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Numerical stability and parameters study of an improved bi-directional evolutionary structural optimization method

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Abstract. This paper presents a modified and improved bi-directional evolutionary structural optimization (BESO) method for topology optimization. A sensitivity filter which has been used in other optimization methods is introduced into BESO so that the design solutions become mesh-independent. To improve the convergence of the optimization process, the sensitivity number considers its historical information. Numerical examples show the effectiveness of the modified BESO method in obtaining convergent and mesh-independent solutions. A study of the effects of various BESO parameters on the solution is then conducted to determine the appropriate values for these parameters.

Keywords: bi-directional evolutionary structural optimization (BESO); numerical stability; mesh-independency; parameters study.

1. Introduction

Topology optimization is a method that helps designers to find a suitable structural layout for the required performances. It has attracted considerable attention in the last three decades. Various techniques have been developed for topology optimization for example the homogenization method (Bendsøe and Kikuchi 1988, Bendsøe and Sigmund 2003) and the evolutionary structural optimization method (Xie and Steven 1993, 1997). In the homogenization method the solution is obtained in the form of perforated composite material. Using Solid Isotropic Material with Penalization (SIMP) method the power-law interpolation penalizes intermediate densities to obtain a solution with nearly 0/1 material distribution (Rietz 2001).

The evolutionary structural optimization (ESO) method can be used to solve a variety of size, shape and topology optimization problems. The basic concept is that by slowly removing inefficient materials, the structure evolves towards an optimum. Bi-directional evolutionary structural optimization (BESO) is an extension of ESO which allows for efficient materials to be added to the structure at the same time as the inefficient ones to be removed (Yang *et al.* 1999, Querin *et al.* 2000). ESO/BESO methods have some attractive features: these methods are very simple to implement as a "post-processor" to commercial finite element analysis (FEA) software packages. The resulting optimal design provides a clear profile of topology (with no "grey" area) and therefore easy to manufacture. Theoretically, it is noted that the sequential linear programming

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(SLP)-based approximate optimization method followed by the Simplex algorithm is equivalent to ESO/BESO if the strain energy rejection criterion is utilized (Tanskanen 2002).

However, ESO/BESO methods with 0/1 formulations are ill-posed and may sometimes lead to non-convergent solutions (Bendsøe and Sigmund 2003). Thus, the steady states have been introduced in ESO (Xie and Steven 1993) and BESO (Querin *et al.* 2000) methods. However, the total iterations would increase significantly to obtain these steady states. The BESO method with perimeter control (Yang *et al.* 2003) is shown to be capable of obtaining somewhat convergent and mesh-independent solutions because of one extra constraint (the perimeter length) on the topology optimization problem. However, predicting the value of the perimeter constraint for a new design problem is a difficult task.

In stiffness optimization using the original BESO method, the volume of the structure is gradually reduced when a full design domain is used as an initial design, or the volume is increased and then decreased when a minimum domain is used as an initial design. The BESO procedure is stopped when the objective volume is reached for the first time or the second time. Obviously, the result from using this type of BESO procedure is problematic when a broken member with no or low strain energy happens to be a part of the final topology (it will be demonstrated in an example of the present paper). If the BESO procedure continues without varying the total volume, no stable convergent solution can be obtained. This is the reason that the evolutionary parameters must be very small to assure that the BESO method produces a reasonable solution. Even so, the selection of parameters leads to much confusion by users. For example, the BESO results from slowly removing or adding materials may actually be worse than those from quickly removing or adding materials for a given optimization problem.

This paper presents a modified and much improved BESO method by extending the work reported in (Yang *et al.* 1999, Querin *et al.* 2000). The new BESO method overcomes the non-convergence and mesh-dependency problems of the original BESO method. Two classical benchmark topology optimization problems: cantilever beam and Messcrschmidt-Bölkow-Blohm (MBB) beam, are used to demonstrate the numerical stability, mesh-independency and effects of evolutionary parameters of the new BESO method.

