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Technical Note

Natural frequencies of a bending-torsion coupled beam supported by in-span linear springs

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1. Introduction

Beams with tip or span-wise constraints have been the main subject of numerous investigations due to their practical importance. Published works tackling free vibration characteristics of beams carrying attachments differ generally in the methods employed (Rayleigh-Ritz, Galerkin, transfer matrix methods, etc.) and the beam models assumed (Timoshenko, Euler-Bernoulli, etc). Among many papers appeared so far the following are worth mentioning: Wu and Chou (1997), Chen and Wu (2002), Oguamanam (2003), Lin and Chang (2005). These and similar studies not cited here assume indirectly that as the main element of system beam undergoes to bending in one of the symmetry planes. However, some beam-like elements such as turbomachinery and helicopter blades, thin-walled open cross section bars, and beams with single cross sectional symmetry experience twisting along with flexural deflections due to shear centre and centroidal axes being non-coincident. The literature on dynamic analysis of such beams is also extensive, and some recently published works on the subject are due to Banerjee (1999), Jun *et al.* (2004), Gökdağ and Kopmaz (2005).

This note, apart from the works cited above, is concerned with the out of symmetry plane free bending vibrations of an open cross-section beam supported by intermediate transversal linear springs. The warping of beam cross section is accounted for, and Euler-Bernoulli beam theory is preferred. Two different approaches, i.e., the dynamic transfer matrix method (DTMM) and the Rayleigh-Ritz method (RRM), are carried out to obtain natural frequencies of the system and to compare the methods used with each other.

2. Theory

The system considered is depicted schematically in Fig. 1, where L is the length of beam, L_i shows the distance between the ith (i = 1, 2, …, $N-1$; N: number of beam spans) spring attachment point

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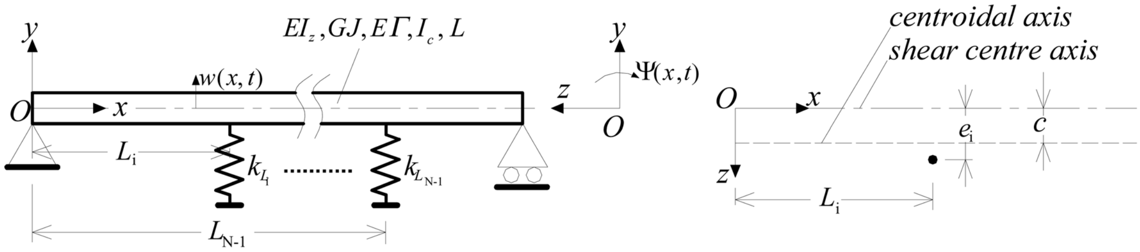

Fig. 1 The system consisting of a beam with non coincident shear and centroidal axes and $(N-1)$ intermediate linear springs

and the beam root point O, k_{L_i} denotes the ith spring stiffness, EI_z, GJ and $E\Gamma$ are, respectively, bending, uniform twisting and warping stiffnesses. I_c is the mass moment of inertia of the cross section per unit length with respect to the centroidal axis. The distance between the centroidal and the shear centre axes and the ith spring attachment point offset are denoted by c and e_i, respectively. The positive directions of twisting angle $\psi(x, t)$ and transversal displacement $w(x, t)$ are as shown.

