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Exact buckling load of a restrained RC column

Nana Krauberger†

Vegrad, d.d., Stari trg 35, 3320 Velenje, Slovenia

Miran Saje‡, Igor Planinc‡† and Sebastjan Bratina‡‡

University of Ljubljana, Faculty of Civil and Geodetic Engineering, Jamova 2, SI-1115 Ljubljana, Slovenia

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Abstract. Theoretical foundation for the buckling load determination in reinforced concrete columns is described and analytical solutions for buckling loads of the Euler-type straight reinforced concrete columns given. The buckling analysis of the limited set of restrained reinforced concrete columns is also included, and some conclusions regarding effects of material non-linearity and restrain stiffnesses on the buckling loads and the buckling lengths are presented. It is shown that the material non-linearity has a substantial effect on the buckling load of the restrained reinforced concrete columns. By contrast, the steel/concrete area ratio and the layout of reinforcing bars are less important. The influence on the effective buckling length is small.

Keywords: reinforced concrete; restrained column; stability; buckling load.

1. Introduction

Stability of structures is an important subject of engineers since a series of disastrous structural collapses took place due to neglecting or misunderstanding theoretical aspects of stability during their design (Bažant and Cedolin 1991). In contrast to steel and timber structures, typical reinforced concrete structures are not slender and that is why an engineer is often prone to assume that the stability check can be omitted. Experiments have, however, revealed that instability is an important mode of collapse even for reinforced concrete structures (ACI-ASCE Committee 441 1966, Bažant and Xiang 1997, Kim and Yang 1995). With certainty, the buckling load must not be overlooked in prefabricated reinforced concrete structures or when structures are subjected to fire (Bratina *et al.* 2005).

The stability analysis in structural mechanics starts in 18th century with Euler's studies on the buckling load of an elastic column (Euler 1774). Experiments on columns, however, did not entirely verify his findings on critical loads. The disagreement was found to be due to presence of

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† Research Assistant
‡ Professor, Corresponding author, E-mail: msaje@fgg.unilj.si
‡† Associate Professor
‡‡ Assistant Professor

geometrical and material imperfections and the eccentricity of applied external loads (Bažant and Cedolin 1991). At the time being, it is believed that the buckling theory of elastic structures is, in principle, well-formed (see Aristizabal-Ochoa 1997, Bažant and Cedolin 1991, Essa 1998, Gadalla and Abdalla 2006, Liu and Xu 2005, Mahini and Seyyedian 2006). Unfortunately, the assumption that the material behaviour is linear elastic is not realistic for reinforced concrete structures (ACI-ASCE Committee 441 1966, Bažant and Xiang 1997, Kim and Yang 1995).

The stability analysis of materially non-linear structures is much more involved compared to the elastic one. The issue of a prime complication is the irreversibility of inelastic deformations. It has been found that the irreversal (plastic) deformations have a significant influence on behaviour of columns subjected to an axial compression force. Two stability modes, which could not be observed in linear elastic columns, can occur: (i) the continuous set of bifurcation points on the primary load-deflection curve, and (ii) not necessarily the loss of column stability at any of the bifurcation points on the primary load-deflection curve. The fact that the column must buckle at the least bifurcation load was established by the elastic buckling analysis (Battini 1999, Hutchinson 1974). But a new feature of elastoplastic buckling is that the column is not at its stability limit at the bifurcaton point, which means that the deflected post-bifurcation states can be stable. Engesser (1889) was the first to study theoretically the stability of elasto-plastic columns. He found that the material non-linearity can substantially lower the critical load. He suggested that the critical load of an inelastic column is obtained from Euler's formulae, in which the elastic modulus is replaced with an inelastic tangent modulus for loading. The further research of the subject lead him to reject the idea of the loading tangent inelastic modulus, and to introduce the reduced modulus instead as a proper combination of loading and unloading moduli. The investigations performed much later showed that Engesser's original proposal of the loading tangent modulus must be taken into consideration in the case of elasto-plastic columns rather than the reduced modulus (Shanley 1947). Today, Engesser's original theory is generally accepted (Bažant and Cedolin 1991, Groper and Kenig 1987, Wang *et al.* 2005).

The present paper provides the application of the standard stability analysis to restrained reinforced concrete columns with symmetric cross-sections. Such restrained columns are often used in practice to model the columns of totally braced, partially braced and unbraced elastic frames (Aristizabal-Ochoa 1997, Gantes and Mageirou 2005, Mageirou and Gantes 2006, Mahini and Seyyedian 2006, Liu and Xu 2005). The geometrically exact beam theory of Reissner (Reissner 1972) is employed as the theoretical basis. Both flexural and axial deformations of the column are taken into account in the analysis, and behaviour of both concrete and steel is assumed non-linear. Shear deformations and imperfections are, however, neglected.

2. Theory

2.1 Basic equations

We consider a restrained straight reinforced concrete column of initial, undeformed length L and constant cross section A. The ends of the column are restrained. ρ^1 and ρ^2 represent the rotational spring constants and μ_V denotes the translational spring constant against lateral translation. The column is centricaly loaded (Fig. 1). If we assume that the x-axis coincides with the centroidal axis of the undeformed column, and employ Reissner's model of the beam (Reissner 1972), yet neglecting shear strains, the following governing equations of the column can be derived (Planinc *et al.* 2001)

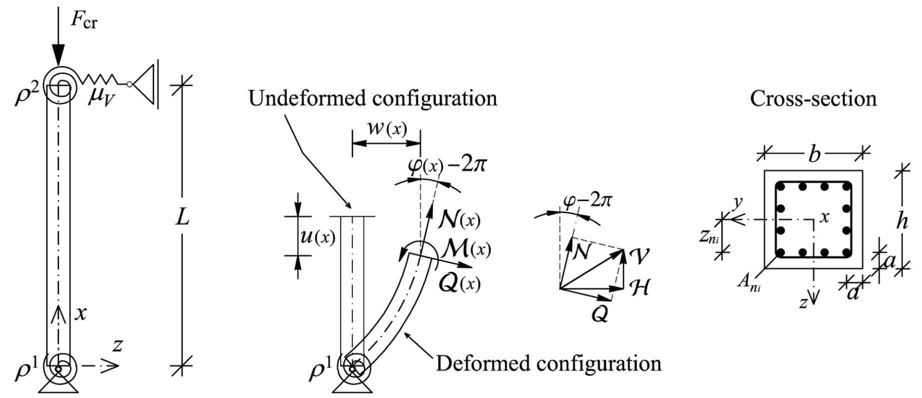


Fig. 1 Deformed and undeformed configuration of a restrained RC column. Reinforced concrete cross-section