2. Sensitivity number

The conventional topology optimization methods often search for the best design of a structure that yields the stiffest structure with given volume of the material. In the evolutionary structural optimization method, a structure can be optimized by removing and adding elements. That is to say that, the element itself, rather than its associated physical parameters, is treated as the design variable. It is known that the removal of an element results in an increase in the mean compliance (Xie and Steven 1997)

$$\alpha_i = \frac{1}{2}\{u_i\}^T[K_i]\{u_i\} \tag{1}$$

where u_i is the nodal displacement vector of the ith element, $[K_i]$ is the element stiffness matrix and α_i is called the sensitivity number of ith element. To seek the stiffest design, elements with the lowest sensitivity numbers should be removed and elements with the highest sensitivity numbers should be added.

However, the sensitivity number for adding material cannot be obtained directly from the FEA results because the candidate elements for addition are not included in the analysis. There are several existing algorithms for adding material: linear extrapolation of displacement field (Yang *et al.* 1999) or adding the elements to over-stressed region directly (Querin *et al*. 2000). Here, a mesh-independent Gaussian-weighted filter (Murio 1993, Bruns and Tortorelli 2003) is used to evaluate sensitivity numbers for candidate elements of addition, although other filters may be implemented, e.g., linearly weighted kernel (Bendsøe and Sigmund 2003, Sigmund and Peterson 1998) and checkerboard suppression filter (Li *et al.* 2001). It will be shown that the Gaussian-weighted filter can effectively solve the mesh-dependency problem of ESO/BESO methods, as it has done in other optimization methods. Using the Gaussian-weighted filter, the sensitivity number for element i is given by

$$\alpha_i = \frac{\sum_{j=1}^{N} \omega(r_{ij})\alpha_i}{\sum_{J=1}^{N} \omega(r_{ij})} \tag{2}$$

where N is the total number of elements in the mesh and $\omega(r_{ij})$ is the weight factor given as

$$\omega(r_{ij}) = \frac{\exp\left(-\left(r_{ij}^2/2\left(\frac{r}{3}\right)^2\right)\right)}{2\pi(r/3)}, \qquad \{i \in N | r_{ij} \le r\}, \quad j = 1, 2, \ldots, N \tag{3}$$

where r_{ij} is defined as the distance between the centers of the elements i and j and r is the filter radius specified by the user. The above sensitivity filter serves two purposes: (1) to extrapolate sensitivity number within the full design domain and (2) to smooth the sensitivity number in the neighbourhood of each element.

To enhance the convergence of the BESO method, it is proposed to further improve the accuracy of the current sensitivity numbers by considering the deformation history of each element. A similar idea can be found in the Method of Moving Asymptotes (MMA) (Svanberg 1987) which uses the information from previous design iterations. A simple way to achieve this is to average the current sensitivity number with that of the previous iteration as

$$\alpha_i = \frac{\alpha_i^n + \alpha_i^{n-1}}{2} \tag{4}$$

where n is the current iteration number. Then let $\alpha_i^n = \alpha_i$ which will be used for next iteration. It can be seen that the updated sensitivity number includes its sensitivity information in the previous iterations.

3. Element removal/addition and convergence criterion

Before we remove and add elements in new design, the target volume for the next iteration (V_{i+1}) needs to be given first. Because the objective volume (V^*) can be more or less than the volume of the initial guess design, the target volume in each iteration may decrease or increase step by step until the objective volume is reached. Therefore

$$V_{i+1} = V_i(1 \pm ER) \qquad (i = 1, 2, 3, \ldots) \tag{5a}$$

where ER is called the evolutionary volume ratio. Once the objective volume is reached, the volume will be kept constant for the remaining iteration as

$$V_{i+1} = V^* \tag{5b}$$

Then sensitivity numbers of all elements, both solid (1) and void (0) are calculated as described in Section 2. The elements are sorted according to their values of the sensitivity number (from the highest to the lowest). For solid element (1), it will be removed (switched to 0) if

$$\alpha_i \le \alpha_{del}^{th} \tag{6a}$$

For void elements (0), it will be added (switched to 1) if

$$\alpha_i > \alpha_{add}^{th} \tag{6b}$$

where α_{del}^{th} and α_{add}^{th} are the threshold sensitivity numbers for removing and adding elements and $\alpha_{del}^{th} \le \alpha_{del}^{th}$. α_{del}^{th} and α_{add}^{th} are determined by the following three simple steps:

