*Structural Engineering and Mechanics, Vol. 26, No. 6 (2007) 725-739* 725
DOI: http://dx.doi.org/10.12989/sem.2007.26.6.725

# A geometrically nonlinear thick plate bending element based on mixed formulation and discrete collocation constraints

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*(Received October 25, 2006, Accepted March 8, 2007)*

**Abstract.** In recent years there are many plate bending elements that emerged for solving both thin and thick plates. The main features of these elements are that they are based on mix formulation interpolation with discrete collocation constraints. These elements passed the patch test for mix formulation and performed well for linear analysis of thin and thick plates. In this paper a member of this family of elements, namely, the Discrete Reissner-Mindlin (DRM) is further extended and developed to analyze both thin and thick plates with geometric nonlinearity. The Von Kármán's large displacement plate theory based on Lagrangian coordinate system is used. The Hu-Washizu variational principle is employed to formulate the stiffness matrix of the geometrically Nonlinear Discrete Reissner-Mindlin (NDRM). An iterative-incremental procedure is implemented to solve the nonlinear equations. The element is then tested for plates with simply supported and clamped edges under uniformly distributed transverse loads. The results obtained using the geometrically NDRM element is then compared with the results of available analytical solutions. It has been observed that the NDRM results agreed well with the analytical solutions results. Therefore, it is concluded that the NDRM element is both reliable and efficient in analyzing thin and thick plates with geometric non-linearity.

**Keywords:** geometric nonlinear thick plate; Discrete Reissner-Mindlin; mixed formulation.

## 1. Introduction

Over the years there are many solutions that have been developed to solve thin and thick plate bending problems. The thin plate theory is based on the assumptions of Kirchhoff in 1850 (Timoshenko and Woinowsky-Krieger 1972). Kirchhoff assumptions were relaxed by Reissner (1945) and further relaxed by Mindlin (1951). Both relaxation is to accommodate thick plates where shear deformation was considered. Classical (analytical) solutions of plate problems were proposed by Navier, Kirchhoff and Levy (Timoshenko and Woinowsky-Krieger 1972, Levy 1942), and

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numerical solutions were derived by Galerkin (Oden 1967). The plate bending was one of the first problems that attracted the finite element researchers as depicted in the work of Zienkiewicz and Cheung (1964), Clough and Fellipa (1969) and Morely (1971). Recent development in this field is given by Zienkiewicz and Taylor (2000) and Bathe (1996).

Although plate bending was one of the first problems to receive considerable attention by the finite element researchers going back to the early sixties, nevertheless, the development of a plate element that accurately represents the behavior of both thin and thick plates remains an active topic of an ongoing research. The major difficulty of using finite element in plate bending was in obtaining the shape functions that satisfy compatibility conditions for convergence of finite element solutions in case of thin plates and do not give poor results due to shear locking in case of thick plate formulation. The major difficulty was not fully appreciated at that time and for these reasons and others finite element plate bending remains a topic of active research to the present day. Independent interpolation of rotations and displacements with discrete constraints (Zienkiewicz *et al.* 1990), the use of Lagrangian multiplier, the use of reduced and selective integration (Pawsey and Clough 1971, Zienkiewicz *et al.* 1971), the use of substitute shear strain fields (Hinton and Huang 1986) and the use of assumed-stress (Darilmaz 2005), among others, are some of the salient approaches used to overcome such major difficulties in formulating plate bending elements. As the result many successful elements have been developed over the years.

Among the large number of plate bending elements developed in recent years is a set of triangular and quadrilateral finite elements that were introduced by Zienkiewicz *et al.* (1990), Onate *et al.* (1992) and Taylor *et al.* (1993) which are based on mixed formulation. These elements managed to overcome the difficulties of continuity and shear locking phenomena. In this set of finite elements, the Discrete Reissner-Mindlin (DRM) element, is found to be of best performance in analyzing both thin and thick plate bending problems. The main features of the DRM element is the mixed interpolation adopted in its formulation and the use of line collocation constraints to substitute for shear in the total potential energy equation in terms of rotations and displacements. The characteristics and performance of the DRM element for both thin and thick elastic plates was introduced by Zienkiewicz *et al.* (1990). The elastic characteristics of the DRM element resting on elastic foundation were given in Abdalla *et al.* (2006). In this paper the element is further developed and extended to solve problems of thin and thick plates with geometric non-linearity.

## 2. Nonlinear plate elements

There are two major source of non-linearity in structural mechanics problems. Mainly: (1) material non-linearity (nonlinear stress-strain relations); and (2) geometric non-linearity (large displacements and/or large strains - nonlinear strain-displacement relations). For accurate determination of displacements and stresses, geometric and material non-linearity need to be considered in some plate bending problems. Also, strains due to membrane actions may significantly affect displacements and stresses in plate bending and should not be ignored.

The nonlinear analysis of plates and shells has been treated by classical (Fourier Series) methods by Chai (1980), Sundara (1966) and Shen (2000), finite difference method by Bhaumik (1967), third-order theory by Reddy (1990), finite strip method by Sheikh (2000), dynamic relaxation by Rushton (1972) and Turvey (1978), perturbation analysis by Thompson (1968) and for restricted classes of problems, by the use of Ritz procedure. However, partially as a result of its greater flexibility, the finite element method appears to be the most popular numerical approach for nonlinear analysis. The pioneering non-linear work on finite element is due to Turner *et al*. (1960). However, most of the earlier analysis related primarily to the linear buckling problems by Crisfield (1973).