$$f_1 = 1 + u' - (1+\varepsilon)\cos\varphi = 0 \tag{1}$$

$$f_2 = w' + (1+\varepsilon)\sin\varphi = 0 \tag{2}$$

$$f_3 = \varphi' - \kappa = 0 \tag{3}$$

$$f_4 = \mathcal{H}' = 0 \tag{4}$$

$$f_5 = \mathcal{V}' = 0 \tag{5}$$

$$f_6 = \mathcal{M}' - (1+\varepsilon)\mathcal{Q} = 0 \tag{6}$$

$$f_7 = \mathcal{N}_c - \mathcal{N} = 0 \tag{7}$$

$$f_8 = \mathcal{M}_c - \mathcal{M} = 0 \tag{8}$$

Here the prime denotes the derivative with respect to x; $u(x)$ and $w(x)$ are the displacements of the centroidal axis in the x and z directions, and $\varphi(x)$ is its cross-section rotation about the y-axis (Fig. 1). The deformation quantities, $\varepsilon(x)$ and $\kappa(x)$, are the extensional strain of the centroidal axis (the membrane deformation) and its pseudocurvature (the flexural deformation), respectively. The extensional strain, D, of an arbitrary fibre $z = const.$ is a function of ε and κ, and it is given by

$$D = \varepsilon + z\kappa \tag{9}$$

The functions $\mathcal{H}(x)$ and $\mathcal{V}(x)$ are the equilibrium stress-resultants in the x and z directions, and $\mathcal{M}(x)$ is the equilibrium bending moment of the cross-section. The equilibrium internal forces $\mathcal{N}(x)$ and $\mathcal{Q}(x)$, which are given with respect to the rotated basis (Fig. 1), are related to $\mathcal{H}$ and $\mathcal{V}$ by the equations

$$\mathcal{N} = \mathcal{H}\cos\varphi - \mathcal{V}\sin\varphi \tag{10}$$

$$\mathcal{Q} = \mathcal{H}\sin\varphi + \mathcal{V}\cos\varphi \tag{11}$$

In addition to the equilibrium internal forces, Eqs. (7)-(8) introduce the constitutive internal axial force, $\mathcal{N}_c(x)$, and the constitutive internal bending moment, $\mathcal{M}_c(x)$. These constitutive quantities depend on the material models of concrete and steel and are given by the equations

$$\mathcal{N}_c = \int_{\mathcal{A}_c} \sigma_c \mathrm{d}A + \sum_{n_i = 1}^{n} \sigma_s A_{n_i} \tag{12}$$

$$\mathcal{M}_c = \int_{\mathcal{A}_c} z\sigma_c \mathrm{d}A + \sum_{n_i = 1}^{n} z_{n_i} \sigma_s A_{n_i} \tag{13}$$

where A_{n_i} denotes areas of reinforcing bars $n_i = 1, 2, \ldots, n$ in the reinforced concrete column, z_{n_i} $(n_i = 1, 2, \ldots, n)$ denote z-coordinates of the centroids of the reinforcing bars with respect to the concrete cross-section centroid (Fig. 1). σ_c is the normal stress in concrete and σ_s in steel; they depend on the corresponding strain, D, and are given by constitutive laws of concrete and steel. $\mathcal{A}_c$ denotes the region occupied by concrete. The material parameters of the constitutive laws must be determined experimentally for the particular material under consideration (Bratina *et al.* 2004). In practice, they are nearly always taken from the database given in the building codes (see Section 3) rather than performing experiments, thus introducing further inaccuracy in the analysis.

The kinematic and static boundary conditions for the columns under consideration are (Fig. 1)

$$u(0) = 0 \tag{14}$$

$$w(0) = 0 \tag{15}$$

$$s_1^1 \mathcal{M}(0) - \rho^1 \varphi(0) = 0 \tag{16}$$

$$\mathcal{H}(L) + F = 0 \tag{17}$$

$$s_1^2 \mathcal{V}(0) + \mu_V w(L) = 0 \tag{18}$$

$$s_2^2 \mathcal{M}(L) + \rho^2 \varphi(L) = 0 \tag{19}$$

A set of parameters s_1^1, s_1^2 and $s_2^2 \in \{0, 1\}$ and end restraints μ_V, ρ^1 and ρ^2 determines different combinations of the boundary conditions for restrained reinforced concrete columns. Eqs. (1)-(11) and (14)-(19) constitute a system of sixteen non-linear algebraic-differential equations for sixteen unknown functions and parameters.

2.2 Linearized buckling analysis

The buckling load of the column is obtained from the linear theory of stability (Keller 1970). In line with this theory, we linearize Eqs. (1)-(8) of the column to obtain

$$\delta f_1 = \delta u' + (1 + \varepsilon)\sin\varphi\delta\varphi - \cos\varphi\delta\varepsilon = 0 \tag{20}$$

$$\delta f_2 = \delta w' + (1 + \varepsilon)\cos\varphi\delta\varphi - \sin\varphi\delta\varepsilon = 0 \tag{21}$$

$$\delta f_3 = \delta\varphi' - \delta\kappa = 0 \tag{22}$$

$$\delta f_4 = \delta\mathcal{H}' = 0 \tag{23}$$

$$\delta f_5 = \delta\mathcal{V}' = 0 \tag{24}$$

$$\delta f_6 = \delta\mathcal{M}' - \mathcal{Q}\delta\varepsilon - (1+\varepsilon)\mathcal{N}\delta\varphi - (1+\varepsilon)\sin\varphi\delta\mathcal{H} - (1+\varepsilon)\cos\varphi\delta\mathcal{V} = 0 \tag{25}$$

$$\delta f_7 = C_{11}\delta\varepsilon + C_{12}\delta\kappa + \mathcal{Q}\delta\varphi - \cos\varphi\delta\mathcal{H} + \sin\varphi\delta\mathcal{V} = 0 \tag{26}$$

$$\delta f_8 = C_{21}\delta\varepsilon + C_{22}\delta\kappa - \delta\mathcal{M} = 0 \tag{27}$$

Here C_{11}, C_{12}, C_{21} and C_{22} are the components of the tangent constitutive matrix of the cross-section, defined by

$$C_{11} = \frac{\partial\mathcal{N}_c}{\partial\varepsilon} \tag{28a}$$

$$C_{12} = C_{21} = \frac{\partial\mathcal{M}_c}{\partial\varepsilon} = 0 \tag{28b}$$

$$C_{22} = \frac{\partial\mathcal{M}_c}{\partial\kappa} \tag{28c}$$

Note that $C_{12} = C_{21} = 0$, which is due to the symmetry of the cross-section. By the help of the tangent moduli of both concrete and steel

$$E_{Tc} = \frac{\partial\sigma_c}{\partial D} \tag{29}$$

$$E_{Ts} = \frac{\partial\sigma_s}{\partial D} \tag{30}$$

and after the introduction of the effective area, A_{eff}, and the effective moment of inertia, J_{eff}, of the composite cross-section of the column, the non-vanishing components of the tangent constitutive matrix can be written as

$$C_{11} = E_{Tc}A_c + E_{Ts}\sum_{n_i=1}^{n} A_{n_i} = E_{Tc}A_{eff} \tag{31}$$

$$C_{22} = E_{Tc}J_c + E_{Ts}\sum_{n_i=1}^{n} z_{n_i}^2 A_{n_i} = E_{Tc}J_{eff} \tag{32}$$