1. Let $\alpha_{add}^{th} = \alpha_{del}^{th} = \alpha_{th}$, thus α_{th} can be easily determined by V_{i+1}. For example, there are 1000 elements in design domain and $\alpha_1 > \alpha_2 > \ldots > \alpha_{1000}$ and if V_{i+1} corresponds to a design with 725 elements then $\alpha_{th} = \alpha_{726}$.
2. Calculate the admission volume ratio (AR), which is defined the number of added elements divided by the total number of elements in the design domain. If $AR \le AR_{max}$ where AR_{max} is a prescribed maximum volume addition ratio, skip step 3. Otherwise recalculate α_{del}^{th} and α_{add}^{th} as in step 3.
3. Calculate α_{add}^{th} by first sorting the sensitivity number of void elements (0). The number of elements to be switched from 0 to 1 will be equal to AR_{max} multiplied by the total number of elements in the design domain. α_{add}^{th} is the sensitivity number of the element ranked just below the last added element. α_{del}^{th} is then determined so that the removed volume is equal to ($V_{i+1} - V_i$ + the volume of the added elements).

It is noted that AR_{max} is introduced to ensure that not too many elements are added in a single iteration. Normally AR_{max} is greater than 1% so that it does not suppress the capability and advantages of adding elements.

The new element removal and addition scheme ranks all elements (void and solid) together, while in the original BESO methods (Yang *et al.* 1999, Querin *et al.* 2000) elements for removal and those for addition are treated differently and ranked separated, which is a bit cumbersome and not very logical.

The cycle of finite element analysis and element removal and addition continues until the objective volume (V^*) is reached and the following convergence criterion is satisfied

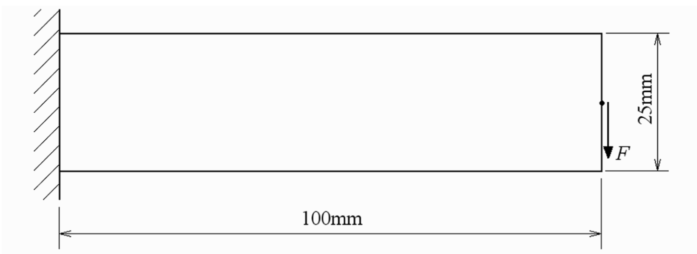
$$error = \frac{\left|\sum_{j=1}^{N} (C_{i-j+1} - C_{i-N-j+1})\right|}{\sum_{j=1}^{N} C_{i-j+1}} \le aerror \tag{7}$$

where C_i is the mean compliance for the structure in ith iteration, N is integral number normally from 2 to 5, and *aerror* is a allowable convergence error normally selected from 0.1% to 0.01%. A strict convergence criterion with N = 5 and *aerror* = 0.01% is used for the following examples, except for those for numerical stability study.

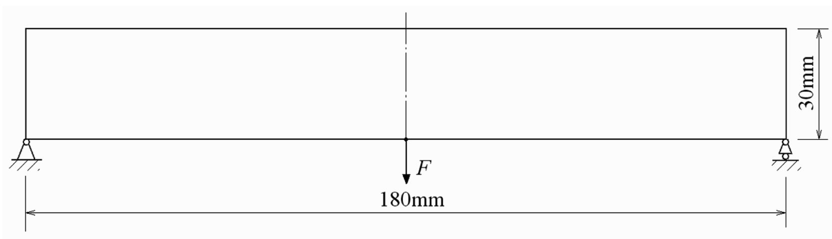
4. Numerical implementation

The evolutionary iteration procedure of the present BESO method is given as follows:

1. Discretize the design domain using an FE mesh for the given boundary and loading conditions. Assign the initial property values (0 or 1) of elements to construct initial design.
2. Perform finite element analysis on the design to obtain sensitivity information as in Eq. (1).
3. Filter the sensitivity number to the design domain and smooth the sensitivity information using the mesh-independent Gaussian-weighted filter as in Eq. (2).
4. Average the sensitivity number using its history information as in Eq. (4) then save the resulting sensitivity number for next iteration.
5. Determine the target volume for the next iteration using Eqs. (5a) or (5b).
5. Reset the property values of elements. For solid elements (1), the property value is switched from 1 to 0 if Eq. (6a) is satisfied. For void elements (0), the property value is switched from 0 to 1 if Eq. (6b) is satisfied. Then construct a new design using elements with property value 1 for the next finite element analysis.
6. Check the boundary and loading conditions for the new design.
7. Repeat steps 2-7 until the objective volume (V^*) is reached and convergence criterion in Eq.(7) is satisfied.