Using the finite element method, Bathe and Bolourchi (1980) presented a displacement based approach for geometric and material nonlinear analysis of plates and shells. Likewise Pica, Wood and Hinton (1980) formulated a Mindlin plate bending element for geometrically nonlinear analysis of plates. Hughes and Lui (1981) and Dvorkin and Bathe (1984) developed nonlinear shell elements. Singh *et al*. (1994) used higher-order shear deformation theory to formulate a 4-node, 14 degrees of freedom $C^1$ continuous rectangular element to solve un-symmetrically laminated rectangular plates with different boundary conditions. Zhu *et al.* (1997) developed a nonlinear triangular thin plate element based on large deformation variational principle. They relaxed the inter-element continuity requirement and the element is formulated based on total Lagrangian assumption. The element is shear-locking free due to introduction of special element displacement function and the element showed good performance. Kim *et al.* (2003) developed a geometric nonlinear 6-node degenerated-shell element and a 12-node solid-shell element that were based on the assumed natural strain sampling scheme. The shear locking phenomena had been avoided and the element performed well.

Zhang *et al.* (2003, 2006) developed a geometric nonlinear non-conforming triangular plate element with drilling degrees of freedom using both total and updated Lagrangian approach and a displacement-based 4-node quadrilateral elements based on first order shear deformation theory and Von-Karman's large deflection theory employing total Lagrarangian approach. The elements were used to analyze plates with geometric nonlinearity and satisfactory results were obtained. By load perturbation of the linear equilibrium equations in their finite element formulation, Levy *et al.* (2001), derived the stiffness of a geometrically nonlinear three-node flat triangular shell element. Leung and Zhu (2004) presented a geometric nonlinear trapezoidal hierarchical Mindlin plate element for nonlinear vibration analyses of plates. They used Legendre orthogonal polynomials to enrich the shape functions and therefore to avoid the shear-locking phenomenon. Kere and Lyly (2005) formulated a plate element based on Reissner-Mindlin-Von Karman for analysis and design of laminated composite structures subjected to large deflection. Andrade *et al.* (2006) developed a 3D 8-node hexahedral isoparametric element for geometric nonlinear analysis of plates and shells. They avoided shear locking by using corotational system stress and strain components.

Filho *et al.* (2004), used the formulation for nonlinear analysis described in Liu *et al.* (1998), for the case of small strain/large rotation, and extended the 8-nodes hexahedral isoparametric finite element developed by Hu and Nagy (1997). They employed a uniform reduced integration method to free the element from shear locking. The modified element was used successfully for geometrically nonlinear static and dynamic analysis of plates and shells undergoing large displacements and rotations. A bibliography about recent development in geometric nonlinear analysis by the finite element method is given by Mackerle (1999). Among all these approaches, the mixed formulation with discrete collocation constraints is used here to develop a nonlinear Discrete Reissner-Mindlin plate element due to its success in solving linear thin and thick plate problems.

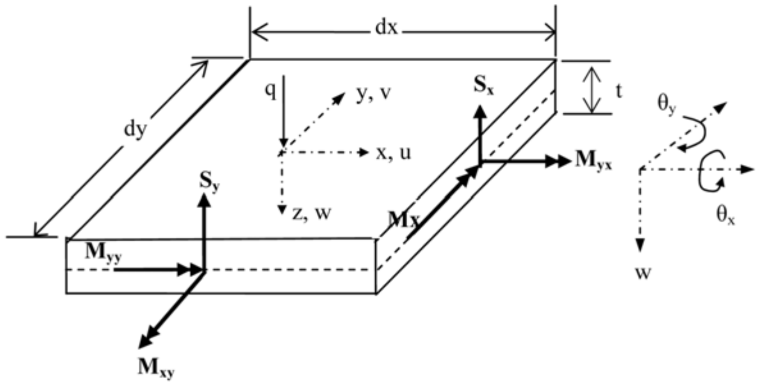


Fig. 1 Stress resultant in a thick plate element

## 3. Basic assumption in the development of DRM element for geometric nonlinearity

### 3.1 Equilibrium equations

The domain of the plate is given by

$$\Omega = \left\{(x, y) \in R^2 \middle| (x, y, z) \in R^3 \middle| z \in \left[-\frac{t}{2}, \frac{t}{2}\right]\right\}$$

where $t$ is the plate thickness.

Fig. 1 shows the stress resultant of a thick plate. The equilibrium equation for large displacement and large strain can be formulated as the sum of internal and external generalized forces and is given by a set of nonlinear equations as follows

$$\int_{\Omega}(\mathbf{B}_0 + \mathbf{B}_L(\hat{a}))^T \sigma d\Omega - \mathbf{f} = 0 \qquad (1)$$

$\mathbf{B}_0$ : strain-displacement transformation matrix for linear infinitesimal strain analysis
$\mathbf{B}_L(\hat{a})$ : strain-displacement transformation matrix for large displacement
$\hat{a}$ : nodal parameters-translation and rotation degrees of freedom
$\sigma$ : is the total stress due to small and large displacement
$\mathbf{f}$ : external generalized forces

Eq. (1) is valid whether displacement and/or strain are large or small.