In order to obtain the perturbed solution, we have to derive the fundamental solution of Eqs. (1)-(8). The solution is easily established by the following argument. The fundamental equilibrium mode of a column whose ends are subjected to centric axial force $F > 0$, is characterized by the condition that the column remains straight, i.e., the rotation φ is equal to zero in all cross-sections. Consequently, $w(x) = 0$ and the solution of Eqs. (1)-(6) reads

$$u(x) = \varepsilon(x - L) \tag{33a}$$

$$w(x) = 0 \tag{33b}$$

$$\varphi(x) = 0 \tag{33c}$$

$$\mathcal{H}(x) = \mathcal{N}(x) = -F \tag{34a}$$

$$\mathcal{V}(x) = \mathcal{Q}(x) = 0 \tag{34b}$$

$$\mathcal{M}(x) = 0 \tag{34c}$$

$$\varepsilon(x) = \text{const.} \tag{35a}$$

$$\kappa(x) = 0 \tag{35b}$$

Only u, $\mathcal{N}$ and ε are not zero. ε is evaluated from Eq. (7), in which we consider Eqs. (12), (34a) and (35b)

$$\mathcal{N}_c(\varepsilon, \kappa = 0) - \mathcal{N} = \mathcal{N}_c(\varepsilon, \kappa = 0) + F = \sigma_c A_c + \sum_{n_i = 1}^{n} \sigma_s A_{n_i} + F = 0 \tag{36}$$

Substituting Eqs. (28a)-(35b) into Eqs. (20)-(27) yields the linearized system of equilibrium equations for the variations of the unknowns of the problem

$$\delta f_1 = \delta u' - \delta\varepsilon = 0 \tag{37}$$

$$\delta f_2 = \delta w' + (1 + \varepsilon)\delta\varphi = 0 \tag{38}$$

$$\delta f_3 = \delta\varphi' - \delta\kappa = 0 \tag{39}$$

$$\delta f_4 = \delta\mathcal{H}' = 0 \tag{40}$$

$$\delta f_5 = \delta\mathcal{V}' = 0 \tag{41}$$

$$\delta f_6 = \delta\mathcal{M}' + (1 + \varepsilon)F\delta\varphi - (1 + \varepsilon)\delta\mathcal{V} = 0 \tag{42}$$

$$\delta f_7 = C_{11}\delta\varepsilon - \delta\mathcal{H} = 0 \tag{43}$$

$$\delta f_8 = C_{22}\delta\kappa - \delta\mathcal{M} = 0 \tag{44}$$

Eqs. (37)-(44) constitute a system of algebraic and differential equations of the first order with constant coefficients. The corresponding set of the boundary conditions is obtained through the linearization of Eqs. (14)-(19) yielding

$$\delta u(0) = 0 \tag{45}$$

$$\delta w(0) = 0 \tag{46}$$

$$s_1^1\delta\mathcal{M}(0) - \rho^1\delta\varphi(0) = 0 \tag{47}$$

$$\delta\mathcal{H}(L) = 0 \tag{48}$$

$$s_1^2\delta\mathcal{V}(0) + \mu_V\delta w(L) = 0 \tag{49}$$

$$s_2^2\delta\mathcal{M}(L) + \rho^2\delta\varphi(L) = 0 \tag{50}$$

Both ε and C_{22} are constants. Thus, from Eqs. (38), (39) and (44) we have $\delta\mathcal{M}' = -C_{22}\frac{\delta w'''}{1 + \varepsilon}$. Inserting the expression into Eq. (42), considering Eq. (38), and differentiating Eq. (42) with respect to x, we obtain a new form of the linearized equilibrium moment equation (42), in which the only unknown function of x is the variation of the transverse displacement, δw

$$C_{22}\delta w^{IV} + (1+\varepsilon)F\delta w'' = 0 \tag{51}$$

Because ε is constant and F a given value, Eq. (51) represents an ordinary homogeneous fourth-order differential equation with constant coefficients for the unknown function $\delta w(x)$. Once the inequalities $C_{22} > 0$ and $(1+\varepsilon)F > 0$ are considered, and the buckling load parameter k is introduced as

$$k^2 = \frac{(1+\varepsilon)F}{C_{22}} \tag{52}$$

Eq. (51) reads

$$\delta w^{IV} + k^2\delta w'' = 0 \tag{53}$$

The solution of the above equation is

$$\delta w(x) = B_1 \sin kx + B_2 \cos kx + B_3 x + B_4 \tag{54}$$

In Eq. (54) $B_i (i = 1, \ldots, 4)$ are unknown integration constants that will be derived from the linearized boundary conditions. By imposing the boundary conditions (46), (47), (49) and (50) to the solution (54), we obtain a homogeneous system of four linear algebraic equations, which is solved for the four unknown constants B_i. The non-trivial solution of the homogeneous system of linear algebraic equations is obtained only if the determinant of the system matrix, $\mathbf{K}_T$, is zero (see, e.g., Planinc and Saje 1999)

$$\det \mathbf{K}_T = 0 \tag{55}$$

Considering the actual form of matrix $\mathbf{K}_T$,

$$\begin{bmatrix} 1 & 0 & 0 & 1 \\ \dfrac{C_{22}k^2 s_1^1}{1+\varepsilon} & \dfrac{k\rho^1}{1+\varepsilon} & \dfrac{\rho^1}{1+\varepsilon} & 0 \\ \mu_V \cos kL & \mu_V \sin kL & -\dfrac{C_{22}k^2 s_2^1}{(1+\varepsilon)^2} + L\mu_V & \mu_V \\ \dfrac{k(C_{22}s_2^2 k \cos kL + \rho^2 \sin kL)}{1+\varepsilon} & \dfrac{k(-\rho^2 \cos kL + C_{22}s_2^2 k \sin kL)}{1+\varepsilon} & -\dfrac{\rho^2}{1+\varepsilon} & 0 \end{bmatrix}$$

we derive

$$\det \mathbf{K}_T = \mathcal{A} + \mathcal{B}\cos kL + k\mathcal{C}\sin kL = 0 \tag{56}$$

where $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$

$$\mathcal{A} = 2(1+\varepsilon)^2 \mu_V \rho^1 \rho^2$$

$$\mathcal{B} = -2(1+\varepsilon)^2\mu_V\rho^1\rho^2 + k^4C_{22}^2s_1^2(s_2^2\rho^1 + s_1^1\rho^2) - k^2L(1+\varepsilon)^2C_{22}\mu_V(s_2^2\rho^1 + s_1^1\rho^2)$$

$$\mathcal{C} = C_{22}s_2^2(C_{22}k^2s_1^1(-C_{22}k^2s_1^2 + L(1+\varepsilon)^2\mu_V) + (1+\varepsilon)^2\mu_V\rho^1) + (C_{22}s_1^1(1+\varepsilon)^2\mu_V + C_{22}k^2s_1^2\rho^1 - L(1+\varepsilon)^2\mu_V\rho^1)\rho^2$$

depend on both k and ε in a complicated way. Finally, for the determination of the buckling load of restrained reinforced concrete columns, the following coupled system of three non-linear algebraic equations for three unknowns k_{cr}, F_{cr} and ε_{cr} must be solved

$$\mathcal{N}_c(\varepsilon_{cr}, \kappa_{cr} = 0) + F_{cr} = \sigma_c A_c + \sum_{n_i=1}^{s} \sigma_s A_{n_i} + F_{cr} = 0 \tag{57}$$

$$\det\mathbf{K}_T = \mathcal{A}_{cr} + \mathcal{B}_{cr}\cos k_{cr}L + k_{cr}\mathcal{C}_{cr}\sin k_{cr}L = 0 \tag{58}$$

$$k_{cr}^2 - \frac{(1+\varepsilon_{cr})F_{cr}}{C_{22}} = 0 \tag{59}$$

This system of simultaneous algebraic equations cannot be solved analytically. The Newton iterative solution method is used here.