Using these notations natural frequencies of the system is first obtained by DTMM According to this method, the equations of motion for the free vibratory motion of the bare beam, i.e., $EI_z w_{xxxx} + m(\ddot{w} + c\ddot{\psi}) = 0$ for bending and, $E\Gamma\,\psi_{xxxx} - GJ\,\psi_{xx} + mc(\ddot{w} + c\ddot{\psi}) + I_c\ddot{\psi} = 0$ for torsion (Gökdağ and Kopmaz (2005)), are solved first with the assumptions $w(x, t) = Y(x)e^{j\omega t}$, $\psi(x, t) = \phi(x)e^{j\omega t}$, and $j = \sqrt{-1}$, where m denotes the mass per unit length of beam, overhead dot and subscript x denote partial derivative with respect to time and spatial coordinate, respectively. $Y(x)$ and $\phi(x)$ are the modal functions related to bending and torsion, respectively, and w is the frequency of harmonic motion. After some algebra analytical forms of $Y(x)$ and $\phi(x)$ can be derived (refer to Banerjee (1999) and Jun *et al.* (2004) for details.) Then one can extract bending slope $\theta(x) = Y'(x)$, derivation of torsion angle $\vartheta(x) = \phi'(x)$, shear force $V(x) = -EI_z Y'''(x)$, bending moment $M_B(x) = EI_z Y''(x)$, warping moment $M_W(x) = E\Gamma\phi''(x)$, and twisting moment $M_T(x) = GJ\phi'(x) - E\Gamma\phi'''(x)$, which can be represented by the state vector $\mathbf{Z}$ as $\mathbf{Z} = [Y(x)\ \ \theta(x)\ \ \phi(x)\ \ \vartheta(x)\ \ V(x)\ \ M_B(x)\ \ M_W(x)\ \ M_T(x)]^T$. Then, for a beam section one can write $\mathbf{Z} = \mathbf{T}_{\mathrm{I}}\mathbf{S}$, where $\mathbf{S}$ is a column vector containing the integration constants C_i ($i = 1, \ldots, 8$), and $\mathbf{T}_{\mathrm{I}}$ is the 8×8 coefficient matrix. It can be verified that the compatibility conditions at the interface of ith and $(i+1)$th beam parts can be written as $\mathbf{Z}_i(L_i) = \mathbf{T}_{\mathrm{II}}\mathbf{Z}_{i+1}(L_i)$, in which $\mathbf{T}_{\mathrm{II}}$ is a square matrix with all diagonal elements equal to 1, and all off-diagonal elements except $\mathbf{T}_{\mathrm{II}}(5,1) = -k_{L_i}$, $\mathbf{T}_{\mathrm{II}}(8,3) = -k_{L_i}e_i^2$, $\mathbf{T}_{\mathrm{II}}(5,3) = \mathbf{T}_{\mathrm{II}}(8,1) = -k_{L_i}e_i$ are zero. For the beam with N spans, i.e., $(N-1)$ attachments, one can obtain the transfer matrix $\mathbf{T}$ as $\mathbf{T} = \left(\prod_{k=N}^{2}(\mathbf{T}_{\mathrm{I}}(L_k)\mathbf{T}_{\mathrm{I}}(L_{k-1})^{-1}\mathbf{T}_{\mathrm{II},k-1}^{-1})\right)\mathbf{T}_{\mathrm{I}}(L_1)\mathbf{T}_{\mathrm{I}}(0)^{-1}$, where $L_N = L$, $\mathbf{T}_{\mathrm{II},k-1}$ is, like in $\mathbf{Z}_i(L_i) = \mathbf{T}_{\mathrm{II}}\mathbf{Z}_{i+1}(L_i)$, calculated for the $(k-1)$th spring attachment point. Introducing the proper end conditions to modify $\mathbf{T}$ yields the characteristic equation giving the natural frequencies of the whole system.

In the RRM the displacement and the torsion angle of beam cross section can be expressed by $Y(x) = \sum_{j=1}^{N_M} a_j\bar{Y}_j(x)$, $\phi(x) = \sum_{j=1}^{N_M} a_j\bar{\phi}_j(x)$, respectively, in which $\bar{Y}_j(x)$ and $\bar{\phi}_j(x)$ are the modal amplitude functions of the bare beam, a_j denotes the modal coefficients, and N_M is the number of modes used. Employing the above expansions in the Rayleigh's quotient and derivating with respect to a_j give $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{A} = \mathbf{0}$, where $\mathbf{A}$ is the vector including a_j coefficients, $\mathbf{K}$ and $\mathbf{M}$ are, respectively, stiffness and mass matrixes comprised of the following entities

$$K_{jk} = \int_0^L (EI_z\overline{Y}_j\overline{Y}_k'''' + E\Gamma\overline{\phi}_j\overline{\phi}_k'''' - GJ\overline{\phi}_j\overline{\phi}_k'')\mathrm{d}x + \sum_{n=1}^{N-1} k_{L_n}(\overline{Y}_j(L_n) + e_n\overline{\phi}_j(L_n))(\overline{Y}_k(L_n) + e_n\overline{\phi}_k(L_n)) \quad (1)$$

$$M_{jk} = \int_0^L (m\overline{Y}_j\overline{Y}_k + mc(\overline{Y}_j\overline{\phi}_k + \overline{Y}_k\overline{\phi}_j) + I_s\overline{\phi}_j\overline{\phi}_k)dx, \quad I_s = mc^2 + I_c \quad j,k = 1,2,\ldots,N_M \quad (2)$$

Then the natural frequencies of the whole system satisfying the equation det $(\mathbf{K} - \omega^2\mathbf{M}) = 0$ are the solution of the above equation, i.e., $(\mathbf{K} - \omega^2\mathbf{M})\mathbf{A} = \mathbf{0}$.