### 3.2 Strain-displacement relations

When the middle surface displacements are large, the rotational strains and the in-plane strains are coupled. The membrane part of Von Kármán's strain-displacement relations for large deflection can be considered as a special case of the Green-Lagrange strains and the resulting strain-displacement relationship, in terms of middle surface strains and change in curvature, will be as given in Eq. (2) below (Zienkiewicz and Taylor 2000).

$$\varepsilon = \begin{Bmatrix} \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \dfrac{\partial v}{\partial y} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 \\ \left(\dfrac{\partial v}{\partial x} + \dfrac{\partial u}{\partial y}\right) + \left(\dfrac{\partial w}{\partial x}\right)\left(\dfrac{\partial w}{\partial x}\right) \end{Bmatrix} - z \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = \{ \varepsilon_0^P + \varepsilon_L^P \} + \{ \varepsilon_0^b \} \tag{2}$$

Where $u$ and $v$ are in-plane displacement components, while $w$ is the displacement component normal to the $x$-$y$ plane. For a plate subjected to out of plane loads only, the in-plane strain terms (due to $u$ and $v$) can be neglected and the total strains will contain strains due to linear curvature ($\varepsilon_0^b$) and strains due to nonlinear rotation ($\varepsilon_L^P$) associated with the out-of-plane (lateral) displacement only as given by Eq. (3).

$$\boldsymbol{\varepsilon} = \varepsilon_L^P + \varepsilon_0^b \tag{3}$$

where

$$\{ \varepsilon_L^P \} = \left\{ \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2, \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2, \left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right) \right\}^T = \frac{1}{2} \begin{bmatrix} \dfrac{\partial w}{\partial x} & 0 \\ 0 & \dfrac{\partial w}{\partial y} \\ \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial x} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{Bmatrix} = \frac{1}{2}\mathbf{A}\boldsymbol{\theta} = \frac{1}{2}\mathbf{A}\nabla \boldsymbol{w} \tag{4}$$

$$\boldsymbol{\varepsilon}_0^b = \begin{Bmatrix} -z\dfrac{\partial^2 w}{\partial x^2} \\ -z\dfrac{\partial^2 w}{\partial y^2} \\ -2z\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix} = -z \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 \\ 0 & \dfrac{\partial}{\partial y} \\ \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} \dfrac{\partial w}{\partial x} \\ \dfrac{\partial w}{\partial y} \end{bmatrix} = -z\mathbf{L}\boldsymbol{\theta} = -z\mathbf{L}\nabla\mathbf{w} \tag{5}$$

where

$$\mathbf{A}^T = \begin{bmatrix} \dfrac{\partial w}{\partial x} & 0 & \dfrac{\partial w}{\partial y} \\ 0 & \dfrac{\partial w}{\partial y} & \dfrac{\partial w}{\partial x} \end{bmatrix}, \quad \nabla^T = \begin{bmatrix} \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} \end{bmatrix}, \quad \mathbf{L}^T = \begin{bmatrix} \dfrac{\partial}{\partial x} & 0 & \dfrac{\partial}{\partial y} \\ 0 & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial x} \end{bmatrix}$$

Therefore, the total strain vector of Eq. (3) that includes the bending and the non-linear in-plane strain can be written in terms of strain transformation matrices and nodal parameters as

$$\boldsymbol{\varepsilon} = (\mathbf{B}_0 + \mathbf{B}_L)\hat{\mathbf{a}} = \varepsilon_0^b + \varepsilon_L^P = -z\mathbf{L}\boldsymbol{\theta} + \frac{1}{2}A\nabla w \tag{6}$$

### 3.3 Stress-strain relations

For linear material behavior the constitutive relation can be written as follows

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\varepsilon} - \boldsymbol{\varepsilon}_0) + \boldsymbol{\sigma}_0$$

where, $\mathbf{D}$ is the set of elastic constants, $\boldsymbol{\varepsilon}$ is the strain vector, $\boldsymbol{\varepsilon}_0$ is the initial strain vector and $\boldsymbol{\sigma}_0$ is the initial stress vector. Ignoring the initial strain and initial stress effect, the remaining stress-strain relationship will be as

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{D}(\varepsilon_0^b + \varepsilon_L^p) \tag{7}$$

## 4. Formulation of the non-linear Discrete Reissner-Mindlin element

The total potential energy including the effect of geometric non-linearity is as follows

$$\Pi = \frac{1}{2}\int_{\Omega}\boldsymbol{\varepsilon}^T\boldsymbol{\sigma}d\Omega + \frac{1}{2}\int_{\Omega}\boldsymbol{\gamma}^T\mathbf{S}d\Omega - \int_{\Omega}\mathbf{w}^T q d\Omega - \int_{\Gamma_t}\boldsymbol{\Theta}^T\tilde{M}d\Gamma - \int_{\Gamma_t}\mathbf{w}^T\tilde{S}d\Gamma \tag{8}$$

Where the terms above represent bending and in-plane strain energy due to linear curvature and non-linear rotation, shear strain energy, work due to applied load ($q$), work due to prescribed moment ($\tilde{M}$) and work due to prescribed shear ($\tilde{S}$), respectively.
Using Eqs. (6), (7) can be written as

$$\boldsymbol{\sigma} = \mathbf{D}\boldsymbol{\varepsilon} = \mathbf{D}\left(-z\mathbf{L}\boldsymbol{\theta} + \frac{\mathbf{A}}{2}\nabla\mathbf{w}\right)$$