3. Buckling load of RC columns according to EC 2 constitutive law

In our numerical examples we assume the constitutive laws of concrete and steel as given by the European standard for concrete (Eurocode 2 2002)

$$\sigma_c(D) = -\chi_c\frac{f_{cm}D(11D_{c1}^2E_{cm} + 10f_{cm}D)}{D_{c1}(-10D_{c1}f_{cm} + 11D_{c1}E_{cm}D + 20f_{cm}D)} \tag{60}$$

$$\sigma_s(D) = \chi_{s1}f_y + \chi_{s2}E_sD + \chi_{s3}E_p(D - D_{y1}) \tag{61}$$

The parameters χ_c, χ_{s1}, χ_{s2} and χ_{s3} describe the domain of functions σ_c and σ_s for concrete and

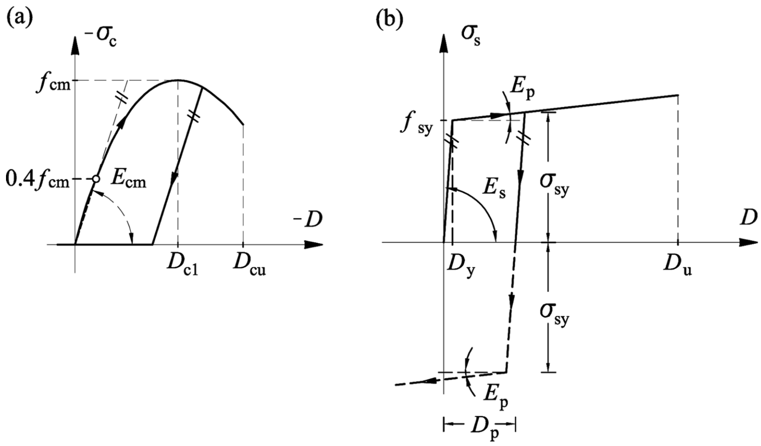


Fig. 2 Constitutive laws of (a) concrete and (b) reinforcing steel according to Eurocode 2 (2002)

steel. If $D_{cu} < D < 0$, then $\chi_c = 1$; otherwise $\chi_c = 0$. Similarly, when $|D| < D_{s1}$, then $\chi_{s1} = \chi_{s3} = 0$ and $\chi_{s2} = 1$. If $D_{s1} < |D| < D_{s2}$, $\chi_{s1} = \chi_{s3} = 1$ and $\chi_{s2} = 0$. The meaning of the remaining parameters in Eqs. (60)-(61) is clear from Fig. 2.

The related tangent moduli are obtained by the differentiation of Eqs. (60)-(61) with respect to D, and read

$$E_{Tc} = \chi_c \frac{10 f_{cm}^2 \varepsilon (D_{c1} - \varepsilon)(11 D_{c1}^2 E_{cm} + 11 D_{c1} E_{cm} \varepsilon + 20 f_{cm} \varepsilon)}{D_{c1}(-10 D_{c1} f_{cm} + 11 D_{c1} E_{cm} \varepsilon + 20 f_{cm} \varepsilon)^2} \tag{62}$$

$$E_{Ts} = \chi_{s2} E_s + \chi_{s3} E_p \tag{63}$$

The material and geometric parameters of the reference cross-section are given in Fig. 3. The material parameters required in the Eurocode 2 rules (60)-(61) are also taken from the Eurocode 2. They are: the mean compressive strength of concrete, f_{cm} = 3.8 kN/cm^2; elastic modulus of concrete, E_{cm} = 3,200 kN/cm^2; yield stress of steel, f_y = 50.0 kN/cm^2; the peak and ultimate compression strains of concrete, respectively: D_{c1} = –2.2‰ and D_{cu} = –3.5‰; the elastic modulus of steel, E_s = 20,000 kN/cm^2; the hardening modulus of steel, E_p = 0 kN/cm^2 (no strain-hardening); and its ultimate strain $D_{y2} = D_u$ = 40‰.

Geometric and material parameters of the reinforced concrete columns are selected in a way, that every reinforced concrete column reaches its sideway buckling before reaching the cross-section ultimate axial bearing capacity $\mathcal{N}_{ult}$ = –4,043.74 kN (Fig. 4(a)). The variation of the normalized

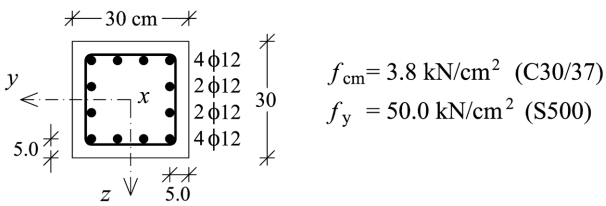


Fig. 3 Reference reinforced concrete cross-section. Geometric and material data

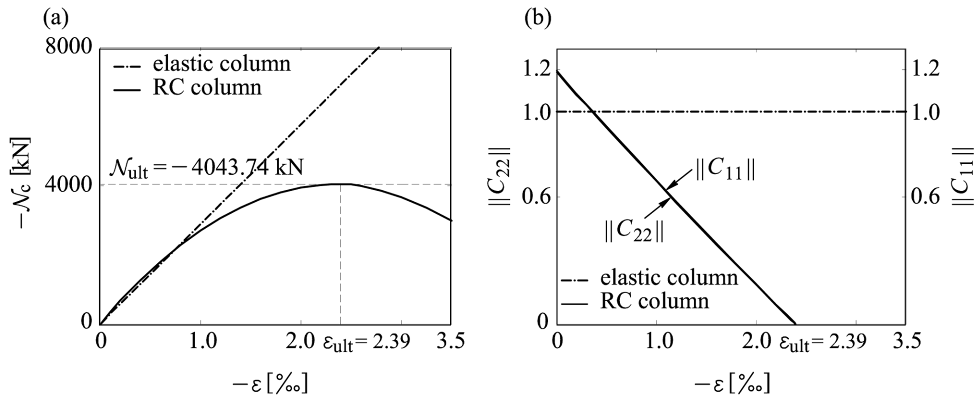


Fig. 4 The variations of (a) constitutive axial force $\mathcal{N}_c$ and (b) normalized flexural tangent stiffness $\|C_{22}\| = C_{22}/E_{cm}J_c$ and normalized axial tangent stiffness $\|C_{11}\| = C_{11}/E_{cm}A_c$ with respect to membrane deformation ε

flexural tangent stiffness ($\|C_{22}\| = C_{22}/E_{cm}J_c$) with respect to membrane deformation ε is shown in Fig. 4(b). For convenience, the variation of the normalized axial tangent stiffness ($\|C_{11}\| = C_{11}/E_{cm}A_c$) vs. ε is also displayed in Fig. 4(b). These relationships are practically linear and coincident. This shows that both the ultimate axial and the ultimate flexural bearing capacity of the cross-section are reached simultaneously.