(a)



(b)

Fig. 1 Dimensions of the design domain and boundary and loading conditions: (a) cantilever beam, (b) MBB beam

5. Examples and discussion

In this section, a series of studies is conducted on two topology optimization problems. The first problem is a cantilever beam with tip load shown in Fig. 1(a). The second one is the MBB beam shown in Fig. 1(b). In both structures, the following parameters are adopted: modulus of elasticity $E = 200$ GPa, Poisson's ratio $v = 0.3$ and applied force $F = 10$ N. Plane stress conditions are assumed and four node elements are used. Due to symmetric properties in the second problem, only half of the structure is modelled and displayed. The objective volume is 50% of the design domain.

5.1 Stability of solution

To improve the convergence of the BESO method, we have modified the algorithm by considering the sensitivity history as shown in Eq. (4). We explore the numerical stability of solutions by applying the BESO method to the following examples and letting the optimization procedure run 150 iterations.

The BESO method without considering the sensitivity history is applied on the cantilever problem first. The BESO parameters are ER = 1%, $AR_{\max}$ = 2% and filter radius r = 1.75 mm. The evolutionary histories for mean compliance and volume are presented in Fig. 2 where the vertical dotted line indicates that the objective volume is satisfied afterward. Several topologies are also shown in Fig. 2. A number of observations can be drawn from this example. First, we notice that the topology obtained when the objective volume is firstly satisfied is not a good design because it has a large mean compliance (0.42 Nmm which is out of the plot range). This bad result is caused by a sudden break of two members. Second, as the BESO procedure run further, the resulting optimal solutions are unstable – they jump from one to another local optimum, despite that Gaussian-weighted filter has been used in this example.

Next we apply the BESO algorithm with the sensitivity history as given in Eq. (4) to the same example using the same BESO parameters as above. Fig. 3(a) shows the new histories of mean compliance and volume and some typical topologies. It can be seen that stable results including topology and mean compliance are obtained after the objective volume is reached. We have also