Mixed interpolation formulation of displacement and rotation are given by

$$\mathbf{w} = \mathbf{N}_w\hat{\mathbf{w}}$$

$$\theta = \mathbf{N}_\theta\hat{\theta}$$

$$\mathbf{S} = \mathbf{N}_s\hat{\mathbf{S}} = \mathbf{N}_s[\mathbf{Q}_\theta\hat{\boldsymbol{\theta}} + \mathbf{Q}_w\hat{\mathbf{w}}]\alpha$$

$$\nabla\mathbf{w} = \left(\frac{\mathbf{S}}{\alpha} + \boldsymbol{\theta}\right) = \mathbf{N}_s\mathbf{Q}_w\hat{\mathbf{w}} + (\mathbf{N}_S\mathbf{Q}_\theta + \mathbf{N}_\theta)\hat{\theta} \quad \text{(shear constraint equation)}$$

where $\mathbf{N}_w$, $\mathbf{N}_\theta$, $\mathbf{N}_s$, $\mathbf{Q}_\theta$, $\mathbf{Q}_w$ are shape functions and transformation matrices, $\alpha$ is the shear rigidity and is given by $\alpha = \beta GtI$, $\beta = 5/6$ for rectangular section, $G$ is the modulus of rigidity and $I$ is an identity matrix.

Substituting these expansions of the variables in the total potential energy expression, Eq. (8), it is obtained

$$\Pi = \frac{1}{2}\int_{\Omega}\left\{\left(\mathbf{LN}_{\theta}\hat{\boldsymbol{\theta}} + \frac{\mathbf{A}}{2}(\mathbf{N}_s\mathbf{Q}_w\hat{\mathbf{w}} + (\mathbf{N}_S\mathbf{Q}_{\theta} + \mathbf{N}_{\theta})\hat{\theta})\right)^T\mathbf{D}\left(\mathbf{LN}_{\theta}\hat{\boldsymbol{\theta}} + \frac{\mathbf{A}}{2}(\mathbf{N}_s\mathbf{Q}_w\hat{\mathbf{w}} + (\mathbf{N}_S\mathbf{Q}_{\theta} + \mathbf{N}_{\theta})\hat{\theta})\right)\right\}d\Omega$$
$$+ \frac{1}{2}\int_{\Omega}\{(\mathbf{N}_S\mathbf{Q}_{\theta}\hat{\boldsymbol{\theta}} + \mathbf{N}_S\mathbf{Q}_w\hat{\boldsymbol{\theta}})^T\alpha(\mathbf{N}_S\mathbf{Q}_{\theta}\hat{\boldsymbol{\theta}} + \mathbf{N}_S\mathbf{Q}_w\hat{\mathbf{w}})\}d\Omega$$
$$- \int_{\Omega}(\mathbf{N}_w\hat{\mathbf{w}})^T q\,d\Omega - \int_{\Gamma_t}(\mathbf{N}_{\theta}\hat{\theta})^T\tilde{M}d\Gamma - \int_{\Gamma_t}(\mathbf{N}_w\hat{\mathbf{w}})^T\tilde{S}d\Gamma \tag{9}$$

The effect of geometric non-linearity assuming that in-plane strains ($\varepsilon_L^P$) are developed due to bending only are now included in Eq. (9). After minimization of the total potential energy of Eq. (9), collecting terms and expressing the equilibrium equations in matrix form, the following system of equations will be obtained.

$$\begin{bmatrix} \mathbf{K}_{\theta\theta} & \mathbf{K}_{\theta w} \\ \mathbf{K}_{w\theta} & \mathbf{K}_{ww} \end{bmatrix}\begin{Bmatrix} \hat{\boldsymbol{\theta}} \\ \hat{\mathbf{w}} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f}_{\theta} \\ \mathbf{f}_w \end{Bmatrix} \tag{10}$$

where

$$\mathbf{K}_{\theta\theta} = \int_{\Omega}\{(\mathbf{LN}_{\theta})^T\mathbf{D}(\mathbf{LN}_{\theta})\}d\Omega + \left(\frac{\mathbf{A}}{2}\right)^2\int_{\Omega}\{(\mathbf{N}_S\mathbf{Q}_{\theta} + \mathbf{N}_{\theta})^T\mathbf{D}(\mathbf{N}_S\mathbf{Q}_{\theta} + \mathbf{N}_{\theta})\}d\Omega$$
$$+ \int_{\Omega}\{(\mathbf{N}_S\mathbf{Q}_{\theta})^T\alpha(\mathbf{N}_S\mathbf{Q}_{\theta})\}d\Omega \tag{11}$$

$$\mathbf{K}_{w\theta} = \mathbf{K}_{\theta w}^T = 2\left(\frac{\mathbf{A}}{2}\right)\int_{\Omega}\{(\mathbf{LN}_{\theta})^T\mathbf{D}(\mathbf{N}_S\mathbf{Q}_w)\}d\Omega + 2\left(\frac{\mathbf{A}}{2}\right)^2\int_{\Omega}\{(\mathbf{N}_S\mathbf{Q}_{\theta} + \mathbf{N}_{\theta})^T\mathbf{D}(\mathbf{N}_S\mathbf{Q}_w)\}d\Omega$$
$$+ \int_{\Omega}\{(\mathbf{N}_S\mathbf{Q}_{\theta})^T\alpha(\mathbf{N}_S\mathbf{Q}_w)\}d\Omega \tag{12}$$

$$\mathbf{K}_{ww} = \left(\frac{\mathbf{A}}{2}\right)^2\int_{\Omega}\{(\mathbf{N}_S\mathbf{Q}_w)^T\mathbf{D}(\mathbf{N}_S\mathbf{Q}_w)\}d\Omega + \int_{\Omega}\{(\mathbf{N}_S\mathbf{Q}_w)^T\alpha(\mathbf{N}_S\mathbf{Q}_w)\}d\Omega \tag{13}$$

$$\mathbf{f}_{\theta} = \int_{\Gamma_t}\mathbf{N}_{\theta}^T\tilde{M}d\Gamma \tag{14}$$

$$\mathbf{f}_w = \int_{\Omega}\mathbf{N}_w^T q\,d\Gamma + \int_{\Gamma_t}\mathbf{N}_w^T\tilde{S}d\Gamma \tag{15}$$

To calculate the non-linear stiffness matrix of the DRM element (Eq. (10)), the integrals are evaluated numerically using three points Gauss Quadrature rule. In the above formulation the load vector is given for the uniformly distributed load $q$. It is clear that the contribution of the nonlinear terms has now been incorporated into Eq. (10).