3. Results and discussions

To check the compatibility of the results from the two methods the physical properties of a semi circular cross section (SCCS) beam in the work of Jun *et al.* (2004) are used. Initially, single intermediate spring is assumed to be mounted to the cantilever beam, so that corresponding numerical values for different spring attachment points L_1 and offsets e_1 are given in Table 1. At first glance one can realise the good agreement between the results in the table. Although the first five modes are considered in RRM, the results of the method agree well with the ones from DTMM, i.e., with the exact natural frequencies. However, the relative error between the fifth natural frequencies from the two methods is noticeable in comparison to the other natural frequencies, which arises from the numerical errors related to ill-conditioned matrix inversion in DTMM procedure. On the other hand, one can verify that the two methods give nearly the same results, i.e., the results with the absolute error lower than 0.01, for the physical properties of the channel cross section (CCS) beam in the work of Jun *et al.* (2004). This fact reveals that although the DTMM procedure gives exact natural frequencies, this method can sometimes yield erroneous results for some beam physical properties due to the foregoing computational deficiency. Meanwhile, in the context of computational time there observed no significant difference between the two methods.

Secondly, Table 2 is introduced in order to investigate the sensitivity of deviation between the results of the methods to increasing attachment number. The numerical values in Table 2 are obtained by using the data of CCS beam. The results are self explanatory; the agreement of the two methods is excellent, which implies RRM can be confidently used by employing just the first five

Table 1 Comparison of the natural frequencies ω (Hz) of the SCCS beam from the two methods for various attachment point locations and offsets

			ω_1	ω_2	ω_3	ω_4	ω_5
I	$L_1 = 0.2L$	RRM	63.9839	137.7774	278.9988	485.3017	664.2076
		DTMM	63.9835	137.7767	278.9731	485.1003	660.6019
	$L_1 = 0.5L$	RRM	67.1342	137.7312	280.0525	485.1363	664.0269
		DTMM	67.1268	137.7308	280.0351	485.1393	654.1348
II	$e_1 = c$	RRM	65.5106	137.8098	280.3421	484.7744	664.2633
		DTMM	65.5068	137.8096	280.3252	484.4564	654.1348
	$e_1 = 2c$	RRM	66.6706	139.7923	283.1390	485.1373	663.8402
		DTMM	66.6545	139.7788	283.1033	485.2447	654.1348

I: $e_1 = c$, $k_{L_1} = 2EI_z/L^3$; **II**: $L_1 = 0.4L$, $k_{L_1} = 2EI_z/L^3$.

Table 2 Natural frequencies ω (Hz) of the CCS beam with several intermediate springs

		ω_1	ω_2	ω_3	ω_4	ω_5
I	RRM	64.9498	98.9679	159.7724	415.5007	615.4365
	DTMM	64.9409	98.9676	159.7678	415.4982	615.4365
II	RRM	83.8875	100.0424	170.2128	419.9383	615.4815
	DTMM	83.8670	100.0394	170.2044	419.9340	615.4814

I: Five intermediate springs; $k_{L_i} = EI_z/L^3$, $L_i = (2i-1)0.1L$, $e_i = 2.0c$, $i = 1, 2, 3, 4, 5$.
II: Ten intermediate springs; $k_{L_i} = EI_z/L^3$, $L_i = (i/11)L$, $e_i = 2.0c$, $i = 1, 2, \ldots, 10$

natural modes of vibration. On the other hand, for the consecutive springs closely located the Timoshenko beam theory should be preferred as pointed out by Lin and Chang (2005). The present models can easily be modified to take shear and rotary effects into account, as well.

According to the above discussion, for a bending-torsion coupled beam with several elastic intermediate supports, one can expect that the two methods will give natural frequencies of extremely close to each other unless the physical properties of beam causes numerical errors previously mentioned. Moreover, RRM can alternatively be employed to obtain the accurate natural frequencies of coupled flexural-torsional beams with multiple intermediate elastic constraints. To the authors' knowledge, the bending-torsion coupled beams with the constraints in this study have not been the subject of a published work so far. Therefore, the present study is expected to guide the works for further investigations in this field.

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