3.1 Reinforced concrete Euler's columns

Four sets of boundary conditions will be analysed, marked as Euler's columns (see Fig. 5). If the column is pinned at both ends, the boundary conditions read: $w(0) = 0$, $\mathcal{M}(0) = 0$, $w(L) = 0$, $\mathcal{M}(L) = 0$. Imposing these conditions to the solution (54), and equating determinant in Eq. (56) to zero, gives $\sin k_{cr}L = 0$ and, consequently, $k_{cr}L = m\pi$ ($m = 0, 1, 2, ...$). The critical load is the least possible force, which here occurs when $m = 1$. Thus, $k_{cr} = \pi/L$. After inserting k_{cr} into Eq. (52), we have $(1 + \varepsilon_{cr})F_{cr} = E_{Tc}J_{eff}\pi^2/L^2$. Adding Eq. (57), we have the following system of two non-linear coupled algebraic equations

$$\mathcal{N}_c(\varepsilon_{cr}, \kappa_{cr} = 0) + F_{cr} = \sigma_c A_c + \sum_{n_i = 1}^{n} \sigma_s A_{n_i} + F_{cr} = 0 \tag{64}$$

$$(1 + \varepsilon_{cr})F_{cr} = \frac{E_{Tc} J_{eff} \pi^2}{L^2} \tag{65}$$

which must be solved for two unknowns F_{cr} and ε_{cr}. When material is linear elastic and incompressible, the axial strain is zero: $\varepsilon_{cr} = 0$. The procedure for deriving the buckling loads for the remaining combinations of the boundary conditions (CC is the fixed-free column, FPC is the fixed-pinned column, FFC is the fixed-fixed column) is similar and need not be given here. The results are presented in Table 1. As indicated in Table 1, the buckling load formulae for F_{cr} of reinforced concrete and elastic columns differ only in the scalar factor $1 + \varepsilon_{cr}$. In elastic columns, where elastic modulus of material is constant, ε_{cr} is a small number compared to 1 and can thus

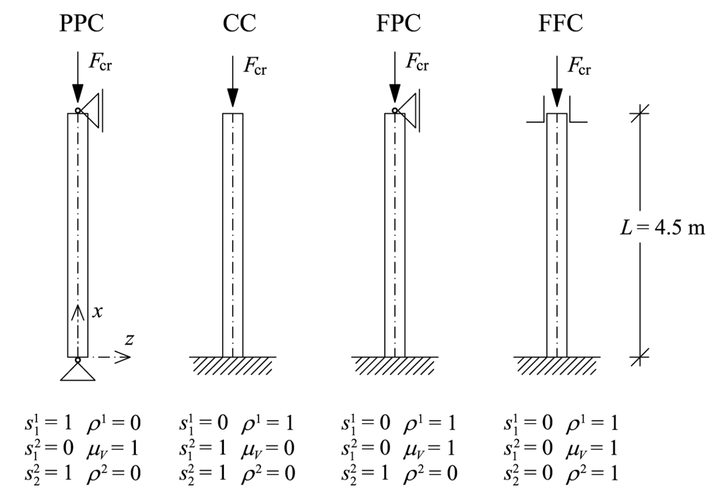


Fig. 5 Euler's reinforced concrete columns

Table 1 Buckling load F_{cr} of reinforced concrete columns

Type of the column	Reinforced concrete column	Elastic column ($\varepsilon = 0$)
PPC	$(1+\varepsilon_{cr})F_{cr} = \dfrac{E_{Tc}J_{eff}\pi^2}{L^2}$	$F_{cr} = \dfrac{E_{cm}J_c\pi^2}{L^2}$
CC	$(1+\varepsilon_{cr})F_{cr} = \dfrac{E_{Tc}J_{eff}\pi^2}{(2L)^2}$	$F_{cr} = \dfrac{E_{cm}J_c\pi^2}{(2L)^2}$
FPC	$(1+\varepsilon_{cr})F_{cr} = \dfrac{E_{Tc}J_{eff}\pi^2}{(0.69915565\ldots L)^2}$	$F_{cr} = \dfrac{E_{cm}J_c\pi^2}{(0.69915565\ldots L)^2}$
FFC	$(1+\varepsilon_{cr})F_{cr} = \dfrac{E_{Tc}J_{eff}\pi^2}{(0.5L)^2}$	$F_{cr} = \dfrac{E_{cm}J_c\pi^2}{(0.5L)^2}$

Table 2 Buckling load F_{cr} of reinforced concrete columns

F_{cr} [kN]	CC	PPC	FPC	FFC
Present exact analysis	2124.270	3668.307	3936.186	4012.639
Elastic analysis	2631.89	10527.58	21534.05	42110.31

Table 3 Critical membrane deformation ε_{cr} of reinforced concrete columns

ε_{cr} [‰]	CC	PPC	FPC	FFC
Present exact analysis	−0.736	−1.659	−2.002	−2.186
Elastic analysis	−0.914	−3.655	−7.477	−14.622

safely be neglected. In reinforced concrete columns, ε_{cr} is small, too, yet its value grossly effects the flexural stiffness $E_{Tc}J_{eff}$ and should not be neglected. Note that if the reinforcement can be neglected, we have $E_{Tc}J_{eff} = E_{cm}J_c$. Then Eq. (65) reduces to Euler's buckling load for the pinned-pinned elastic column, $F_{cr} = E_{cm}J_c\pi^2/L^2$.

The numerical values of the buckling load are presented in Table 2. For the sake of comparison, the related exact buckling loads for elastic columns are also given in Table 2. It is clear that the non-linearity of material has a substantial impact on the buckling load. The buckling load of a RC column is always smaller than that of the corresponding elastic column. This is particularly true for the fixed-fixed column (FFC), which has the smallest buckling length (L_u = 0.5L) of the four columns. Table 3 shows the related values of the critical membrane deformation at F_{cr}. Note that the critical membrane deformation is smaller than the peak compression strain of concrete. Observe also that it is smaller than the corresponding critical deformation of the elastic column.

The effect of the non-linearity of concrete and steel on the buckling load is further shown in Fig. 6, where the plot of the normalized buckling load, F_{cr}/F_{ult}, versus the column slenderness, $\lambda = L_u\sqrt{A_c/J_c}$, is shown for all types of the columns in one curve for both linear elastic and reinforced concrete columns. Fig. 6 shows that buckling occurs, if the slenderness is bigger than $\lambda_{ult} = 17.32$. The figure indicates that the material non-linearity has no influence on the buckling load, if the slenderness is larger than about 150.