## 5. The Discrete Reissner-Mindlin element

The Discrete Reissner-Mindlin Element was developed by Zienkiewicz *et al.* (1990) and Papadopoulos *et al.* (1990) for linear plate bending. It is a triangular element with three corner

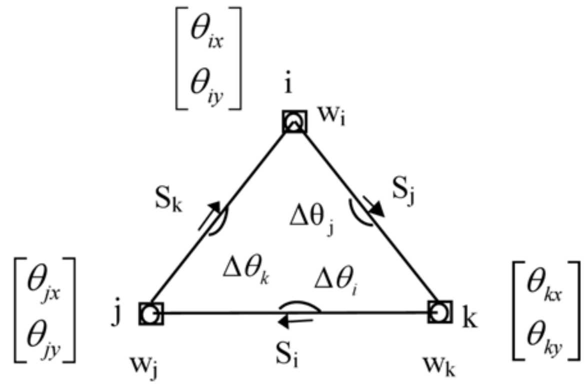

Fig. 2 The DRM element variables

nodes each has three degrees of freedom (T3D3), mainly, plane rotations, $\theta_x$, $\theta_y$ and a lateral translation $w$, i.e., no in-plane degrees of freedom are considered. A single shear variable $S$ and one hierarchical rotational degree of freedom $\Delta\theta$ were introduced on each of the element sides as shown in Fig. 2.

The element variables can be written in terms of nodal degrees of freedom and shape functions using triangular coordinate system as shown below

$$\boldsymbol{\theta} = \sum_{i=1}^{3} L_i \bar{\theta}^i + \sum_{i=1}^{3} 4 L_i L_j \Delta\bar{\theta}^k e_k = \mathbf{N}_\theta \hat{\theta} \tag{16}$$

where $e_k$ is a tangent vector

$$\mathbf{w} = \sum_{i=1}^{3} L_i \bar{w}^i = \mathbf{N}_w \hat{\mathbf{w}} \tag{17}$$

$$\mathbf{S} = \sum_{i=1}^{3} L_i \bar{S}^i = \mathbf{N}_s \hat{\mathbf{S}} = \mathbf{N}_s [\mathbf{Q}_\theta \hat{\boldsymbol{\theta}} + \mathbf{Q}_w \hat{\mathbf{w}}]\alpha \tag{18}$$

Where $\hat{\mathbf{w}}^T = [w_i \; w_j \; w_k]$, $\hat{\boldsymbol{\theta}}^T = [\theta_{ix} \; \theta_{iy} \; \theta_{jx} \; \theta_{jy} \; \theta_{kx} \; \theta_{ky} \; \Delta\theta_i \; \Delta\theta_j \; \Delta\theta_k]$ and $\mathbf{N}_\theta$, $\mathbf{N}_w$, $\mathbf{N}_s$, $\mathbf{Q}_\theta$ and $\mathbf{Q}_w$ are shape functions and transformation matrices. These matrices (Eqs. (16), (17) and (18)) were substituted in Eq. (11) through Eq. (13) to compute the elements of the NDRM stiffness matrix and in Eqs. (14) and (15) to compute the equivalent nodal load vectors.

## 6. Exact solutions

The Von Kármán equations for large displacement of isotropic plate as shown in Chai (1980) are given by

$$D\left(\frac{\partial 4w}{\partial x^4} + 2\frac{\partial 4w}{\partial x^2 \partial y^2} + \frac{\partial 4w}{\partial y^4}\right) = q + t\left(\frac{\partial 2w}{\partial x^2}\frac{\partial 2F}{\partial y^2} + \frac{\partial 2w}{\partial y^2}\frac{\partial 2F}{\partial x^2} - 2\frac{\partial 2w}{\partial x \partial y}\frac{\partial 2F}{\partial x \partial y}\right) \tag{19}$$

$$\left(\frac{\partial 4F}{\partial x^4} + 2\frac{\partial 4F}{\partial x^2 \partial y^2} + \frac{\partial 4F}{\partial y^4}\right) = E\left[\left(\frac{\partial 2w}{\partial x \partial y}\right)^2 - \left(\frac{\partial 2w}{\partial x^2}\right)\left(\frac{\partial 2w}{\partial y^2}\right)\right] \tag{20}$$

Where $F$ is the Airy stress function, $q$ is the applied uniformly distributed load, $D$ is the flexural rigidity, $E$ is the modulus of elasticity and $t$ is the plate thickness. Using generalized double Fourier Series (Sandra *et al.* 1966), it can be shown that for simply supported square plate subjected to uniformly distributed transverse loading, the lateral deflection $w$, after some manipulation, is given by Chai (1980)

$$0.06492\left(\frac{w}{t}\right)^3 + \frac{1}{3(1-\nu^2)}\left(\frac{w}{t}\right) - \frac{16q}{\pi^6 E}\left(\frac{a}{t}\right)^4 = 0 \tag{21}$$

For fixed edges square plate the lateral deflection $w$ is given by

$$0.06232\left(\frac{w}{t}\right)^3 + \frac{1}{10}\left(\frac{w}{t}\right) - 0.0016\frac{(1-\nu^2)q}{E}\left(\frac{a}{t}\right)^4 = 0 \tag{22}$$

This exact solution is used as a bench mark for comparison with the finite element solution.