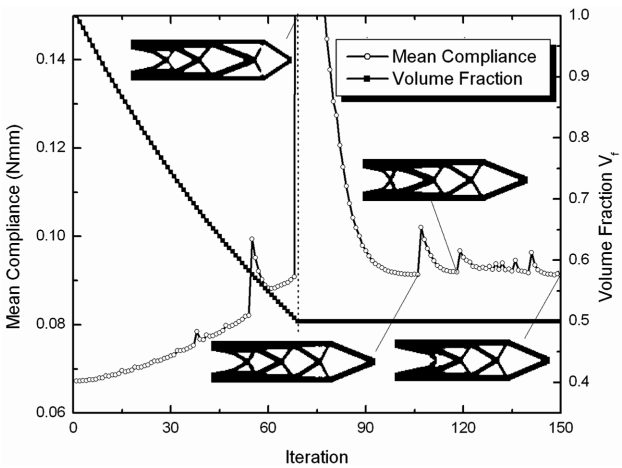
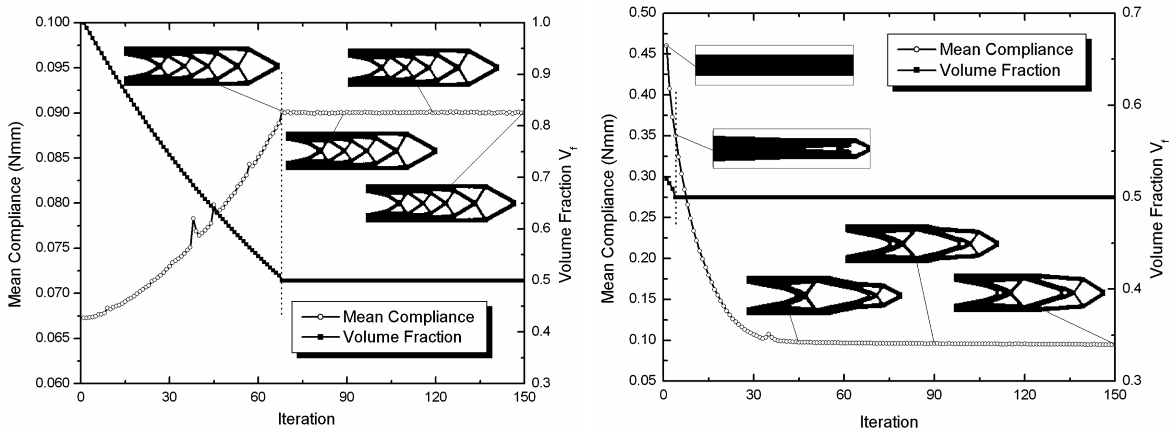
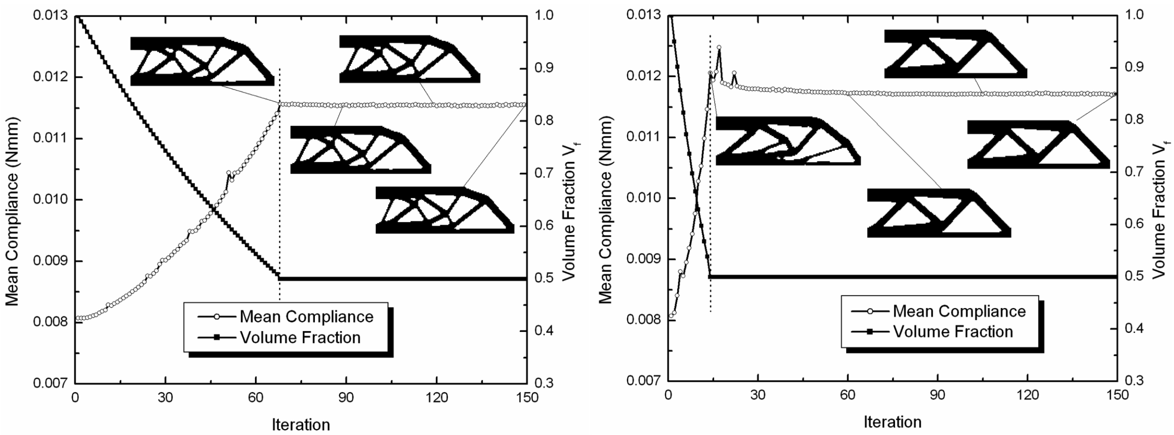


Fig. 2 Demonstration of an unstable solution using BESO without considering sensitivity history



(a) (b)

Fig. 3 Demonstration of stable solutions for cantilever beam: (a) BESO starting from full domain, (b) BESO starting from the middle-half design



(a) (b)

Fig. 4 Demonstration of stable solutions for MBB beam: (a) BESO with small evolutionary parameters, (b) BESO with large evolutionary parameters

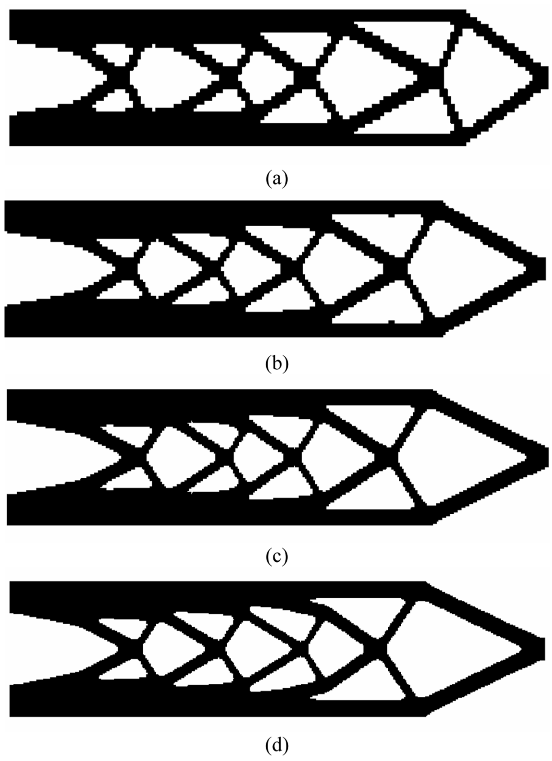
applied the revised BESO method to this structure with a different initial design which has a volume close to the objective volume. Fig. 3(b) shows the evolutionary histories of mean compliance, volume and topology. It can be seen that a convergent mean compliance is obtained after about 45 iterations. The topology evolves to a final solution stably but slowly (due to a small admission ratio). Details about the effect of admission ratio will be discussed later.