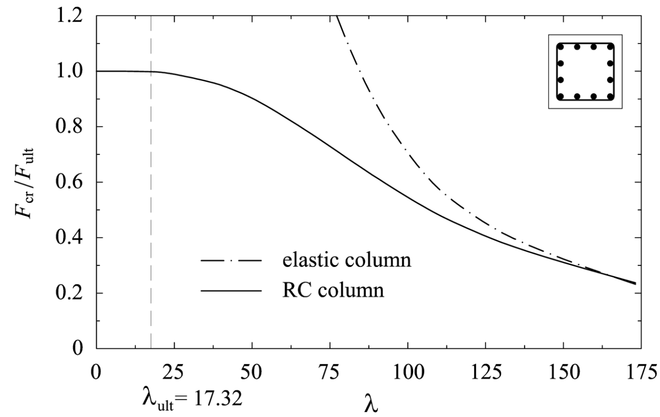
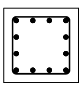
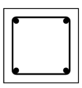


Fig. 6 The variation of normalized buckling load F_{cr}/F_{ult} vs. column slenderness λ

In order to study the effect of the steel/concrete area ratio on the buckling load, we analysed five different ratios (μ = 0.5%, 1.0%, 1.5%, 2.0%, 3.0%), for two different reinforcement settings (4ϕ, 12ϕ) in the cross-section. The results are presented in Table 4 and in Fig. 7. We can observe a relatively small increase in the buckling load with the reinforcement increase for any type of columns and any layout of reinforcing bars (4ϕ, 12ϕ). The buckling load at the ratio as high as 3.0% is only about 20% larger than the one obtained with the 0.5% reinforcement. Fig. 7 shows the graphs of F_{cr} versus μ for a series of low to high-strength concretes for the pinned-pinned (Fig. 7(a)) and fixed-fixed columns (Fig. 7(b)) for the 12ϕ-layout. The graphs clearly indicate that the relations are linear functions of μ with the inclination being independent both of the type of concrete and the percentage of the reinforcement. This rule holds only if the column collapses due to buckling. If, in contrast, the collapse is due to fracture of material, F_{cr} is virtually independent on μ. As expected, the number of reinforcing bars (4ϕ vs. 12ϕ) at a given μ only slightly effects the critical load. It is interesting that the 4ϕ-columns seem to exhibit slightly higher values of the buckling load than the 12ϕ-columns, provided, however, that the local buckling of each individual bar is prevented by the correct placement of stirrups. The effect of compression strength of concrete

Table 4 The effect of the reinforcement layout and the steel/concrete area ratio on the buckling load F_{cr} [kN]. Concrete C30/37

		μ [%]	0.5	1.0	1.5	2.0	3.0
		$\lvert\mathcal{N}_{ult}\rvert$ [kN]	3 620.97	3 827.92	4 043.74	4 259.77	4 709.03
12ϕ		PPC	3 320.86	3 491.79	3 665.96	3 843.28	4 207.07
		CC	1 977.25	2 050.17	2 123.29	2 196.61	2 343.77
		FPC	3 537.48	3 732.78	3 933.01	4 138.14	4 562.99
		FFC	3 597.89	3 801.05	4 009.82	4 224.20	4 669.77
4ϕ		PPC	3 340.24	3 532.00	3 728.45	3 929.55	4 345.49
		CC	1 997.33	2 090.89	2 185.18	2 280.15	2 472.00
		FPC	3 548.87	3 756.32	3 969.51	4 188.45	4 643.71
		FFC	3 603.89	3 813.18	4 028.24	4 249.10	4 708.35

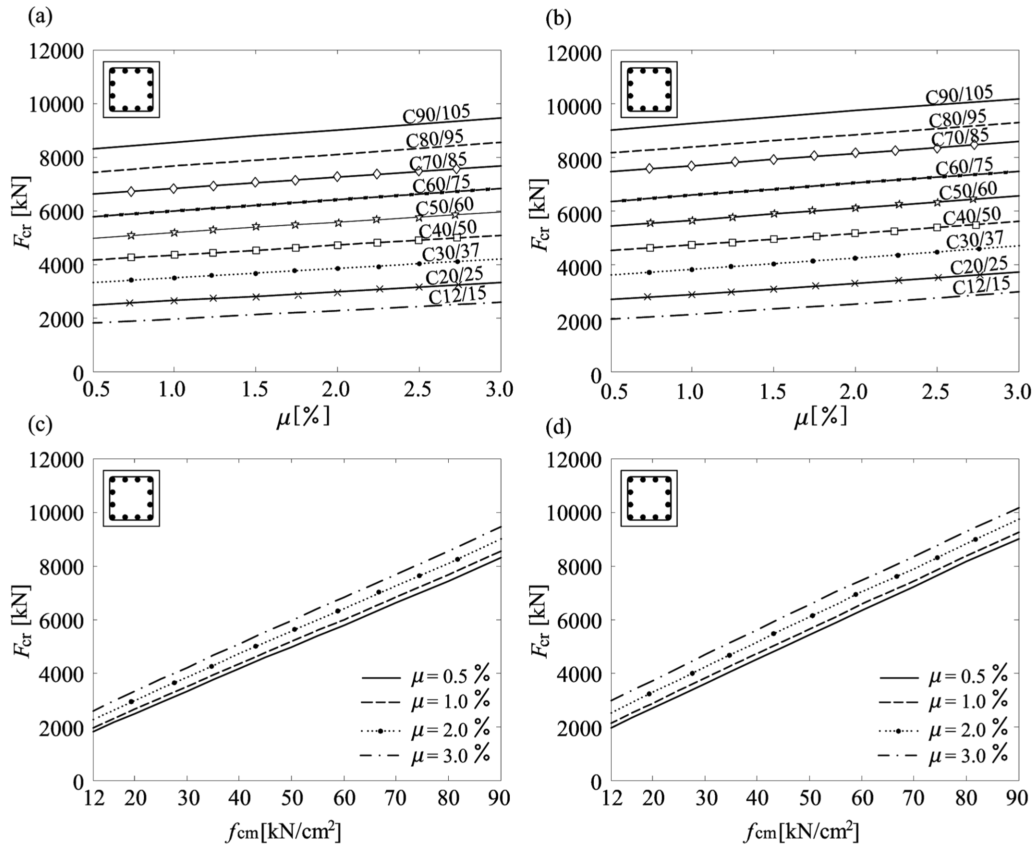


Fig. 7 The effect of the steel/concrete area ratio and strength of concrete on buckling load F_{cr}. (a) and (c) pinned-pinned column (PPC); (b) and (d) fixed-fixed column (FFC)

on the buckling load is depicted in Figs. 7(c) and 7(d). As expected, the effect is substantial. Note that the relation F_{cr} vs. f_{cm} is linear.

3.2 Restrained reinforced concrete columns

We consider the reinforced concrete columns with end restraints, as shown in Fig. 8. Columns can be utilized in the stability analysis of unbraced (RCA), partially braced (RCB) and totally braced (RCC) frames (see, e.g., Liu and Xu 2005, Mahini and Seyyedian 2006).