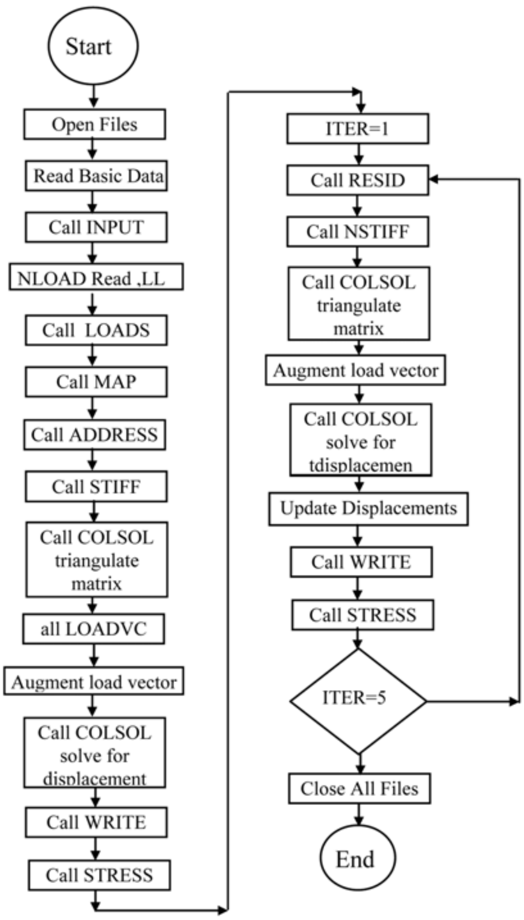


Fig. 3 Flow Chart for the NDRM computer program

## 7. Computer implementation and numerical results

A computer program to generate the stiffness matrix of the NDRM element, together with details needed for the assembly of the structure stiffness matrix and the solution of the nonlinear system of equation, is written as detailed in Ibrahim (1998). The integration of the terms of the element stiffness matrix is computed numerically using Gauss Quadrature and Radau integration constant with third order integration process. For the nonlinear solution the program computes the non-linear terms of the element stiffness matrix using the residual load. For each iteration process, a line search is conducted using the norm of the residual load vector in order to achieve convergence of solution in a finite number of iterations. The program is capable of solving linear and nonlinear plate bending problems. A detailed flow chart diagram for the program is shown in Fig. 3.

The input to the program is the number of elements in the $x$-direction, the number of elements in the $y$-direction, plate boundary conditions indicator for clamped or simply supported edges, young modulus, plate thickness, Poison's ratio and the incrementally applied load. The mesh is then generated automatically and the finite element analysis is carried out. The mesh sizes, number of elements and number of nodes used for testing and verification of the finite element solution is shown in Table 1. Fig. 4 shows samples of 2 × 2 and 4 × 4 finite element mesh.

The output of the program contains the elastic solution and the solutions of the different iterations of the non-linear calculations. The solutions consist of nodal displacements, mainly $x$-rotations, $y$-

Table 1 Elements and nodes of the different meshes

| Mesh | Number of elements | Number of nodes |
|---|---|---|
| 2 × 2 | 8 | 25 |
| 4 × 4 | 32 | 81 |
| 8 × 8 | 128 | 289 |
| 10 × 10 | 200 | 441 |