The revised BESO method has been further verified using other optimization problems such as the MBB beam. The BESO starts from the full design as shown in Fig. 1(b) and other optimization parameters are ER = 1%, $AR_{\max}$ = 2% and filter radius r = 2.3 mm. Fig. 4(a) shows the results of mean compliance, volume and topologies for 150 iterations. As expected, a stable topology and convergent mean compliance are obtained. To demonstrate that the convergent results do not come from low evolutionary values ER and $AR_{\max}$, we raise both ER and $AR_{\max}$ to 5%. The evolutionary results are shown in Fig. 4(b). The same conclusion can be drawn although the topology is not very good when the objective volume is first reached. But after the mean compliance oscillates for several iterations it converged to a stable solution.

It is not surprising that different solutions are obtained for a same problem because most problems in topology optimization design are not convex. The non-convexity typically means that one can find several different solutions to the same discretized problem when choosing different initial designs or different parameters of the algorithms. No qualitative difference could be found in these solutions in terms of their structural performance (e.g., the mean compliance). All following studies are based on the convergent solutions by incorporating Eqs. (4) and (7).

5.2 Mesh-independent solutions

One problem associated with topology optimization is that the optimal solution depends on the density of the finite element mesh, as observed in many FEA-based applications. The perimeter control technique in the BESO method has be investigated by Yang *et al.* (2003). However, it is difficult to pre-determine the appropriate value of the perimeter. The filter technique for solving the mesh-dependency problem has been widely used in optimization methods (Bendsøe and Sigmund 2003). In this paper, the Gaussian-weighted filter is used for the BESO method. Here the numerical tests are conducted on two examples with several different mesh densities for the finite element discretization.



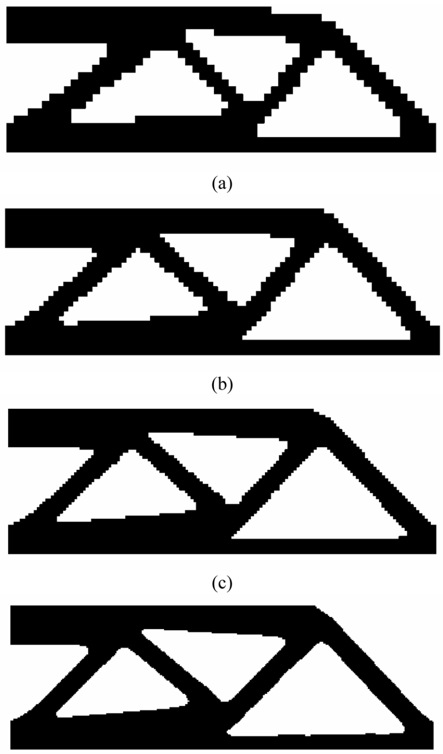
(a)

(b)

(c)

(d)

Fig. 5 Mesh-independent solutions for the cantilever beam: (a) 144 × 36, (b) 200 × 50, (c) 280 × 70, (d) 400 × 100



(a) (b) (c) (d)

Fig. 6 Mesh-independent solutions for the MBB beam: (a) 20 × 60, (b) 30 × 90, (c) 50 × 150, (d) 100 × 300

The first example is the cantilever beam with four different meshes: 144 × 36, 200 × 50, 280 × 70, 400 × 100. The same values of parameters ER = 1%, $AR_{\max}$ = 2% and r = 1.75 mm are used. The optimal topologies found are shown in Fig. 5, in which all meshes converge to the same optimal solution in terms of numbers of internal members. Next, four meshes are used for the MBB beam: 20 × 60, 30 × 90, 50 × 150, 100 × 300. The BESO parameters are ER = 5%, $AR_{\max}$ = 5% and r = 2.3 mm. Fig. 6 shows the convergent topologies for these different meshes. It can be seen that the revised BESO method can successfully obtain a mesh-independent design.

