\mathbf{V} = \mathbf{V}\mathbf{J}^H \tag{A.5}$$

where $\mathbf{A}^H$ and $\mathbf{J}^H$ are the conjugate and transpose of **A** and **J** respectively, **V** is the generalized modal matrix of the $\mathbf{A}^H$.

Eq. (A.5) can be also written as follows

$$\left.\begin{aligned} (\mathbf{A}^H-\tilde{\lambda}\mathbf{I})\mathbf{v}_i &= \mathbf{v}_{i+1}, \quad i = 1, 2, \ldots, m-1 \\ (\mathbf{A}^H-\tilde{\lambda}\mathbf{I})\mathbf{v}_m &= \mathbf{0}, \\ (\mathbf{A}^H-\tilde{\lambda}_j\mathbf{I})\mathbf{v}_j &= \mathbf{0}, \quad j = m+1, \ldots, n \end{aligned}\right\} \tag{A.6}$$

where $\mathbf{u}_i$ and $\mathbf{v}_i$ are the right and the left generalized modes, respectively.

The right generalized modal matrix $\mathbf{U}$ and the left generalized modal matrix $\mathbf{V}$ satisfy the following orthogonal condition

$$\mathbf{V}^H\mathbf{U} = \mathbf{I} \tag{A.7}$$

Appendix B: The nearly defective system

The numerical analysis results show that if some changes of parameters in the defective system are made, the system with defective repeated eigenvalues can be perturbed into one with close eigenvalues and the corresponding eigenvectors to near parallel, which is called a nearly defective system. For such a special case from the viewpoint of mathematics, although the close eigenvalues are distinct, the dynamic characteristic of the system is still defective. Thus, the formula for obtaining the gain matrix in Eq. (23) of systems with the distinct eigenvalues cannot be used for the case of the nearly defective system. In addition, the formula for obtaining the gain matrix in Eq. (17) of system with repeated eigenvalues of defective system as discussed in the main text cannot be also used directly to deal with a nearly defective system.

Assume that n eigenvalues of $\mathbf{A}$ are close. The right modal matrix $\mathbf{U} = [\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_n]$, and the left modal matrix $\mathbf{V} = [\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n]$, satisfy the following equations

$$\mathbf{AU} = \mathbf{UJ}, \quad \mathbf{A}^H\mathbf{V} = \mathbf{VJ}^H \tag{B.1}$$

and the orthogonal condition

$$\mathbf{U}^H\mathbf{V} = \mathbf{VU}^H = \mathbf{I} \tag{B.2}$$

where $\mathbf{J} = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$.

Taking the algebra average of $\lambda_1, \lambda_2, \ldots, \lambda_n$

$$\lambda_0 = \frac{1}{n}\sum_{i=1}^{n}\lambda_i \tag{B.3}$$

and letting

$$\mathbf{J}_0 = \begin{bmatrix} \lambda_0 & 1 & & \\ & \lambda_0 & \ddots & \\ & & \ddots & 1 \\ & & & \lambda_0 \end{bmatrix} \tag{B.4}$$

$$\delta\mathbf{J}_0 = \begin{bmatrix} \lambda_1 - \lambda_0 & -1 & & \\ & \lambda_2 - \lambda_0 & \ddots & \\ & & \ddots & -1 \\ & & & \lambda_n - \lambda_0 \end{bmatrix} \tag{B.5}$$

the matrix $\mathbf{J}$ in Eq. (B.1) can be written in the following form

$$\mathbf{J} = \mathbf{J}_0 + \delta\mathbf{J}_0 \tag{B.6}$$

Considering the orthogonal condition (B.2), and substituting Eq. (B.5) into Eq. (B.1), yields

$$\begin{aligned}\mathbf{AU} &= \mathbf{U}\mathbf{J}_0\mathbf{V}^H + \mathbf{U}\,\delta\mathbf{J}_0\mathbf{V}^H \\ &= \mathbf{A}_r + \delta\mathbf{A}\end{aligned} \tag{B.7}$$

where

$$\mathbf{A}_r = \mathbf{U}\mathbf{J}_0\mathbf{V}^H, \quad \delta\mathbf{A} = \mathbf{U}\,\delta\mathbf{J}_0\mathbf{V}^H \tag{B.8}$$

If $\lambda_1, \lambda_2, \ldots, \lambda_n$ are the close eigenvalues and $\delta = \max|\lambda_i - \lambda_0|$, it can be shown that the error matrix, $\delta\mathbf{A} = \mathbf{U}\,\delta\mathbf{J}_0\mathbf{V}^H$, is a small perturbational one and its norm satisfies

$$\begin{aligned}\|\delta\mathbf{A}\|_2 &\le \|\mathbf{U}\|_2\|\delta\mathbf{J}_0\|_2\|\mathbf{V}^H\|_2 \\ &\le \|\delta\mathbf{J}_0\|_2 \le \delta^2\end{aligned} \tag{B.9}$$

Since the eigenvalues of $\mathbf{J}_0$ cannot be changed by the orthogonal transformation, the eigenvalues of $\mathbf{A}$ are identical with those of $\mathbf{J}_0$. Eq. (B.7) indicates that the matrix $\mathbf{A}$ is equal to the sum of the defective matrix $\mathbf{A}_r$ with n repeated eigenvalues and the perturbed matrix $\delta\mathbf{A}$, and the right and the left modal matrices $\mathbf{U}$ and $\mathbf{V}^r$ of $\mathbf{A}_r$ are the same as those of $\mathbf{A}$. Therefore, Eq. (B.1) can be written as follows

$$\mathbf{AU} = \mathbf{U}(\mathbf{J}_0 + \delta\mathbf{J}_0), \quad \mathbf{A}^H\mathbf{V} = \mathbf{V}(\mathbf{J}_0 + \delta\mathbf{J}_0)^H \tag{B.10}$$

Using the modal transformation

$$\mathbf{x}(t) = \mathbf{U}\xi(t) \tag{B.11}$$

if δ is sufficiently small, the control Eq. (B.10) can be approximated by

$$\dot{\xi}(t) = \mathbf{J}_0\xi(t) + \mathbf{V}^H\mathbf{Bz}(t) \tag{B.12}$$

Eq. (B.12) shows that the control problem of the nearly defective system with close eigenvalues can be transformed into one of the defective system with repeated eigenvalues, which are equal to the average value of the close eigenvalues.