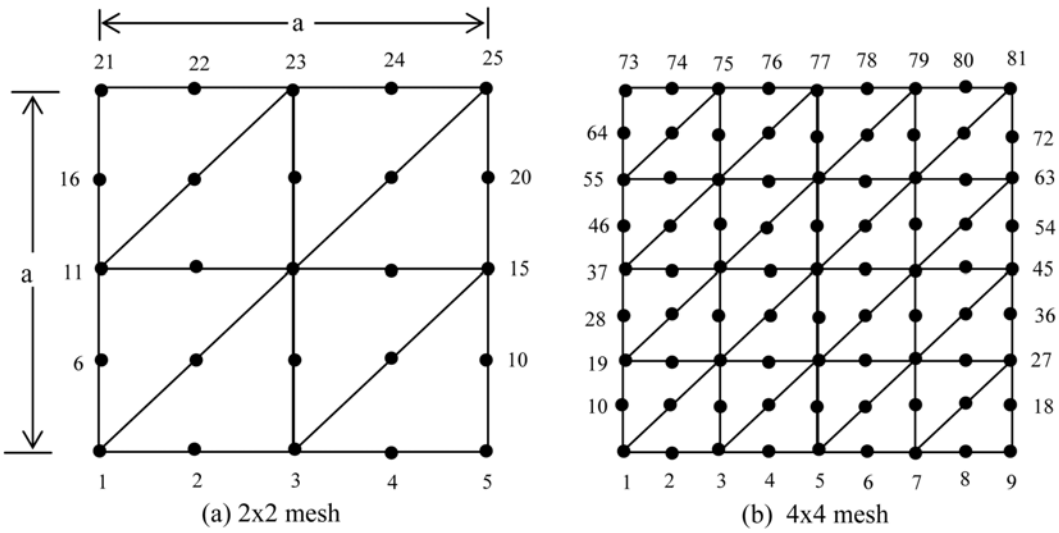


Fig. 4 2 × 2 and 4 × 4 finite element meshes

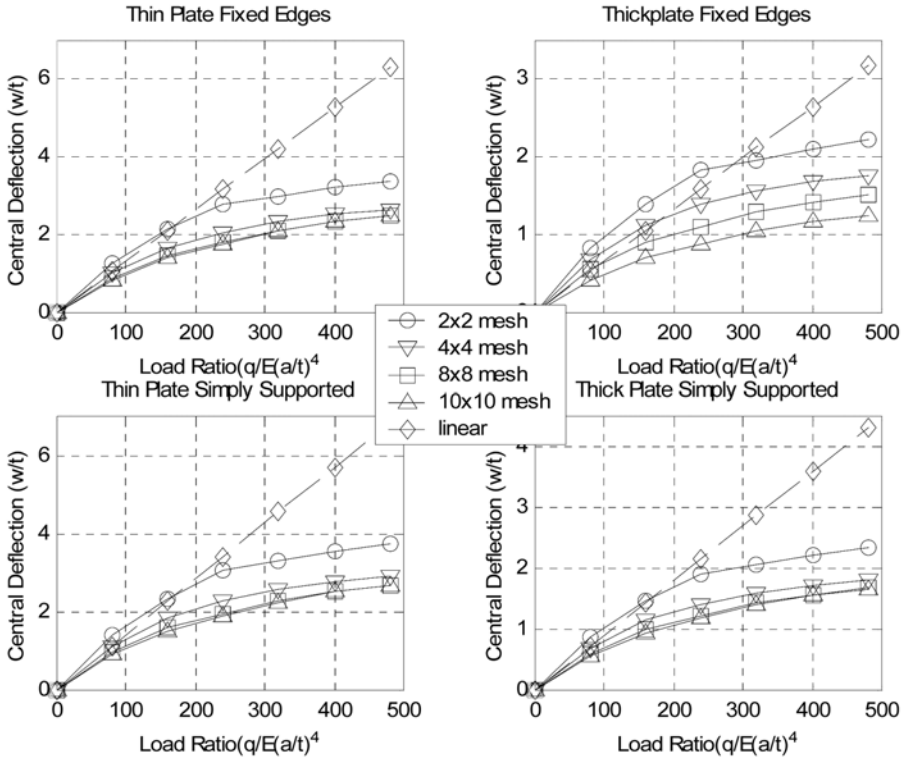


Fig. 5 Load-central deflection ($w/t$) of thin and thick plates (thin plate ($t/a$ = 0.05) and thick plate ($t/a$ = 0.5))

rotations and $z$-translations, the bending moments about $x$ and $y$ axes in each nodal point, and the shear forces.

The NDRM element has been tested for square plates with clamped edges and simply supported edges subjected to uniformly distributed loads. The uniformly distributed load ($q$) has been applied incrementally. The non-linear responses such as deflections and stress resultants have been calculated. The performance of the NDRM element for both thin and thick plates where the thickness of the plate was varied from $t/a = 0.05$ representing very thin plate to $t/a = 0.5$ representing very thick plates was investigated. An analytical solution using Von Kármán equations (Chai 1980, Sandra 1966) with Double Fourier Series method was used for verification of results.

Different finite element meshes were used ranging from a 2 × 2-mesh to a 10 × 10-mesh. Fig. 5 shows the linear and non-linear central deflections for thin and thick plates with clamped edges and simply supported edges for different mesh sizes. As the mesh is refined the non-linear solution departs from the linear solution. Fig. 6 shows the variation of central deflection response for a clamped edges plate with various plate thickness. Fig. 7 shows the variation of central deflection response for a simply supported edges plate with various plate thickness.

Table 2 shows a comparison between the result of the NDRM element and the analytical solution. Comparison of performance of the NDRM and that of the analytical solution in central deflection for simply supported edges plate is graphically shown in Fig. 8. It is observed that the NDRM element result is identical to that of the analytical solution for the linear part for small loads and deviate slightly for the nonlinear part as the load increases. Comparison of performance of the

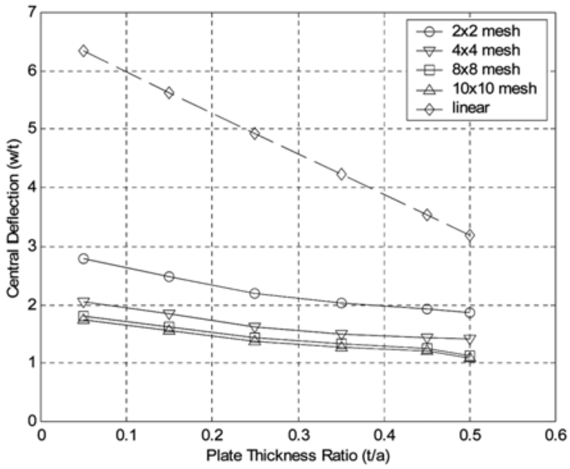


Fig. 6 Variation of central deflection with plate thickness for fixed edges plate ($q = 150$ KN/m$^2$)