5.3 Effect of evolutionary ratio (ER) and maximum volume addition ratio (AR_{max})

Obviously a different evolutionary ratio (ER) defines a different BESO search path and may lead to a different solution. To investigate the effect of the evolutionary ratio, the cantilever beam problem is solved with various values of ER while keeping other BESO parameters as $AR_{\max}$ = 2%, r = 1.75 mm and the full design as the initial design. Fig. 7(a-d) shows the topologies with ER = 1%, 2%, 5% and 8% respectively. By inspection, it is noted that larger evolutionary ratios result in less members in the final topology. The corresponding mean compliances are 0.0900, 0.0904, 0.092 and 0.0926 Nmm. The performance of the solution is improved slightly with a small evolutionary ratio. It has been realized that the introduction of more holes, without changing the structure volume, will generally increase the efficiency of a given structure (Bendsøe and Sigmund 2003). The total iterations are 85, 50, 71 and 47 respectively. This means that a larger evolutionary ratio

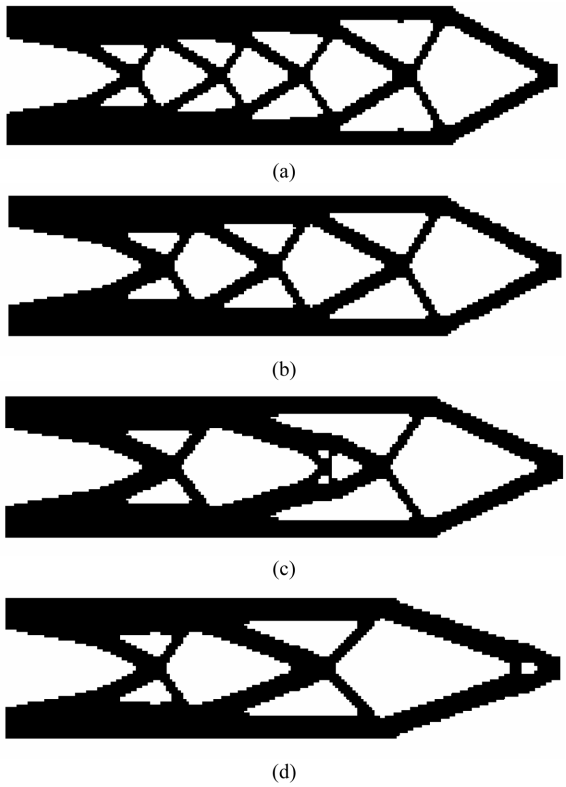


Fig. 7 Effect of evolutionary volume ratio *ER*: (a) *ER* = 1%; (b) *ER* = 2%, (c) *ER* = 5%, (d) *ER* = 8%

may not necessarily save the computational time because with a larger evolutionary ratio more iterations may be required to correct a larger deviation from an optimum after the objective volume is satisfied.

When BESO starts from the full design, the maximum volume addition ratio has no significant effect on the optimal solution except that if it is too small it could limit the recovery ability of the BESO method. $AR_{max} \geq 1\%$ is recommended.

It should be pointed out that the effects of *ER* and AR_{max} are also related to other BESO parameters such as the initial design and the filter radius. For example, when BESO starts from a minimum domain or initial design with a relatively low material fraction, the selection of the maximum addition ratio and evolutionary ratio can be rather tricky. A large evolutionary ratio or maximum addition ratio may cause singularity in the FEA model. For instance, if AR_{max} = 3% is used for the example shown in Fig. 3(b), singularity would occur in the FEA model. However, a small maximum addition ratio may slow down the convergence speed. Therefore a compromise needs to be found. Similarly, the filter radius also has an influence on the effect of *ER* and AR_{max}. For example, when a large filter radius is used, even a small *ER* would not result in a structure with much detail (which might have a high efficiency).

5.4 Effect of initial design and filter radius

It will be seen that the effect of the initial design depends on the specific optimization problem.