The consideration of the specific boundary conditions of the RCA column in Eq. (56) gives

$$\det\mathbf{K}_T = \frac{1}{(1+\varepsilon_{cr})^4}(C_{22,cr}k_{cr}^4(C_{22,cr}k_{cr}(\rho^1+\rho^2)\cos k_{cr}L + (-C_{22,cr}^2k_{cr}^2+\rho^1\rho^2)\sin k_{cr}L)) = 0 \quad (66)$$

Likewise, for the RCB column we obtain

$$\det\mathbf{K}_T = \frac{1}{(1+\varepsilon_{cr})^4}(\mathcal{A} + \mathcal{B}\cos k_{cr}L + k_{cr}\mathcal{C}\sin k_{cr}L) = 0 \quad (67)$$

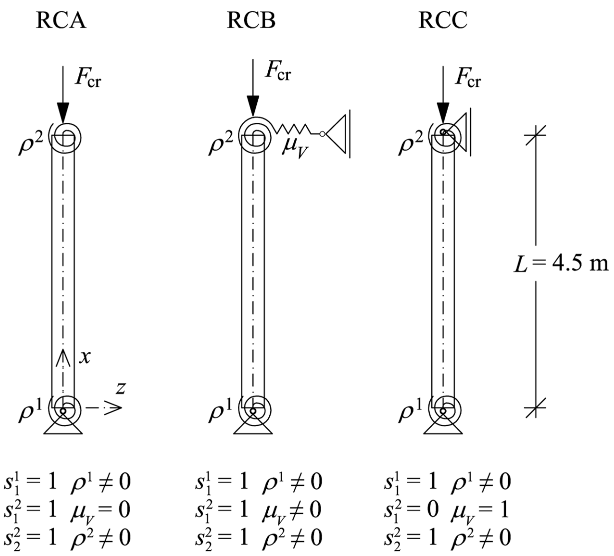


Fig. 8 Reinforced concrete columns with end restraints

where

$$\mathcal{A} = 2(1+\varepsilon_{cr})^2\mu_V\rho^1\rho^2$$

$$\mathcal{B} = -2(1+\varepsilon_{cr})^2\mu_V\rho^1\rho^2 + k_{cr}^4C_{22,cr}^2(\rho^1+\rho^2) - k_{cr}^2L(1+\varepsilon_{cr})^2C_{22,cr}\mu_V(\rho^1+\rho^2)$$

$$\begin{aligned}\mathcal{C} = {} & C_{22,cr}(C_{22,cr}k_{cr}^2(-C_{22,cr}k_{cr}^2 + L(1+\varepsilon_{cr})^2\mu_V) + (1+\varepsilon_{cr})^2\mu_V\rho^1) + \\ & (C_{22,cr}(1+\varepsilon_{cr})^2\mu_V + C_{22,cr}k_{cr}^2\rho^1 - L(1+\varepsilon_{cr})^2\mu_V\rho^1)\rho^2)\end{aligned}$$

For the RCC column we have

$$\begin{aligned}\det\mathbf{K}_T = {} & \frac{1}{(1+\varepsilon_{cr})^2}(k_{cr}(2\rho^1\rho^2 - (2\rho^1\rho^2 + C_{22,cr}k_{cr}^2L(\rho^1+\rho^2))\cos k_{cr}L + \\ & k_{cr}(C_{22,cr}(C_{22,cr}k_{cr}^2L + \rho^1) + (C_{22,cr} - L\rho^1)\rho^2)\sin k_{cr}L)) = 0 \end{aligned} \tag{68}$$

If linear elastic material is considered along with the inextensibility condition $\varepsilon_{cr} = 0$, and if the reinforcement is neglected, Eqs. (66)-(68) reduce to the expressions as given by Wang and co-workers (Wang *et al.* 2005).

In what follows we study effects of various parameters and types of end restraints on the buckling load. In particular, we are interested in the effective length factor α, defined as

$$L_u = \alpha L \tag{69}$$

In this case the condition $\det\mathbf{K}_T = 0$ can conveniently be replaced with

$$(1+\varepsilon_{cr})F_{cr} = \frac{E_{Tc}J_{eff}\pi^2}{(\alpha L)^2} \tag{70}$$

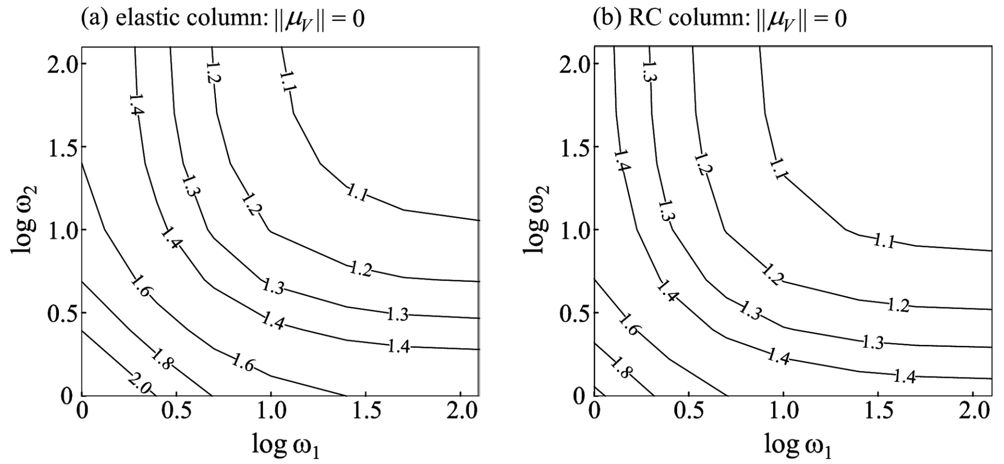


Fig. 9 Variation of effective length factor α with respect to logω_1 and logω_2 for reinforced concrete columns RCA ($\mu_V = 0$): (a) elastic constitutive relations for concrete, with reinforcement being neglected, and (b) non-linear constitutive relations for both concrete and steel

Table 5 The effect of the non-linear constitutive relations of concrete and steel on buckling load parameters F_{cr}, ε_{cr} and α (logω_1 = logω_2 = 1).

Type of column	Material model	F_{cr} [kN]	ε_{cr} [‰]	α
RCA ($\|\mu_V\|$ = 0)	elastic column RC column	7 383.92 3 489.78	−2.564 −1.500	1.20 1.13
RCB ($\|\mu_V\|$ = 0.005)	elastic column RC column	20 211.67 3 970.24	−7.018 −2.071	0.72 0.63
RCC $\|\mu_V\|$ = 1)	elastic column RC column	30 365.71 3 994.99	−10.544 −2.132	0.59 0.56

Fig. 9 shows the effect of material non-linearity of concrete and steel and the spring stiffnesses on the effective length factor for RCA columns. In Fig. 9, ω_1 and ω_2 denote the normalized stiffnesses, $\omega_i = \|\rho^i\| = \rho^i L/(E_{cm}J_c)$ $(i = 1, 2)$, and $\|\mu_V\| = \mu_V L/(E_{cm}A_c)$ is the normalized translational spring stiffness.