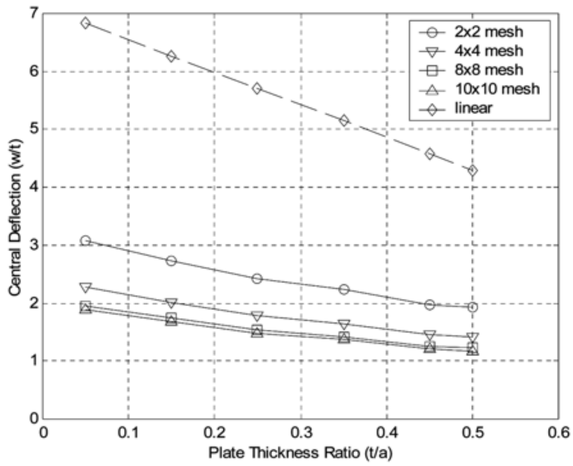


Fig. 7 Variation of central deflection with plate thickness for simply supported edges plate ($q = 150$ KN/m$^2$)

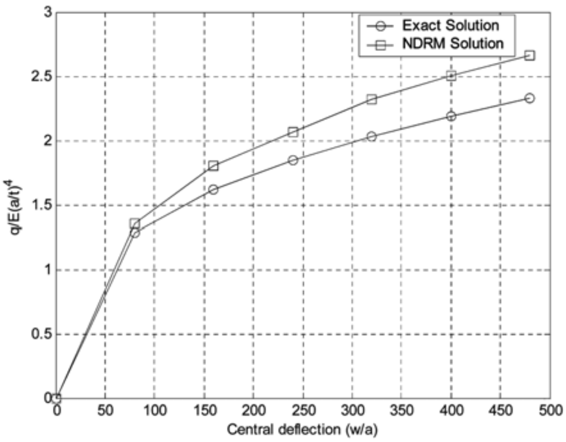


Fig. 8 Comparison of central deflection exact and NDRM-simply supported thin plate

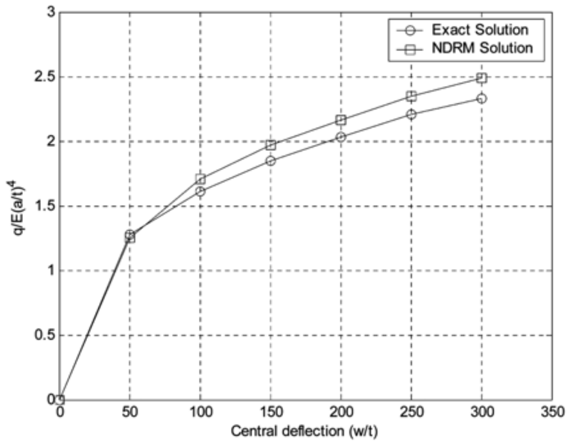


Fig. 9 Comparison of central deflection exact and NDRM-fixed edges thin plate

Table 2 Comparison of NDRM solution and exact Von Kármán solution for thin plates

| | Simply supported edges | | Fixed edges | |
|---|---|---|---|---|
| Load intensity $\frac{q}{E}\left(\frac{a}{t}\right)^4$ | Exact solution | NDRM solution (10 × 10) | Exact solution | NDRM solution (10 × 10) |
| 0 | 0 | 0 | 0 | 0 |
| 80 | 1.2818 | 1.3539 | 1.2258 | 1.2536 |
| 160 | 1.6150 | 1.8036 | 1.5494 | 1.7103 |
| 240 | 1.8488 | 2.0641 | 1.7759 | 1.9662 |
| 320 | 2.0349 | 2.3203 | 1.9561 | 2.1584 |
| 400 | 2.1920 | 2.5034 | 2.1081 | 2.3454 |
| 480 | 2.3294 | 2.6628 | 2.2409 | 2.4840 |

NDRM and that of the analytical solution in deflection for clamped edges plate is shown in Fig. 9. Similar trends to that of the simply supported plate were observed for the fixed edges plate.

## 8. Conclusions

In this study a non-linear thick plate element based on mixed formulation with discrete collocation constraints was developed. The element has been tested for thin and thick square plates with clamped edges and simply supported edges subjected to uniformly distributed transverse loads. The non-linear responses such as deflections have been calculated. The deflections obtained from the finite element solution were then compared with that of the analytical solutions. The following points have been concluded:

- The performance of the NDRM element, which has been tested for geometric nonlinear plates subjected to uniformly distributed load, was found to be quite acceptable as compared to the analytical solution based on Von Kármán. As expected, when the number of element increases the NDRM solution approaches the Von Kármán solution for nonlinear plates, i.e., it converges.
- As the load increases the discrepancy between the NDRM solution and the exact solution becomes apparent. This could be attributed to the assumption made regarding the exclusion of the in-plane strains due in-plane deformation ($u$, $v$) which becomes more significant as the load increases.
- The finite element analysis using the NDRM element handles plates of varying thickness ranging from $t/a$ = 0.05 (for very thin plates) to $t/a$ = 0.5 (for very thick plates). Therefore, the NDRM element can be use to analyze both thin and thick plates with no shear locking.
- In the formulation of the NDRM element discrete constraints are used to express the shear stress resultants parameters in terms of displacements parameters. Therefore, shear locking was avoided and no singularity was observed.
- As expected, the nonlinear displacement is smaller than the linear displacement due to membrane action where stress due to membrane action reduces deflection due to flexure.
- The nonlinear characteristics of the NDRM element have been formulated in this study, further extension of the DRM element to plates subjected to combined lateral and in-plane loading and plates with material non-linearity can be formulated in a similar manner.

## Acknowledgements

The authors would like to acknowledge Dr. Y. E-A. Mohamedzein for his input, valuable discussions and support during the thesis writing.

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