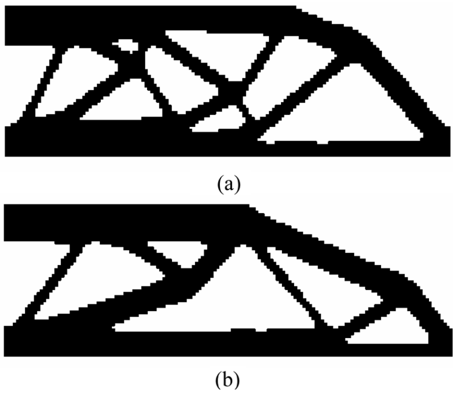


Fig. 8 Effect of the initial design: (a) starting from full domain, (b) starting from bottom half design

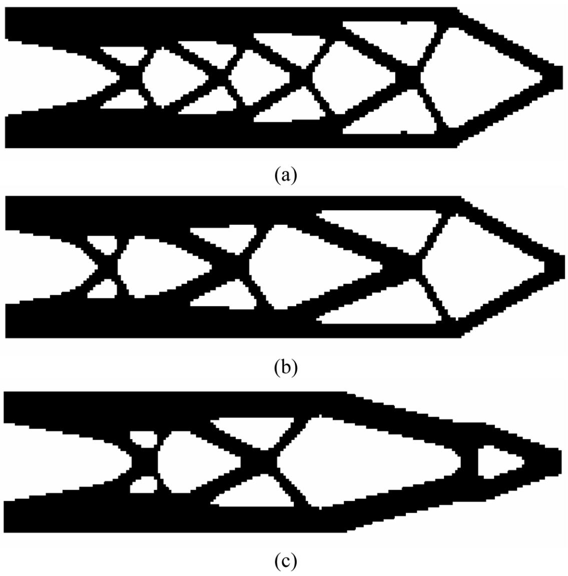


Fig. 9 Effect of filter radius r: (a) r = 1.75 mm, (b) r = 2.25 mm, (c) r = 2.75 mm

Figs. 3(a) and 3(b) already showed the effect of initial design on the cantilever beam. The total iterations are 85 for Fig. 3(a) and 118 for Fig. 3(b) when the convergence criterion with *aerror* = 0.01% is satisfied. It can be seen that BESO starting from the full design leads to a quick convergence and a low mean compliance design (0.0900 Nmm vs. 0.0954 Nmm). For MBB beam, the BESO topologies starting from the full design and the bottom half design are shown in Figs. 8(a) and (b) when ER = 1%, $AR_{\max}$ = 2% and r = 2.3 mm are used. The total iteration numbers are 90 vs. 67, mean compliance 0.0115 Nmm vs. 0.01187 Nmm, respectively. It is noted that BESO starting from the bottom half design can save the computational time significantly but at the expense of slightly lower performance of the final solution.

The filter in other optimization methods is only used to smooth sensitivity number or density to ensure mesh-independency. In the present BESO method, the filter is also used for creating sensitivity information for the candidate elements to be added. So the smallest filter radius should be at least the size of one element, otherwise, the present BESO become ESO without the capability of adding elements. For the cantilever problem, topology with less and less members can be found while the filter radius gradually increases as shown in Fig. 9.

5.5 Further discussion

As BESO uses a binary representation of structure (either 0 or 1), it is worth noting the following two points although the revised BESO method is far more sophisticated than original ESO/BESO.

1. Any isolation of a boundary or load that changes the nature of the problem should be detected and avoided during the optimization process, otherwise ESO/BESO may lead to a incorrect solution (Zhou and Rozvany 2001). This problem can easily detected and solved by FEA or BESO procedure in step 6. When such a problem is detected, using a finer mesh or replacing removed elements with "soft" elements to preserve the boundary or load can effectively fix the problem.
2. Improper parameters may causes non-convergence of the BESO solution such as using too coarse a mesh, extremely large ER, $AR_{\max}$ or filter radius.

6. Conclusions

The capability of the revised BESO method has been demonstrated through two examples. There are many new features in the present BESO method: averaging sensitivity number with its history, introducing mesh-independent filter, new element addition and removal algorithms, and a convergence criterion. Most importantly, it obtains convergent and mesh-independent solutions which the original BESO method could not.

Based on the convergent solutions, parameters studies show that large evolutionary ratio (ER) and filter radius (r) introduce less "holes" in the final topology and slightly reduce the efficiency of the design. BESO starting from the full domain always leads to better designs than from other initial designs although the computational time may be higher. Based on the authors' experience, the following values are suggested: $1\% \le ER \le 5\%$; $AR_{\max} \ge 1\%$; $r >$ one element length.

Acknowledgements

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