Comparing Figs. 9(a) and 9(b), we can observe that the material non-linearity has a relatively small influence on the effective length of a RCA column. The numerical values of the effective length factor for log ω_1 = log ω_2 = 1 are also displayed in Table 5. If log ω_1 and log ω_2 are big, α approaches 1, and if log ω_1 and log ω_2 are small, α approaches ∞ (Fig. 9). Unlike the effective buckling length, the buckling load is very much affected by the non-linearity of material. For example, for the RCA column with log ω_1 = log ω_2 = 1 and $\|\mu_V\| = 0$, the buckling load is 3,489.78 kN, in contrast to 7,383.92 kN of the elastic column (Table 5).

The effect of the material non-linearity of RCB and RCC columns on the buckling load is shown in Figs. 10 and 11 and in Table 5. As in the RCA columns, the material non-linearity effects F_{cr} and ε_{cr}, while its effect on the effective length factor is small. The effective length factor of the RCB column varies from $\alpha = 1$ for large ω_1 and ω_2 to a large α for log ω_1 = log ω_2 = 0. For the RCC columns (Fig. 11), the effective length factor α varies from $\alpha = 0.5$ for large ω_1 and ω_2 (fixed-fixed column) to $\alpha = 1$ for log ω_1 = log ω_2 = 0 (pinned-pinned column).

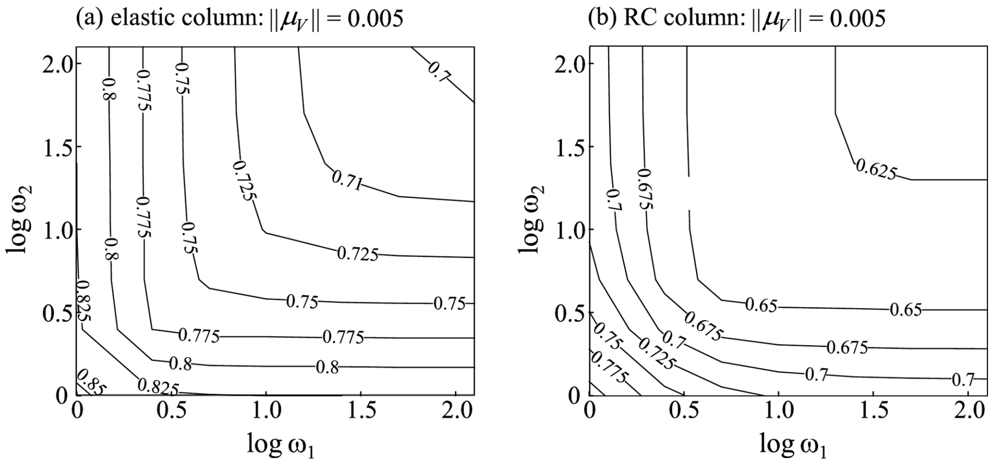


Fig. 10 Variation of effective length factor α with respect to log ω_1 and log ω_2 for reinforced concrete columns RCB ($\|\mu_V\|$ = 0.005): (a) elastic constitutive relations for concrete, with reinforcement being neglected, and (b) non-linear constitutive relations for both concrete and steel

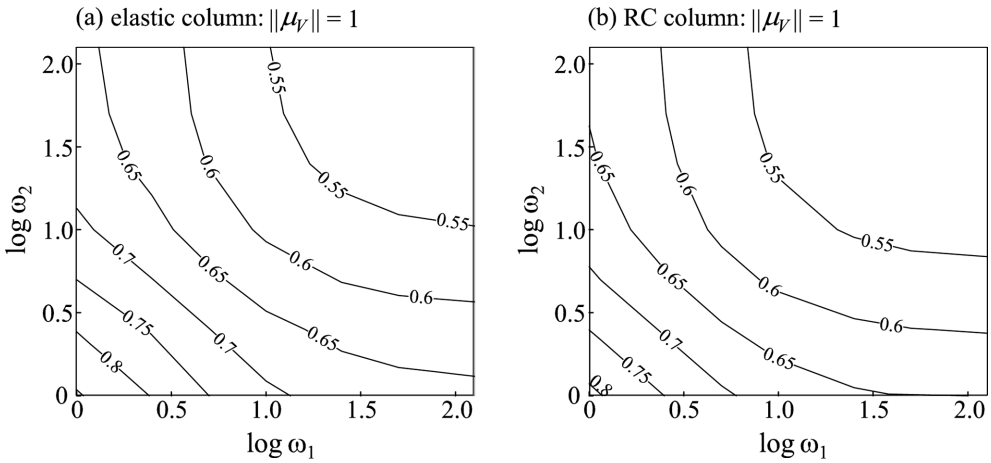


Fig. 11 Variation of effective length factor α with respect to log ω_1 and log ω_2 of reinforced concrete columns RCC ($\|\mu_V\|$ = 1): (a) elastic constitutive relations for concrete, with reinforcement being neglected, and (b) non-linear constitutive relations for both concrete and steel

4. Conclusions

This paper has provided the theoretical foundation and the exact solutions for the buckling load of centrically loaded restrained reinforced concrete columns. Based on the analytical results and the parametric studies undertaken, the following conclusions can be drawn:

- The non-linearity of concrete and steel has a substantial effect on the buckling load and the corresponding membrane deformation. For the columns with a short effective buckling length, the effect appears to be the largest; e.g., for the fixed-fixed column, the ratio $F_{cr}^{elastic}/F_{cr}^{RC_{column}}$ is about 10 (Table 2). If the slenderness of the column is large, the effect of the material non-linearity is small; e.g., for fixed-free column, the ratio $F_{cr}^{elastic}/F_{cr}^{RC\,column}$ is only about 1.2 (Table 2).

- Unlike in the linear elastic columns, the effect of membrane deformations on the buckling load of restrained reinforced concrete columns is very important and should not be neglected in the analysis.
- The buckling load of the restrained reinforced concrete columns linearly increases with strength of concrete and the steel/concrete area ratio, μ. The increase rate is somewhat bigger for small ratio. For example, if μ = 0.5%, the ratio $F_{cr}^{C90/105}/F_{cr}^{C12/15}$ is about 4.6, while for μ = 3.0%, this ratio is about 3.6 (Figs. 7(a) and 7(b)).
- For a given concrete, the buckling load magnitude changes linearly with the steel/concrete area ratio. This ratio has only a moderate influence on the buckling load. For example, $F_{cr}^{\mu = 3.0\%}/F_{cr}^{\mu = 0.5\%}$ is about 1.30 (Table 4). The layout of the steel bars is rather unimportant. If the bars are gathered solely at the corners of the cross-section rather than placed along the sides, the buckling load is only negligibly higher (Table 4).
- The non-linearity of concrete and reinforcing steel has a relatively small influence on the effective length factor. This holds true for both unrestrained and restrained columns.

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